Cluster synchronization in a network of non-identical dynamic systems*

Wu Jian-She(吴建设)[a)†], Jiao Li-Cheng(焦李成)[a)‡], and Chen Guan-Rong(陈关荣)[b)§]

[a)] *Key Laboratory of Intelligent Perception and Image Understanding of Ministry of Education of China, Xidian University, Xi'an* 710071, *China*

[b)] *Department of Electronic Engineering, City University of Hong Kong, Hong Kong, SAR, China*

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Cluster synchronization in a network of non-identical dynamic systems is studied in this paper, using two-cluster synchronization for detailed analysis and discussion. The results show that the common intercluster coupling condition is not always needed for the diffusively coupled network. Several sufficient conditions are obtained by using the Schur unitary triangularization theorem, which extends previous results. Some numerical examples are presented for illustration.

Keywords: complex network, cluster synchronization, diffusive couplings, Schur's theorem

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1. Introduction

The phenomenon of dynamic synchronization exists ubiquitously in nature, which has a great potential application in diverse areas of science and technology.[1−4] Theoretical studies of the phenomenon include the synchronization of two identical or different systems[5−8] and the synchronization in various types of complex networks.[2,9−14] In particular, the synchronization in a complex network composed of identical dynamic systems has been extensively studied in the past decade.[15−18] Among several important and interesting topics, the question of how the synchronizability of a network is affected by the network topology has been carefully studied,[19−24] which is usually described by a constant coupling matrix[25−31] or more generally by a Laplacian matrix.[32,33]

Complete synchronization of a dynamic complex network can take the form of the global synchronization[34−37] or the cluster synchronization.[38−44] In the global synchronization, all the nodes of the network are mutually synchronized. The network may be split into several groups by certain partition rules. When the dynamic nodes synchronize each other in each group but no synchronization appears between any two different groups, cluster synchronization is observed. In Ref. [38], the existence and the stability of unconditional clusters in some networks with identical nodes were studied. An unconditional cluster is defined as the cluster that its existence does not depend on the state of the other nodes. The authors of Ref. [39] studied the transition from incoherent behaviour to global synchronization via a cluster synchronization regime as coupling strength increased. The problem of driving a general network to a selected cluster synchronization pattern by pinning control was studied in Ref. [40], where sufficient conditions were obtained to guarantee the realization of the cluster synchronization. The projective cluster synchronization of a drive-response dynamical network with coupled partially linear chaotic systems was studied in Ref. [41], in which it is shown that the projection cluster synchronization can be realized by controlling only one node in each cluster. In most of those studies considered was the cluster synchronization in dynamic networks with identical nodes. Noticeably, cluster synchronization in a network of non-identical dynamic systems is substantially different from and indeed much more difficult than in iden-

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†Corresponding author. E-mail: jshwu@mail.xidian.edu.cn

‡E-mail: lchjiao@mail.xidian.edu.cn

§E-mail: eegchen@cityu.edu.hk

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tical systems. On the other hand, the cluster synchronization in a network composed of non-identical systems is especially phenomenal, such as those in community networks with non-identical nodes.[44] Cluster synchronization in a network of non-identical systems is even more complex and difficult. Assuming that all the identical dynamical systems as nodes in a network form a sub-network, cluster synchronization means that all of the different sub-networks, each of which consists of identical dynamic systems, achieve synchronization individually within each sub-network, but typically the synchronous states of these sub-networks are mutually different, which is quite different from the "outer synchronization" of two coupled complex networks.[45] It is shown in Ref. [46] that cluster synchronization regimes persist when the chaotic oscillators have slightly different parameters. An effective pinning control scheme to realize the cluster synchronization in community networks was proposed in Ref. [44]. In Ref. [47], the relation between cluster synchronization and the unweighted graph topology for non-identical nodes is investigated and conditions guaranteeing cluster synchronization were provided, showing that the common intercluster coupling condition and the intracluster communication play key roles in cluster synchronization. Based on their analyses, adaptive feedback algorithms which can synchronize any bi-directed networks satisfying the conditions of common intercluster coupling and intracluster communication was proposed.[47]

In the present paper, the cluster synchronization of non-identical systems is further studied, showing that the common intercluster coupling condition is not needed for the diffusively coupled networks for cluster synchronization, this is quite different from the case of the networks studied in Ref. [47]. Several sufficient conditions for cluster synchronization are obtained by using the Schur unitary triangularization theorem, which extends the previous results. In particular, for a network consisting of two different types of dynamical systems, its sub-network cluster synchronization is called two-cluster synchronization. N-cluster (or multi-cluster) synchronizations are similarly defined. Since two-cluster synchronization is simple in notation and yet representative, in this paper the focus is on the two-cluster synchronization in a network composed of two different kinds of dynamic systems. The analytical results can be easily extended to the cases of N-cluster synchronization.

The rest of this paper is organized as follows. In Section 2, the concerned network model of two clusters of different dynamic systems is presented. In Section 3, we analyse the phenomenon of two-cluster synchronization, some criteria for two-cluster synchronization are obtained by using the Schur unitary triangularization theorem and some simulation examples are given for illustration.

2. Two-cluster network of non-identical systems

Cluster synchronization is studied in this paper, but for simplicity of notation and presentation, only a network with two clusters of different dynamic systems is considered, where the two different systems are named system-1 and system-2, respectively.

The network discussed here is assumed to be connected with totally $M+N$ nodes including M nodes of system-1, $x_1(t), \ldots, x_M(t)$ and N nodes of system-2, $x_{M+1}(t), \ldots, x_{M+N}(t)$, described respectively by

$$\dot{x}_i(t) = f_1(x_i(t)) + \sigma \sum_{j=1}^{M+N} g_{ij} A x_j(t), \quad i = 1, \ldots, M, \tag{1}$$

and

$$\dot{x}_i(t) = f_2(x_i(t)) + \sigma \sum_{j=1}^{M+N} g_{ij} A x_j(t), \quad i = M+1, \ldots, M+N, \tag{2}$$

where $x_i(t) \in R^n$ is the state vector of node i, $\dot{x}_i(t) = f_1(x_i(t))$ and $\dot{x}_i(t) = f_2(x_i(t))$ describe the dynamics of system-1 and system-2, respectively, with $f_1 \neq f_2$, $A = (a_{ij}) \in R^{n\times n}$ is a constant inner-coupling matrix, σ is a constant coupling strength, and $G = (g_{ij}) \in R^{(M+N)\times(M+N)}$ is a constant outer-coupling matrix with $g_{ij} \in R$ defined as follows:

(i) $1 \leq i \leq M$: if there is a connection from node j to node i, $j \neq i$ and $1 \leq j \leq M$, then $g_{ij} > 0$, while the whole sub-network-1 is diffusively connected in the sense of having internal energy balance:

$$g_{ii} = -\sum_{j=1, j\neq i}^{M} g_{ij}, \quad i = 1, \ldots, M; \tag{3}$$

and, for $M+1 \leq j \leq M+N$,

$$\sum_{j=M+1}^{M+N} g_{ij} = 0, \quad i = 1, \ldots, M; \tag{4}$$

(Note that equations (3) and (4) together imply that $\sum_{j=1}^{M+N} g_{ij} = 0$, i.e., matrix G has zero row-sums when

$1 \le i \le M$). All the nodes of system-1 and their couplings together form a sub-network, called sub-network-1 and all the nodes of system-2 and their couplings together form another sub-network, called sub-network-2. In physics, equation (4) indicates external energy balance of sub-network-1, that is, the sum of energy transmission from sub-network-2 to sub-network-1 is zero, though positive or negative energy may be transmitted from nodes of sub-network-2 to those of sub-network-1.

(ii) $M+1 \le i \le M+N$: if there is a connection from node j to node i, $i \ne j$ and $M+1 \le j \le M+N$, then $g_{ij} > 0$ and the whole sub-network is diffusively connected in the sense of having internal energy balance:

$$g_{ii} = -\sum_{j=M+1, j\ne i}^{M+N} g_{ij}, \quad i = M+1, \ldots, M+N; \tag{5}$$

and, for $1 \le j \le M$,

$$\sum_{j=1}^{M} g_{ij} = 0, \quad i = M+1, \ldots, M+N; \tag{6}$$

(Similarly, equations (5) and (6) together imply that $\sum_{j=1}^{M+N} g_{ij} = 0$ i.e., matrix G has zero row-sums when $M+1 \le i \le M+N$). Similar to Eq. (4), equation (6) indicates the external energy balance of sub-network-2, that is, the sum of energy transmission from sub-network-1 to sub-network-2 is zero, though positive or negative energy may be transmitted from nodes of sub-network-1 to those of sub-network-2.

The objective here is to achieve the network complete cluster synchronization, in the sense that all the nodes of system-1 completely synchronize to each other and all nodes of system-2 completely synchronize to each other, while the synchronous states of the two clusters of nodes are generally different, which are denoted by $s_1(t)$ and $s_2(t)$ respectively, with $s_1(t) \ne s_2(t)$. Clearly, if $s_1(t) = s_2(t)$ then this network complete cluster synchronization reduces to the regular network complete synchronization.

In a more precise description, if $||x_i(t) - s_1(t)|| \to 0$ for $i = 1, \ldots, M$ and $||x_i(t) - s_2(t)|| \to 0$ for $i = M+1, \ldots, M+N$, as $t \to \infty$, where $||\cdot||$ is the Euclidean norm of vectors, then sub-network-1 of system-1 and the sub-network-2 of system-2 are said to achieve two-cluster complete synchronization.

Note that, in general, these two synchronous states $s_1(t)$ and $s_2(t)$ can have equilibrium, periodic or even chaotic orbits. However, if they are both in equilibrium, namely $s_1(t) = s_1^* \in R^n$ and $s_2(t) = s_2^* \in R^n$ with $s_1^* \ne s_2^*$, then by taking time limit in Eq. (1) and then using the zero row-sum properties (3) and (4), one has $\dot{s}_1^* = f_1(s_1^*) = 0$; similarly, from Eqs. (2), (5) and (6), one has $\dot{s}_2^* = f_2(s_2^*) = 0$. For simplicity of notation and discussion, this particular case will be considered in the following, although with a little more effort most results below can be extended to the more general case with non-trivial orbits of $s_1(t)$ and $s_2(t)$.

Next, therefore, we denote the synchronous state vector of the entire two-cluster network by $S = (s_1^{*\mathrm{T}}, \ldots, s_1^{*\mathrm{T}}, s_2^{*\mathrm{T}}, \ldots, s_2^{*\mathrm{T}})^{\mathrm{T}} \in R^{n(M+N)}$, which is the synchronization manifold of the two-cluster network. Since the outer-coupling matrix G has zero row-sums, zero is an eigenvalue of G, denoted as $\lambda_1 = 0$, which is associated with eigenvector $(1, \ldots, 1)^{\mathrm{T}} \in R^{M+N}$. The outer-coupling matrix G may have complex eigenvalues in general ($\lambda_i \in C$) and Theorem 1 in Section 3 is general for cluster synchronization without requiring the eigenvalues to be real. However, the special case where G has only real eigenvalues ($\lambda_i \in R$) is very interesting . If $\lambda_i \in R$, without loss of generality they can be ranged in descending order as[48,49]

$$0 = \lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_{M+N}.$$

Note that in Eqs. (4) and (6) there is no requirement of nonnegative couplings due to the fact that negative energy transmission is allowed in our network, so the matrix G may not be irreducible in a general discussion. Therefore G may have more than one zero eigenvalue. Denote the largest non-zero eigenvalue of G by $\lambda_{\max}$ in the following.

To this end, by combining Eqs. (1) and (2) together, the entire network can be described by the following system of differential equations:

$$\dot{x}_i(t) = f(x_i(t)) + \sigma \sum_{j=1}^{M+N} g_{ij} A x_j(t), \quad i = 1, \ldots, M+N, \tag{7}$$

where $f(x_i(t)) = f_1(x_i(t))$. Equation (7) actually is Eq. (1) for $1 \le i \le M$, while $f(x_i(t)) = f_2(x_i(t))$ so equation (7) actually is Eq. (2) for $M+1 \le i \le M+N$. Furthermore, by using the Kronecker product, network (7) can be equivalently expressed in a more compact form as

$$\dot{X}(t) = f(X(t)) + \sigma(G \otimes A)X(t), \tag{8}$$

where $G \otimes A$ denotes the Kronecker product of matrices G and A,

$$X(t) = (x_1^{\mathrm{T}}(t), \ldots, x_{M+N}^{\mathrm{T}}(t))^{\mathrm{T}} \in R^{n(M+N)},$$

$$f(X(t)) = (f_1^{\mathrm{T}}(x_1(t)), \ldots, f_1^{\mathrm{T}}(x_M(t)), f_2^{\mathrm{T}}(x_{M+1}(t)), \ldots, f_2^{\mathrm{T}}(x_{M+N}(t)))^{\mathrm{T}} \in R^{n(M+N)}.$$

The coupling matrix of sub-netowrk-1 is denoted by G_{11} which is a sub-matrix of the outer-coupling matrix G. Similarly, the coupling matrix of sub-netowrk-2 is denoted by G_{22} which is also a sub-matrix of the outer-coupling matrix G. As a result, the outer-coupling matrix G can be decomposed into four blocks as follows:

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \tag{9}$$

where $G_{11} \in R^{M\times M}$, $G_{12} \in R^{M\times N}$, $G_{21} \in R^{N\times M}$ and $G_{22} \in R^{N\times N}$, with G_{12} describing the couplings from sub-network-2 to sub-network-1 and G_{21} denoting the couplings from sub-network-1 to sub-network-2 and both being sub-matrices of the outer-coupling matrix G. In the case of $G_{12} = G_{21} = 0$, the entire network is split into two separated self-connected sub-networks.

The eigenvalues of G_{11} are denoted by λ_i^{11}, $i = 1, \ldots, M$. Hereafter, assume that the off-diagonal elements of G_{11} are nonnegative, thus it is irreducible. According to Lemma 2 in Refs. [48] and [49], the real parts of the eigenvalues of G_{11} are non-positive and zero is an eigenvalue of G_{11} with multiplicity one associated with eigenvector $(1, \ldots, 1)^{\mathrm{T}} \in R^M$. For simplicity, let $\lambda_1^{11} = 0$. If the coupling matrix G_{11} has only real eigenvalues, then they are denoted as

$$\lambda_i^{11} = 0 > \lambda_2^{11} \geq \ldots \geq \lambda_M^{11}. \tag{10}$$

Similarly, the eigenvalues of G_{22} are denoted as λ_i^{22}, $i = 1, \ldots, N$. Also assume that the off-diagonal elements of G_{22} are nonnegative, thus it is irreducible. Then, the real parts of the eigenvalues of G_{22} are non-positive and zero is an eigenvalue of G_{22} with multiplicity one associated with eigenvector $(1, \ldots, 1)^{\mathrm{T}} \in R^N$. Likewise, always assume $\lambda_1^{22} = 0$. If G_{22} has only real eigenvalues, then they are denoted as

$$\lambda_i^{22} = 0 > \lambda_2^{22} \geq \cdots \geq \lambda_N^{22}. \tag{11}$$

There are no requirements that the off-diagonal elements of G_{12} and G_{21} are nonnegative, thus the outer-coupling matrix G of the entire network still may not be irreducible.

3. Analyses about two-cluster synchronization

As mentioned above, since two-cluster synchronization is simple in notation, it is representative. Only two-cluster synchronization in network (7) is discussed. We analyse the phenomenon of two-cluster synchronization in several cases.

Case 1 The simplest case. Sub-network-1 and sub-network-2 are completely synchronizable individually to s_1^* and s_2^*, respectively, and $G_{21} = 0$ or $G_{12} = 0$. In this case, analysis shows that network (7) is of two-cluster completely synchronization also to s_1^* and s_2^*.

Suppose that $G_{12} = 0$, then sub-network-2 will not affect the synchronizability of sub-network-1. By assumption, sub-network-1 is completely synchronizable therefore will self-synchronize to its equilibrium s_1^*. No matter whether sub-networ-1 affects sub-network-2 in the synchronization process, as soon as sub-network-1 achieves its complete synchronization to s_1^*, it follows from Eq. (6) that the complete synchronization of sub-network-2 to s_2^* will also be achieved, which is independent of the effect of sub-network-1. The case of $G_{21} = 0$ is similar to that of $G_{12} = 0$.

Case 2 The general case. Whether the network is two-cluster synchronizable depends on the non-zero eigenvalues of the outer-coupling matrix. We have the following criterion for two-cluster synchronization.

Theorem 1 The synchronization manifold S of the two-cluster network (7) is asymptotically stable if the zero solutions of the following two time-varying single-node systems are asymptotically stable for both cases of $f(s^*) = f_1(s_1^*)$ and $f(s^*) = f_2(s_2^*)$:

$$\dot{w}(t) = (Df(s^*) + \sigma\lambda_i A)\, w(t),\ i = 2, \ldots, M+N, \tag{12}$$

where $Df_1(s_1^*)$ is the Jacobian of $f_1(x_i(t))$ at $x_i(t) = s_1^*$ and $Df_2(s^*)$ is the Jacobian of $f_2(x_i(t))$ at $x_i(t) = s_2^*$, with $w(t) \in C^n$.

The proof of the Theorem 1 is standard but lengthy. We shall present it in Appendix A of this paper for convenience of readers but here we only discuss its physical meanings. Theorem 1 indicates that for a network featuring reducible or irreducible and symmetric or asymmetric outer-coupling matrix, if the zero solution of Eq. (12) is asymptotically stable for all the non-zero eigenvalues of G, then the network is of two-cluster synchronization. In the general case, the eigenvalues may be scattered in the complex plane;

we need to examine all the evolutionary dynamics of Eq. (12) for non-zero eigenvalues. In the case that the outer-coupling matrix has only real values, the situation can be dramatically simplified.

Case 3 The outer-coupling matrix has only real eigenvalues.

A complex network with a symmetric outer-coupling matrix is very important and interesting, and has only real eigenvalues. If the outer-coupling matrix of network (7) has only real eigenvalues and the symmetric form of A is positive-definite, namely $A^s = \frac{1}{2}(A + A^{\mathrm{T}}) \succ 0$, then it is not necessary to verify all the systems of Eq. (12) with non-zero eigenvalues. Only the stability of the one system of Eq. (12) with the largest non-zero eigenvalue is enough.

Corollary 1 If $\lambda_i \in R$, $i = 2, \ldots, M+N$, and $A^s \succ 0$, then the two-cluster synchronization manifold of network (7) is asymptotically stable if the zero solution of the following time-varying system is asymptotically stable for both $f(s^*) = f_1(s_1^*)$ and $f(s^*) = f_2(s_2^*)$:

$$\dot{w}_{\max}(t) = (Df(s^*) + \sigma\lambda_{\max}A)\, w_{\max}(t), \qquad (13)$$

where $\lambda_{\max}$ is the largest non-zero eigenvalue of matrix G, and $w_{\max}(t) \in R^n$. The detailed proof of Corollary 1 is presented in Appendix B.

Corollary 1 indicates that the two-cluster synchronizability of network (7) is determined by the largest non-zero eigenvalue $\lambda_{\max}$ of matrix G, if all $\lambda_i \in R$ and $A^s \succ 0$. This is due to the fact that the stability of the zero solution of Eq. (13) with the largest non-zero eigenvalue is sufficient to guarantee the stability of all the other equations in expression (12) with non-zero eigenvalues.

On the other hand, as to the case of $A^s \prec 0$ (negative-definite), similar to Corollary 1, it is also not necessary to verify all the systems in expression (12) with non-zero eigenvalues. Only the stability of the one system in expression (12) with the smallest non-zero eigenvalue can guarantee the stability of all the other systems in expression (12).

Corollary 2 If $\lambda_i \in R$, $i = 2, \ldots, M+N$, and $A^s \prec 0$, then the two-cluster synchronization manifold of network (7) is asymptotically stable if the zero solution of the following time-varying system is asymptotically stable for both $f(s^*) = f_1(s_1^*)$ and $f(s^*) = f_2(s_2^*)$:

$$\dot{w}_{M+N}(t) = (Df(s^*) + \sigma\lambda_{M+N}A)\, w_{M+N}(t),$$
$$w_{M+N}(t) \in R^n. \qquad (14)$$

The proof is similar to that of Corollary 1 and hence is omitted here.

Corollary 2 indicates that the two-cluster synchronizability of network (7) is determined by the smallest eigenvalue λ_{M+N} of matrix G.

In general, regardless of the definiteness of A^s, if all the eigenvalues of the matrix G are real, it is also not necessary to verify the stability of all the systems in expression (12). It will suffice to verify the stability of the zero solution of system (12) for only the largest non-zero eigenvalue $\lambda_{\max}$ and the smallest eigenvalue λ_{M+N} of matrix G. In other words, if the zero solution of system (12) is stable for both $\lambda_i = \lambda_{\max}$ and $\lambda_i = \lambda_{M+N}$, then it will be stable for all the other eigenvalues of matrix G.

Corollaries 1 and 2 are similar to Corollaries 1 and 2 in Ref. [50], which states that complete synchronization in network of identical systems is determined by the largest non-zero eigenvalue when $A^s \succ 0$ and is determined by the smallest eigenvalue when $A^s \prec 0$.

It can be easily noticed from Eqs. (4) and (6) that after network (7) achieves two-cluster synchronization, the coupling from all nodes of a sub-network to any node of another sub-network becomes zero. In other words, after network (7) achieves two-cluster synchronization, the two sub-networks achieves complete synchronization, so mutual influence no longer exists. Thus the evolving process to two-cluster synchronization is a process from two mutually coupled sub-networks to two "uncoupled" sub-networks and the two sub-networks are completely synchronized themselves. The external energy balance conditions (4) and (6) are necessary to guarantee the stability of the state that two sub-networks synchronized individually but no influence on each other. However, the converse may not be true: the individual complete synchronization of the two sub-networks does not necessarily guarantee the two-cluster synchronization of the entire network.

In Theorem 1 and its corollaries, the synchronizability of the network is estimated by the eigenvalues of the outer-coupling matrix. Eigenvalue estimation is the open research problems for many matrices.[51] From Corollaries 1 and 2 of Theorem 1, the eigenvalues of a matrix can be estimated by the two-cluster synchronizability of the network described by the matrix as follows.

Corollary 3 If $\lambda_i \in R$, $i = 2, \ldots, M+N$, then

$$\lambda_{\max} \geq \lambda_2^{\max}, \lambda_{M+N} \leq \lambda_{M,N}^{\min},$$
$$\lambda_2^{\max} = \max\left\{\lambda_2^{11}, \lambda_2^{22}\right\},$$

$$\lambda_{M,N}^{\min} = \min\left\{\lambda_M^{11}, \lambda_N^{22}\right\}, \tag{15}$$

where $\lambda_{\max}$ and λ_{M+N} are the largest non-zero eigenvalue and the smallest eigenvalue of matrix G, respectively, λ_2^{11} and λ_2^{22} are the second largest eigenvalues of G^{11} and G^{22}, respectively and λ_M^{11} and λ_N^{22} are the smallest eigenvalues of G^{11} and G^{22}, respectively.

The correctness of Corollary 3 is due to the fact that the two-cluster synchronization of the entire network guarantees the complete synchronization of the two sub-networks, but the converse may not be true.

Specially, if $G_{21} = 0$ or $G_{12} = 0$, the complete synchronization of the two sub-networks is sufficient to ensure the two-cluster synchronization of the entire network by the discussion of Case 1. Therefore, by Theorem 1 and the corollaries, one has

$$\lambda_{\max} = \lambda_2^{\max}, \quad \lambda_{M+N} = \lambda_{M,N}^{\min}. \tag{16}$$

To illustrate these conclusions for eigenvalue estimation, a simple example of a network with two kinds of different systems is shown in Fig. 1, where the coupling matrix G_{11} of sub-network-1 is

$$G_{11} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

which has eigenvalues $\lambda_1^{11} = 0$, $\lambda_2^{11} = \lambda_3^{11} = -3$. The coupling matrix G_{22} of sub-network-2 is

$$G_{22} = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix},$$

which has eigenvalues $\lambda_1^{22} = 0$, $\lambda_2^{22} = \lambda_3^{22} = -2$ and $\lambda_4^{22} = -4$. Thus, the overall outer-coupling matrix of the entire network is

$$G_{=} \begin{pmatrix} -2 & 1 & 1 & 1 & 0 & -1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -0.7639$, $\lambda_4 = -2$, $\lambda_5 = \lambda_6 = -3$ and $\lambda_7 = -5.2361$. Here, it is noticed that $\lambda_2^{\max} = -2$, $\lambda_{3,4}^{\min} = -4$, $\lambda_{\max} = \lambda_3 > \lambda_2^{\max}$ and $\lambda_{M+N} = \lambda_7 < \lambda_{3,4}^{\min}$, consistent with Corollary 3.

In the network shown in Fig. 1, if the couplings are changed such that $G_{12} = 0$, with G_{11}, G_{21} and G_{22} unchanged, then the outer-coupling matrix has eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = -2$, $\lambda_5 = \lambda_6 = -3$, and $\lambda_7 = -4$. In this case, $\lambda_{\max} = \lambda_3 = \lambda_4 = \lambda_2^{\max}$ and $\lambda_{M+N} = \lambda_7 = \lambda_{3,4}^{\min}$, consistent with expression (16).

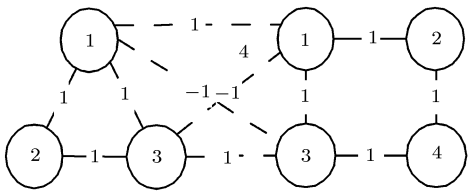


Fig. 1. A network with two kinds of different systems, where sub-network-1 has three nodes of system-1 and sub-network-2 has four nodes of system-2.

It can be similarly verified that if the outer-coupling matrix is changed such that $G_{21} = 0$, with G_{11}, G_{12}, and G_{22} unchanged, then the result is consistent with expression (16) too.

This example also motivates the following concept and study.

Definition 1 Given two sub-networks: sub-network-1 of system 1 with coupling matrix G_{11} and sub-network-2 of system 2 with coupling matrix G_{22}. By exchanging the two systems at the nodes of the network, two new sub-networks are obtained: sub-network-3 of system 1 with coupling matrix G_{22} and sub-network-4 of system 2 with coupling matrix G_{11}. If the synchronization manifolds of all the four sub-networks are asymptotically stable, then sub-network-1 and sub-network-2 are said to be exchangeably synchronizable.

Based on the exchangeably synchronizable concept, another sufficient condition for two-cluster synchronization is derived based on the complete synchronization of the two sub-networks of identical systems.

Theorem 2 If two sub-networks are exchangeably synchronizable, with $\lambda_i \in R$, $i = 2, \ldots, M+N$ and expression (16) holds, then the two-cluster synchronization manifold of network (7) is asymptotically stable.

Detailed mathematical proof of Theorem 2 can be found in Appendix C of this paper, here, we only discuss its physical meaning. Because of expression (16), the range of eigenvalues of the outer-coupling matrix of the entire network is not enlarged compared with that of the two sub-networks, so the two-cluster

synchronization of entire network can be guaranteed by the synchronization of all the four sub-networks. Without expression (16), the two-cluster synchronization of the entire network can guarantee the synchronization of its two sub-networks, but conversely, even the synchronization of the four sub-networks cannot guarantee the two-cluster synchronization of the entire network.

In the following, a simple example is given to illustrate and verify Theorem 2.

Consider a 20-node network of two different systems, i.e., the Lorenz system[52] and the Chen system.[53] The Lorenz system is described by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - xz - y, \\ \dot{z} = xy - bz, \end{cases}$$

which has a chaotic attractor when $a = 10$, $b = 8/3$, and $c = 28$; the Chen system is described by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = xy - bz, \end{cases}$$

which has a chaotic attractor when $a = 35$, $b = 3$ and $c = 28$.

Sub-network-1 is a nearest-neighbour coupled 10-node network with the coupling matrix

$$G_{11} = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 1 \\ 1 & -2 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \ddots & \ddots & 1 & -2 & 1 \\ 1 & 0 & \dots & 0 & 1 & -2 \end{pmatrix},$$

which has eigenvalues $\lambda_1^{11} = 0$, $\lambda_2^{11} = \lambda_3^{11} = -0.382$, $\lambda_4^{11} = \lambda_5^{11} = -1.382$, $\lambda_6^{11} = \lambda_7^{11} = -2.618$, $\lambda_8^{11} = \lambda_9^{11} = -3.618$, and $\lambda_{10}^{11} = -4$.

Sub-network-2 is a star-shape 10-node network with the coupling matrix

$$G_{22} = \begin{pmatrix} -9 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 0 & \ddots & 0 & 0 \\ 1 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 & 0 \\ 1 & \vdots & \ddots & 0 & -1 & 0 \\ 1 & 0 & \dots & 0 & 0 & -1 \end{pmatrix},$$

which has eigenvalues $\lambda_1^{22} = 0$, $\lambda_2^{22} = \cdots = \lambda_9^{22} = -1$, and $\lambda_{10}^{22} = -10$. It is clear that $\lambda_2^{\max} = \max\{\lambda_2^{11}, \lambda_2^{22}\} = -0.382$ and $\lambda_{N,M}^{\min} = \min\{\lambda_{10}^{11}, \lambda_{10}^{22}\} = -10$.

The 10×10 matrices G_{12} and G_{21} are given as follows:

$$G_{12} = \begin{pmatrix} 0 & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 \end{pmatrix},$$

$$G_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The outer-coupling matrix G has only real eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -0.382$, $\lambda_4 = -0.4679$, $\lambda_5 = -0.5858$, $\lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = -1$, $\lambda_{13} = -1.382$, $\lambda_{14} = -1.6527$, $\lambda_{15} = -2.618$, $\lambda_{16} = -3$, $\lambda_{17} = -3.4142$, $\lambda_{18} = -3.618$, $\lambda_{19} = -3.8794$, and $\lambda_{20} = -10$. Here, it is noticed that $\lambda_{\max} = \lambda_3 = \lambda_2^{\max} = -0.382$, $\lambda_{20} = \lambda_{10,10}^{\min} = -10$.

Numerical simulations show that

1) sub-network-1 (consisting of the Lorenz system with coupling matrix G_{11}) synchronizes when $\sigma \geq 9.9$ and desynchronizes when $\sigma \leq 9.8$ (see Fig. 2);

2) sub-network-2 (consisting of the Chen system with coupling matrix G_{22}) synchronizes when $\sigma \geq 2.3$ and desynchronizes when $\sigma \leq 2.2$;

3) sub-network-3 (consisting of the Lorenz system with coupling matrix G_{22}) synchronizes when $\sigma \geq 2.7$ and desynchronizes when $\sigma \leq 2.6$;

4) sub-network-4 (consisting of the Chen system with coupling matrix G_{11}) synchronizes when $\sigma \geq 5.4$ and desynchronizes when $\sigma \leq 5.3$.

From the above simulations, it is obvious that the four sub-networks are completely synchronizable when $\sigma \geq 9.9$. Simulations also show that two-cluster complete synchronization of the entire network is achieved when $\sigma \geq 9.9$ but is not achievable when $\sigma \leq 9.8$ (see Fig. 3).

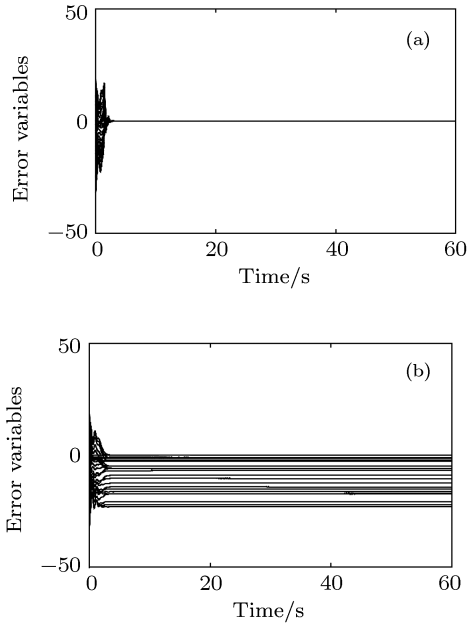


Fig. 2. Numerical results of sub-network-1 for complete synchronization. Panel (a) shows synchronization errors when $\sigma = 9.9$ and panel (b) indicates synchronization errors when $\sigma = 9.8$.

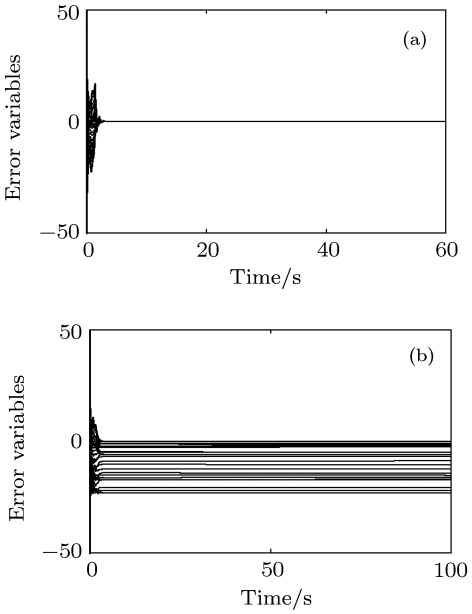


Fig. 3. Numerical results of the entire network for two-cluster synchronization at $\sigma = 9.9$ (a) and 9.8 (b).

From Corollary 3 and Theorem 2, one can see that the two-cluster synchronizability of a network with two different systems is never better than its two individual subnetworks. The structure which satisfies condition (16) is optimal for two-cluster synchronization.

In Ref. [47], the cluster synchronization in a network without the diffusive coupling conditions (4) and (6) was studied, showing that the common intercluster coupling condition (Eq. (7) in Ref. [47]) is necessary for cluster synchronization in such networks. The common intercluster coupling condition implies that if a node of sub-network 1 is connected to a node of sub-network-2, then each node of sub-network-1 should be connected to a certain node of sub-network-2 and *vice versa*. In this paper, the common intercluster coupling condition is not required for cluster synchronization, which is due to the diffusive coupling conditions (4) and (6) (see Fig. 1 for an example). So in this paper the result and analyses in Ref. [47] are somewhat extended.

From the analysis of Case 1, one can also see that if a number of synchronizable sub-networks are connected together by unidirectional couplings from one sub-network to the other, whose structure is shown in Fig. 4, then the entire network is of N-cluster synchronization.

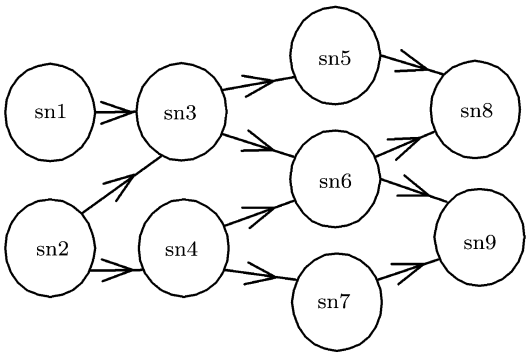


Fig. 4. A network composed of different systems connected from several sub-networks by unidirectional couplings.

Appendix A: Proof of Theorem 1

Proof Let the synchronization error vectors be

$$e_i(t) = x_i(t) - s^*, \quad i = 1, \ldots, M+N, \tag{A1}$$

where s^* is an equilibrium solution of an isolated node system, $s^* = s_1^*$ for $1 \le i \le M$ and $s^* = s_2^*$ for $1+N \le i \le M+N$. Thus, the problem of two-cluster synchronization in network (7) can be converted into the stabilization problem for systems (A1).

Substituting Eq. (A1) into Eq. (7) and noticing that $f(x_i(t)) - f(s^*) = Df(s^*)e_i(t)$, one obtains

$$\dot{e}_i(t) = Df(s^*)e_i(t) + \sigma \sum_{j=1}^{M+N} g_{ij} A e_j(t), \quad i = 1, \ldots, M+N. \tag{A2}$$

By using the Kronecker product, equation (A2) can be written as

$$\dot{E}(t) = \left(I_{M+N} \otimes Df(s^*) + \sigma G \otimes A\right)_E (t), \quad \text{(A3)}$$

where $E(t) = (e_1^{\mathrm{T}}(t), \ldots, e_{M+N}^{\mathrm{T}}(t))^{\mathrm{T}} \in R^{n(M+N)}$ is the error vector between $X(t)$ and S.

According to the Schur triangularization in matrix theory,[54] for any $\varepsilon > 0$ there is a nonsingular matrix $P_\varepsilon \in C^{(M+N)\times(M+N)}$ such that $P_\varepsilon^{-1} G P_\varepsilon = H_\varepsilon = (h_{ij}(\varepsilon))$ is upper triangular and $|h_{ij}(\varepsilon)| < \varepsilon$ for $1 \leq i < j \leq M+N$, with the diagonal entries $h_{ii} = \lambda_i$, $i = 1, \ldots, M+N$, namely,

$$H_\varepsilon = \begin{pmatrix} \lambda_1 & h_{1,2} & \ldots & h_{1,M+N} \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_{M+N-1,M+N} \\ 0 & \ldots & 0 & \lambda_{M+N} \end{pmatrix},$$

in which $|h_{ij}(\varepsilon)| < \varepsilon$. Thus, $G = P_\varepsilon H P_\varepsilon^{-1}$. Let

$$E(t) = (P_\varepsilon \otimes I)\tilde{E}(t), \quad \text{(A4)}$$

where $\tilde{E}(t) = (\tilde{e}_1^{\mathrm{T}}, \ldots, \tilde{e}_{M+N}^{\mathrm{T}})^{\mathrm{T}} \in R^{n(M+N)}$. It follows from Eq. (A4) that $\dot{E}(t) = (P_\varepsilon \otimes I)\dot{\tilde{E}}(t)$.

Now, by substituting Eq. (A4) into Eq. (A3) and then performing some algebraic computation, one obtains

$$\dot{\tilde{E}}(t) = I_{M+N} \otimes Df(s^*) + \sigma(H_\varepsilon \otimes A)\tilde{E}(t). \quad \text{(A5)}$$

The stability of the zero solution of Eq. (A5) is equivalent to that of Eq. (A3). Rewrite Eq. (A5) as follows:

$$\begin{cases} \dot{\tilde{e}}_1(t) = \left(Df(s^*) + \sigma\lambda_1 A\right)\tilde{e}_1(t) + \sigma h_{1,2} A\tilde{e}_2(t) \\ \qquad\qquad + \cdots + \sigma h_{1,M+N} A\tilde{e}_{M+N}(t), \\ \vdots \\ \dot{\tilde{e}}_{M+N-1}(t) = \left(Df(s^*) + \sigma\lambda_{M+N-1} A\right)\tilde{e}_{M+N-1}(t) \\ \qquad\qquad + \sigma^h_{M+N-1,M+N} A\tilde{e}_{M+N}(t), \\ \dot{\tilde{e}}_{M+N}(t) = \left(Df(s^*) + \sigma\lambda_{M+N} A\right)\tilde{e}_{M+N}(t). \end{cases} \quad \text{(A6)}$$

Since $|h_{ij}| < \varepsilon$ and ε can be arbitrarily small, the effects of $\tilde{e}_i(t)$ on the stability of $\dot{\tilde{e}}_{i-1}(t), \ldots, \dot{\tilde{e}}_1(t)$ can be arbitrarily small, $i = 2, \ldots, M+N$. More importantly, from Eq. (12) and the last equation in expression (A6), one can see that $\tilde{e}_{M+N}(t)$ will converge to zero asymptotically. Thus, in the second equation from the bottom in expression (A6), since $\tilde{e}_{M+N}(t)$ asymptotically decays to zero, one can see that $\tilde{e}_{M+N-1}(t)$ will also converge to zero asymptotically. More precisely, one has

$$\begin{aligned} \|\tilde{e}_{M+N-1}(t)\| =& \Big\| \mathrm{e}^{\Phi t}\tilde{e}_{M+N-1}(0) \\ &+ \int_0^t \mathrm{e}^{\Phi(t-\tau)}\Psi\tilde{e}_{M+N}(\tau)\mathrm{d}\tau \Big\| \\ \leq& \left\|\mathrm{e}^{\Phi t}\right\| ||\tilde{e}_{M+N-1}(0)|| \\ &+ \int_0^t \mathrm{e}^{\|\Phi(t-\tau)\|} \|\Psi\| \, ||\tilde{e}_{M+N}(\tau)|| \mathrm{d}\tau \\ \leq& \left\|\mathrm{e}^{\Phi t}\right\| ||\tilde{e}_{M+N-1}(0)|| \\ &+ \left\| \int_0^t \mathrm{e}^{\Phi\tau}\mathrm{d}\tau \right\| ||\tilde{e}_{M+N}(t)|| \to 0, \end{aligned}$$

where $||\cdot||$ is the super-norm,[54] $\Phi = (Df(s^*) + \sigma\lambda_{M+N-1}A)$ is asymptotically stable by assumption, $\|\tilde{e}_{M+N-1}(0)\|$ and $||\psi|| = ||\sigma^h_{M+N-1,M+N}A||$ are constants, $\left\|\int_0^t \mathrm{e}^{\Phi\tau}\mathrm{d}\tau\right\| \leq 1$, with $\left\|\mathrm{e}^{\Phi t}\right\| \to 0$ and $\|\tilde{e}_{M+N}(t)\| \to 0$ as $t \to \infty$.

Continuing to proceed this way, one can verify that all $\tilde{e}_i(t)$ functions will converge to zero asymptotically, $i = 1, 2, \ldots, M+N$. Finally, since $\dot{\tilde{e}}_1(t) = Df(s^*)\tilde{e}_1(t)$ corresponds to one of the two-cluster synchronous state, it follows from Eq. (A4) that $\tilde{E}(t)$ converges to zero asymptotically as well, implying that $E(t)$ converges to zero asymptotically. This completes the proof.

Appendix B: Proof of Corollary 1

Proof Define a Lyapunov function as $V(t) = \frac{1}{2}\|w(t)\|^2$ and, for simplicity and let F denote

$$F = (Df(s^*) + \sigma\lambda_{\max}A)^{\mathrm{T}} + Df(s^*) + \sigma\lambda_{\max}A.$$

From

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_{\max}(t)\|^2 < 0,$$

one has $w_{\max}^{\mathrm{T}}(t)Fw_{\max}(t) < 0$. Then, from Eq. (12), one obtains

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}\|w(t)\|^2}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}(w^{\mathrm{T}}(t)w(t))}{\mathrm{d}t}$$

$$
\begin{aligned}
&= w^{\mathrm{T}}(t)\left(\frac{(Df(s^*)+\sigma\lambda_i A)^{\mathrm{T}}+Df(s^*)+\sigma\lambda_i A}{2}\right)w(t)\\
&= w^{\mathrm{T}}(t)\left(\frac{(Df(s^*)+\sigma\lambda_{\max} A)^{\mathrm{T}}+\sigma(\lambda_i-\lambda_{\max})A^{\mathrm{T}}+Df(s^*)+\sigma\lambda_{\max}A+\sigma(\lambda_i-\lambda_{\max})A}{2}\right)w(t)\\
&\le \frac{1}{2}w^{\mathrm{T}}(t)Fw(t)+\sigma(\lambda_i-\lambda_{\max})w^{\mathrm{T}}(t)\left(\frac{A^{\mathrm{T}}+A}{2}\right)w(t)<0,
\end{aligned}
$$

for all $i=2,\ldots,M+N$. By Theorem 1 and the Lyapunov stability theory, Corollary 1 is proved.

Appendix C: Proof of Theorem 2

Proof Since sub-network-1 itself is synchronizable, the synchronization errors $e_i^1(t)=x_i^1(t)-s_1^*$, $i=1,\ldots,M$, will converge to zero, where $x_i^1(t)$ is the n-dimensional state vector of the i-th node of sub-network-1.

On the other hand, one has

$$
\begin{aligned}
\dot{e}_i^1(t) &= f_1(x_i)-f_1(s_1^*)+\sigma\sum_{j=1}^{M}g_{ij}^{11}Ae_j^1(t)\\
&= Df_1(s_1^*)e_i^1(t)+\sigma\sum_{j=1}^{M}g_{ij}^{11}Ae_j^1(t),\ i=1,\ldots,M.
\end{aligned}
$$

First, let $E^1(t)=(e_1^{1\mathrm{T}},\ldots,e_N^{1\mathrm{T}})^{\mathrm{T}}\in R^{nM}$ be the error vector between $X^1(t)$ and S^1, where $X^1(t)=(x_1^{1\mathrm{T}},\ldots,x_N^{1\mathrm{T}})^{\mathrm{T}}\in R^{nM}$ and $S^1=(s_1^{\mathrm{T}},\ldots,s_1^{\mathrm{T}})^{\mathrm{T}}\in R^{nM}$. Then, for an arbitrary $\varepsilon>0$, triangularize the coupling matrix by $G^{11}=P_\varepsilon H_\varepsilon U_\varepsilon^{-1}$, where $H_\varepsilon=(h_{ij})\in C^{M\times M}$ is upper triangular, with $|h_{ij}|<\varepsilon$ for $1\le i<j\le M$ and P_ε is nonsingular one has

$$
E^1(t)=(P_\varepsilon\otimes I)\tilde{E}(t), \tag{A7}
$$

where $\tilde{E}(t)=(\tilde{e}_1^{\mathrm{T}},\ldots,\tilde{e}_M^{\mathrm{T}})^{\mathrm{T}}\in R^{nM}$. Clearly, $E^1(t)\to 0$ implies $\tilde{E}(t)\to 0$. In a similar way as that from Eq. (A1) to Eqs. (A6), one obtains

$$
\begin{cases}
\dot{\tilde{e}}_1(t)=\left(Df_1(s_1^*)+\sigma\lambda_1^{11}A\right)\tilde{e}_1(t)\\
\qquad\qquad +\sigma h_{1,2}A\tilde{e}_2(t)+\cdots+\sigma h_{1,M}A\tilde{e}_M(t),\\
\vdots\\
\dot{\tilde{e}}_{M-1}(t)=\left(Df_1(s_1^*)+\sigma\lambda_{M-1}^{11}A\right)\tilde{e}_{M-1}(t)\\
\qquad\qquad +\sigma h_{M-1,M}A\tilde{e}_M(t),\\
\dot{\tilde{e}}_M(t)=\left(Df_1(s_1^*)+\sigma\lambda_M^{11}A\right)\tilde{e}_M(t).
\end{cases} \tag{A8}
$$

Since $|h_{ij}(\varepsilon)|<\varepsilon$ and ε can be arbitrarily small, the effects of $\dot{\tilde{e}}_i(t)$ on the stability of $\dot{\tilde{e}}_{i-1}(t),\ldots,\dot{\tilde{e}}_1(t)$ can be arbitrarily small, $i=2,\ldots,M$. More importantly, in a similar way to that for the proof of Theorem 1, it can be easily verified that the asymptotic stability of the zero solution of system (A8) is equivalent to the asymptotic stability of the zero solution of the following system:

$$
\begin{cases}
\dot{\tilde{e}}_1(t)=\left(Df_1(s_1^*)+\sigma\lambda_1^{11}A\right)\tilde{e}_1(t),\\
\vdots\\
\dot{\tilde{e}}_{M-1}(t)=\left(Df_1(s_1^*)+\sigma\lambda_{M-1}^{11}A\right)\tilde{e}_{M-1}(t),\\
\dot{\tilde{e}}_M(t)=\left(Df_1(s_1^*)+\sigma\lambda_M^{11}A\right)\tilde{e}_M(t).
\end{cases} \tag{A9}
$$

Likewise, this same argument can be applied to sub-network-2, sub-network-3, and sub-network-4.

Now, comparing Eq. (A9) with Eq. (12), for the two cases of $f(s^*)=f_1(s_1^*)$ and $f(s^*)=f_2(s_2^*)$, it is obvious that the zero solution of Eq. (12) is asymptotically stable for both cases with the largest non-zero eigenvalue $\lambda_{\max}$ and the smallest eigenvalue λ_{M+N}, respectively. This completes the proof.

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