Parameter estimation for chaotic systems with and without noise using differential evolution-based method*

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We present an approach in which the differential evolution (DE) algorithm is used to address identification problems in chaotic systems with or without delay terms. Unlike existing considerations, the scheme is able to simultaneously extract (i) the commonly considered parameters, (ii) the delay, and (iii) the initial state. The main goal is to present and verify the robustness against the common white Guassian noise of the DE-based method. Results of the time-delay logistic system, the Mackey–Glass system and the Lorenz system are also presented.

Keywords: chaotic system, differential evolution, noise, parameter estimation

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1. Introduction

Parameter esimation of chaotic systems is an important issue in nonlinear systems, which has attracted much attention over the past few decades. Recently, the analysis of chaotic systems and the parameter estimation in these systems have become a hot topic.[1−18] Meanwhile many techniques have been proposed and implemented for such estimation problems. In optical chaotic communication systems, a great deal of research has been completed when the parameters are known in advance for drive-response schemes.[1,2] However, Rontani *et al.*[3] demonstrated that the time-delay, in the transmitter side, could be successfully identified using an autocorrelation function and a mutual information technique, where the time-delay was only the unknown parameter.[3] Then Wu *et al.*[4] extended the scheme and found some more general information about the extraction of delay. In other chaotic systems, the parameter estimation is more active. Some researches focused on parameter estimation using the synchronization-based methods. Parlitz and Junge,[5] and Parliz[6] estimated the parameters of chaotic systems by minimizing the average synchronization error. In Ref. [7], the authors formulated a new approach to the parameter estimation of a delay dynamical system using the idea of least square and found that the method was quite effective for the estimation of the delay. In Ref. [8] the authors presented the successful determining of the parameters in the region of generalized synchronization using the proposed methods. In Ref. [9], Quinn *et al.* explored the extraction of parameter and state information from experimental systems using synchronization, where the chaotic Colpitts oscillator was taken as an example. Moreover, both the experiments and simulations were included. Alvarez *et al.*[10] conducted the parameter estimation from the two-valued symbolic sequences with the quadratic map when its initial value was known in advance. Following that Wu *et al.*[11] improved the study for such a kind of symbolic sequence and parameter estimation, where the initial value was unknown. In addition, a novel method called minimization of relative singular value (MRSV) was ultilized to estimate the parameters of chaotic signal.[12] On the other hand, parameter estimation for chaotic systems was formulated as a multidimensional optimization problem. Piccardi[13] estimated the parameters of chaotic systems using symbolic time-series analysis and the standard genetic algorithm (GA) was employed to minimize the defined cost function, so as to extract the unknown parameters. More recently, many optimization methods have been adopted to estimate parameters for chaotic systems: GA in Ref. [14], particle swarm optimization

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(PSO) in Refs. [15] and [16], chaotic ant swarm based method in Ref. [17] and differential evolution (DE)-based method in Refs. [18] and [19].

Unlike other existing methods, DE has many fascinating features. It operates with real numbers and avoids complicated search process, additionally, it only uses a simple differential operator to create new candidates during the search procedures. Detailed comparison among GA, PSO and DE was presented in Ref. [19], where the feasibility, the easy implementation and the fast convergence of the DE-based method were testified. Nonetheless, most of the optimization problems assumed the initial state of the chaotic systems were known beforehand. Nevertheless, in real applications, the initial state may be unavailable and then the initial state is an additional unknown parameter. So the extraction of the initial state is needed. As we know, the existence of a unique minimum may eventually be true for the search for two or three parameters.[19,20] However, if the parameter space becomes larger, it may not be true anymore. In other words, it is necessary and profound to explore the values of the initial state and other parameters at the same time, i.e., the initial state, the delay and the commonly considered parameters.

Another motivation in this study is that we also explore the noise robustness of the DE-based approach to the parameter estimation for chaotic systems. Despite the fact that the parameter estimation has been extensively studied for chaotic systems, few studies have addressed the issue of how the noise affects the estimation process.[7,13,14] When a certain amount of noise is introduced, the claimed generality may be lost. Noise will affect the output of the chaotic system and bias the extraction of the true parameters. In this respect, we carry out the parameter estimation for chaotic systems with and without noise.

The remainder of this paper is organized as follows. In Section 2, the problem formulation and differential evolution-based method are illustrated. In Section 3 we present the simulation results for the time-delay logistic system, the Mackey–Glass system and the Lorenz system, in which the noise-free case and noisy case are both explored. Finally, a basic conclusion is summarized in Section 4.

2. Problem formulation and differential evolution-based method

In this section, an n-dimensional chaotic system is modeled by a set of differential equations with the corresponding delay terms:

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = f(X(t),\ X(t-\tau),\ X_0,\ \gamma), \tag{1}$$

or by those without delay terms:

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = f(X(t),\ X_0,\ \gamma), \tag{2}$$

where $X(t) \in R^n$ denotes the state vector, $X(t-\tau)$ describes the delay vector (τ is the delay), X_0 represents the initial state, and $\gamma \in R^D$ is a set of the original parameters. For convenience, we define the one-dimensional state $X_i(t)$ as the measurable output time series, that is,

$$y(t) = X_i(t) + w_t, \tag{3}$$

where $y(t)$ denotes the system output. It is worthwhile to note that both the noise-free data ($w_t = 0$) and noisy data ($w_t \neq 0$) are considered. Here w_t is the output of common white Guassian noise.

In this consideration, the estimation of the initial state and the delay is also studied, i.e., the unknown parameters consist of the commonly considered parameters γ, the delay τ and the initial state X_0. Suppose that the structure of system (1) or (2) is known and the output time series calculated with the actual parameters are also given, then the problem of parameter estimation will be converted into the problem of evaluating the values of the parameters mentioned above. For convenience, we define $\theta = [\gamma,\ \tau, X_0]$ as the set of original parameters and $\overline{\theta} = [\overline{\gamma},\ \overline{\tau}, \overline{X_0}]$ as that of estimated parameters. Substituting $\overline{\theta}$ into Eq. (1) or Eq. (2), the estimated output $\overline{y(t)}$ can be obtained. Consequently, an optimization problem is formulated as follows:

$$\begin{cases} \theta^* = \arg\min\limits_{\overline{\theta}\in H} J, \\ J = 1 - \dfrac{\langle [y(t) - \langle y(t)\rangle][\overline{y(t)} - \langle \overline{y(t)}\rangle]\rangle}{\{\langle [y(t) - \langle y(t)\rangle]^2\rangle \langle [\overline{y(t)} - \langle \overline{y(t)}\rangle]^2\rangle\}^{1/2}}, \end{cases} \tag{4}$$

where J is the objective function (or called cost function), $\langle\cdot\rangle$ is the time average and H is the searching domain. Our aim is to search for appropriate θ^* by

minimizing the objective function J. However, solving Eq. (4) can be hard by using traditional optimization methods, because the chaotic system (1) or (2) output unstable dynamics and multiple local optima may occur in the landscape of J.

On the other hand, the parameter estimation for chaotic systems is a multi-dimensional continuous optimization problem. The DE algorithm is used to solve the well-defined objective function (4) with quick convergence. The algorithm has many variants and the variant implemented in our scheme is DE/rand-to-best/1/bin (see Ref. [21] for detail). Moreover, the control variables used in the simulations are listed in Table 1. To be valid, the searching domain H is also specified in this table.

Table 1. Control variables used in the simulations.

Variables	Values
Population size (S)	40
Mutation factor (F)	0.6
Crossover rate (CR)	0.6
Total number of output	300
Maximum generation	500
Searching domain (H)	$\{\overline{\theta}\|0.5\theta \le \overline{\theta} \le 1.5\theta\}$

To better evaluate the efficiency of the proposed scheme, two indexes are introduced, i.e., the noise-to-signal ratio (NSR) $R_{\rm NSR}$ and the error evaluator factor (EEF) $F_{\rm EEF}$. And the detailed definitions are presented in the following.

(i) Noise-to-signal ratio $R_{\rm NSR}$

The $R_{\rm NSR}$ is proposed to show the robustness of our scheme in the presence of noise. The deviation of the signal (chaotic output) in Eq. (3) is $\mathrm{var}(X_i(t))$ and the deviation of the white Guassian noise equals $\mathrm{var}(w_t)$. Therefore the $R_{\rm NSR}$ index is expressed as the following mathematical form:

$$R_{\rm NSR} = \left[\frac{\mathrm{var}(w_t)}{\mathrm{var}(X_i(t))}\right]^{1/2} \times 100\%. \qquad (5)$$

(ii) Error evaluator factor $F_{\rm EEF}$

The $F_{\rm EEF}$ is introduced to evaluate the performance of the whole estimation procedure, it is written as

$$F_{\rm EEF} = \left\|\frac{\overline{\theta}-\theta}{\theta}\right\|_2 \times 100\%. \qquad (6)$$

3. Numerical results

As can be seen above, the unknown parameters comprise the commonly involved parameter (γ), the delay time (τ) and the initial state (X_0). The DE-based approach is employed to solve the formulated optimization problem. To show the fitness and feasibility of the proposed method, two kinds of chaotic systems are included, i.e., chaotic systems with or without delay terms. In a way different from that in the previous papers, we introduce a suitable objective function with fast convergence and the values of the initial state are also identified without any beforehand knowledge. Most importantly, certain levels of noise are considered and implemented with three chaotic systems and the corresponding results are presented in this section.

3.1. Application to delay differential chaotic systems

3.1.1. Time-delay logistic system

The first example is related to the logistic chaotic system with a delay term, which has the following form (e.g. Ref. [20]):

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\lambda x(t) + rx(t-\tau)(1-x(t-\tau)), \qquad (7)$$

where the unknown parameters involve λ, r, τ and the value of the initial state x_0. For our choice of $\lambda = 26$, $r = 104$, $\tau = 0.5$ and $x_0 = 0.02$, the system displays chaotic fluctuations. All the control variables related to the DE algorithm, the length of the output time series and the search domain are given in Table 1.

Figure 1 shows that the objective function converges to zero very quickly, and the critical generation is approximately 100, which is slightly different from the result in Ref. [20]. The reason is that more unknown parameters are involved in this study and our exploration is in a bigger search range (H). Moreover,

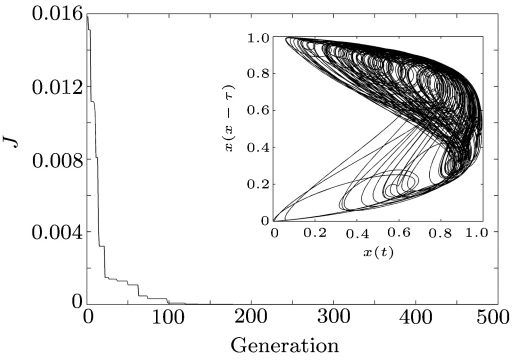


Fig. 1. Convergence trajectory of the objective function J. Inset: the chaotic attractor for logistic system.

the inset in Fig. 1 displays the logistic chaotic attractor, which also indicates that the logistic system works in a chaotic regime.

Figure 2 shows the evolutionary process for each unknown parameter. The four panels of Fig. 2 exhibit the detailed evolutionary procedures of four relevant parameters: τ [Fig. 2(a)], λ [Fig. 2(b)], r [Fig. 2(c)], and x_0 [Fig. 2(d)]. It is clear that they almost reach the actual values of the original parameters after 100 generations, apart from the delay. In addition, it is worthwhile noting that the estimate of τ always has subtle errors within 500 generations, that is, the evolutionary line in Fig. 2(a) is not strictly straight, which also shows some difference from that of Ref. [20]. The numerical results indicate that the relevant parameters are successfully detected by using the DE-based method.

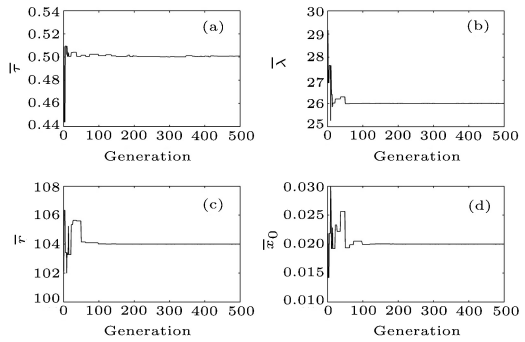


Fig. 2. Evolution of parameter estimation for logistic chaotic system: (a) a searching process for τ, (b) a searching process for λ, (c) a searching process for r, (d) a searching process for x_0.

To assess the effect of the noise on the performance of parameter estimation, we also test the estimation procedure for three specific $R_{\rm NSR}$, i.e., 0%, 10% and 25% and the $F_{\rm EEF}$ is calculated for each case. Specifically, 20 independent realizations for each case are conducted and these results are averaged. In Table 2 the corresponding results are summarized (the same processes are also performed for other systems in the remainder of the work, unless otherwise specified). The performance of the DE-based approach is overall satisfactory and comparable, in particular, an extremely good outcome in the noise-free case ($R_{\rm NSR} = 0\%$) is obtained. Moreover, comparing the results of the three cases, we conclude that fairly good results, with $F_{\rm EEF} < 4.58\%$, can be attained if $R_{\rm NSR}$ is smaller than 10%. However, if $R_{\rm NSR}$ is substantially greater than 10%, for example, $R_{\rm NSR} = 25\%$, the performance of the method is a bit poor, with $F_{\rm EEF} = 25.06\%$.

Table 2. Results of the parameter estimation procedures for logistic chaotic system. The true values of the parameters and the initial state are $\gamma = [26, 104]$, $\tau = 0.5$ and $x_0 = 0.02$.

$R_{\rm NSR}$	0%	10%	25%
$\overline{\gamma}$	[26.000, 104.000]	[26.0381, 104.2900]	[26.2548, 104.1500]
$\overline{\tau}$	0.5009	0.5048	0.5056
$\overline{x}_0$	0.0200	0.0209	0.0250
$F_{\rm EEF}$	0.18%	4.58%	25.06%

3.1.2. Time-delay Mackey–Glass system

The second example is the Mackey–Glass system, which initially has been introduced as a model of blood generation for patients with leukemia. However, recently, it has been employed to produce high-dimensional chaos in the area of chaos time series analysis, or chaos control, etc.[22−24] The equation is given as follows:

$$\frac{{\rm d}x(t)}{{\rm d}t} = -ax(t) + \frac{bx(t-\tau)}{1+x(t-\tau)^{10}}, \tag{8}$$

where the parameters τ, a, b and the initial state x_0 are unknown. The system is chaotic when $\tau = 17$, $a = 0.1$, $b = 0.2$ and $x_0 = 1.2$. Unless otherwise explicitly stated, the search problems in the remainder of the article are solved with the parameter values given in Table 1.

Figure 3 shows the simulation results for the evolutionary process of the objective function value. It can be seen that the value converges to zero very fast, after about 50 generations. The inset shows the chaotic behaviour of such a system. The evolutionary process of the objective function value is different from that in Fig. 1. From the viewpoint in physics,

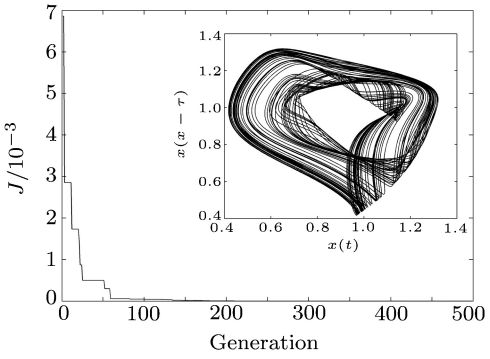


Fig. 3. Convergence trajectory of the objective function J. Inset: the chaotic behaviour for Mackey–Glass system.

one possible reason is that the different chaotic time series derive from two chaotic systems.

Figure 4 shows a typical search process for each unknown parameter. The results indicate that all the unknown parameters have been successfully identified: τ [Fig. 4(a)], a [Fig. 4(b)], b [Fig. 4(c)] and x_0 [Fig. 4(d)]. More details of the search procedures are given in Table 3, where the noisy data cases are also included. Likewise, if $R_{\rm NSR}$ is smaller than 10%, the simulation results are overall approving, with $F_{\rm EEF} < 8.06\%$.

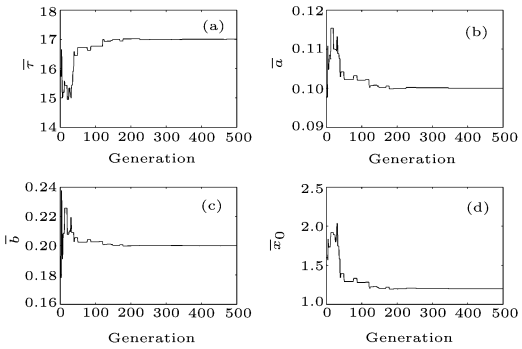


Fig. 4. Evolution of parameter estimation for Mackey–Glass system: (a) a searching process for τ, (b) a searching process for a, (c) a searching process for b, and (d) a searching process for x_0.

Table 3. Results of the parameter estimation procedures for Mackey–Glass system. The true values of the parameters and the initial state are $\gamma = [0.1, 0.2]$, $\tau = 17$ and $x_0 = 1.2$.

$R_{\rm NSR}$	0%	10%	25%
$\overline{\gamma}$	[0.1000, 0.2000]	[0.10601, 0.2099]	[0.1219, 0.2113]
$\overline{\tau}$	17.0034	16.9133	17.0409
$\overline{x}_0$	1.2000	1.2335	1.1814
$F_{\rm EEF}$	0.02%	8.06%	22.67%

3.2. Application to chaotic systems without delay terms

A typical chaotic system without delayed terms is the Lorenz system,[25] which is described by

$$\begin{cases} \dfrac{{\rm d}x_1(t)}{{\rm d}t} = a(x_2 - x_1), \\ \dfrac{{\rm d}x_2(t)}{{\rm d}t} = bx_1 - x_1x_3 - x_2, \\ \dfrac{{\rm d}x_3(t)}{{\rm d}t} = x_1x_2 - cx_3, \end{cases} \tag{9}$$

where the unknown parameters contains a, b, c and the initial state $X_0 = [x_{10}, x_{20}, x_{30}]$. When they take values of $a = 10$, $b = 28$, $c = 8/3$, and $X_0 = [1, 2, 3]$, respectively, the system has chaotic behaviour. The inset in Fig. 5 exhibits the proof of chaotic dynamics.

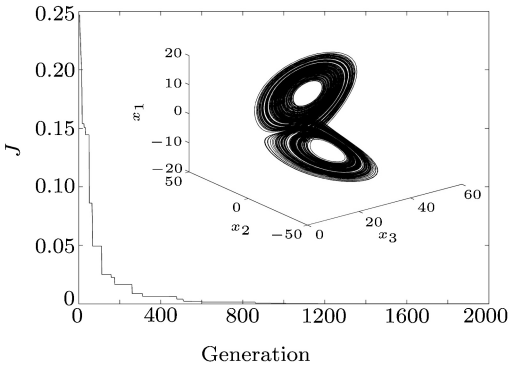


Fig. 5. Convergence trajectory of the objective function J. Inset: the chaotic behavior for Lorenz system.

Figure 5 shows the convergence trajectory of the objective function J versus the generation, from which we can see that the efforts for seeking the true values of the unknown parameters are successful. It is clearly seen that the objective function value decreases to zero gradually and finally reaches zero after around 1000 generations.

The evolutionary procedure for each unknown parameter and the initial state of Lorenz system is given in detail as shown in Fig. 6: a [Fig. 6(a)], b [Fig. 6(b)], c [Fig. 6(c)], x_{10} [Fig. 6(d)], x_{20} [Fig. 6(e)], and x_{30} [Fig. 6(f)]. It is demonstrated that the relevant parameters evolve to their own true values after enough generations. More importantly, it should be stressed that in the Lorenz system, the control parameters in the DE-based algorithm are set differently from those in the former systems. In this simulation, a six-dimensional optimization problem is formulated. To

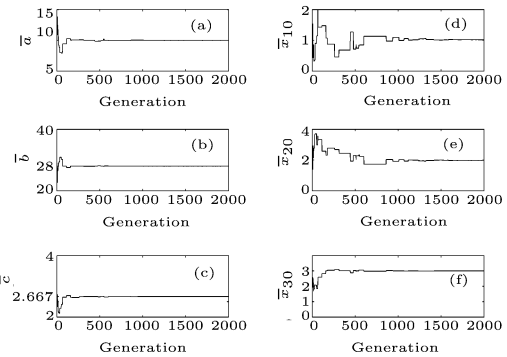


Fig. 6. Evolution of parameter estimation for Lorenz system: (a) a searching process for a, (b) a searching process for b, (c) a searching process for c, (d) a searching process for x_{10}, (e) a searching process for x_{20}, and (f) a searching process for x_{30}.

obtain better results, the population size and the maximum generation were 60 and 2000, respectively. For others, readers may refer to Table 1. Also, almost all the relevant parameters can be figured out after a slightly larger generation (about 1000 generations) compared with the cases in the former systems, where only four unknown parameters are involved.

In Table 4, the results in our numerical experiment of noise-free and noisy cases for Lorenz system are also presented. In particular, for the ideal case ($R_{\rm NSR} = 0\%$), all the true values are identified and the $F_{\rm EEF}$ comes only from the floating point representation in MATLAB. In the same way, the search performance of the method is pleasing if the output time series are not badly corrupted by noise ($R_{\rm NSR} < 10\%$).

Table 4. Results of the parameter estimation procedures for Lorenz system. The true values of the parameters and the initial state are $\gamma = [10,\ 28,\ 8/3]$ and $X_0 = [1,\ 2,\ 3]$.

$R_{\rm NSR}$	0%	10%	25%
$\overline{\gamma}$	[10.000, 28.000, 2.6667]	[10.0328, 27.9773, 2.6536]	[10.2727, 27.5300, 2.7635]
$\overline{X}_0$	[1.000, 2.000, 3.000]	[1.0113, 1.9874, 2.9022]	[0.9738, 1.9130, 2.8794]
$F_{\rm EEF}$	0.0013%	3.61%	8.06%

4. Conclusion

The proposed DE-based approach is shown to be a convenient and powerful tool for identifying the unknown parameters for different chaotic systems. Unlike existing considerations, we have also considered the extraction of the initial state. To claim the generality and show the noise immunity, the white Guassian noise is introduced. Finally, the numerical experiments demonstrate the efficiency and robustness of the method. As the future work, we shall consider the application of the method to other high-dimensional chaotic systems, e.g., an external-cavity semiconductor laser system, to further verify the efficiency and fitness of the proposed scheme.

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