# Impact of neutron star crust on gravitational waves from the axial $w$-modes*

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The imprints of the neutron star crust on the gravitational waves emitted from the axial $w$-modes are investigated by adopting two typical equations of state (EOSs) of the crust matter and two representative EOSs of the core matter. It is shown that there is a significant effect of the crust EOSs on the gravitational waves from the axial $w$-mode oscillation for a stiff core EOS.

**Keywords:** equation of state, neutron star, gravitational waves

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## 1. Introduction

A stable neutron star is balanced by the gravity and the pressure of the high density matter. Commonly, its density decreases from the centre to the surface. To the neutron star matter with density greater than the density of normal nuclear density, the matter is believed to exist as a uniform liquid of neutrons, protons, electrons and muons (near the centre the exotic particles may become abundant).[1−7] This region is called the core of the neutron star. To the density region above the point of neutron drip ($4\times10^{14}$ kg·m$^{-3}$) and below about the normal nuclear density, the matter is supposed to be composed of a matrix of nuclei in a sea of neutrons and electrons,[8] and this region is called the inner crust.[9] The region with a density below the dripping point is called the outer crust.[9] In recent years, both nuclear physics scientists and astrophysics scientists pay their attention to the outer layer of neutron stars, including both the inner crust and the outer crust.[8−15] It is crucial to investigate the neutron star crust for understanding the observational phenomena of neutron stars, such as the glitches, the thermal evolution, the gravitational radiation from the stellar oscillations, etc. But it is still a great challenge though the density of the crust matter is subnuclear because of the strong gravity and the strong magnetic field environment in the crust. For example, the location of the inner crust edge (the core–crust interface), which separates the liquid core from the solid crust of neutron stars, is still not determined very precisely up to now.[11−15] On the other hand, the pulsars with high masses have been reported recently, such as the pulsar in binary PSR J1614-2230 with a mass of $(1.97\pm0.04)\ M_{\odot}$,[16] the pulsar in PRS J1748-2021B with an extremely high mass of $(2.74\pm0.22)\ M_{\odot}$,[17] the pulsar in EXO 0748-676 with a mass constrained to $M\geq(2.10\pm0.28)\ M_{\odot}$,[18] etc. These observations suggest that a stiff equation of state (EOS) is needed to describe the supra-dense matter in the core of neutron star because only such an EOS of the core can support a larger maximum mass of the neutron star.

Gravitational waves are expected to be a powerful tool for probing the neutron star properties and the EOS of dense matter in the future. However, until now there is still no direct proof to show that the gravitational waves signals have been detected.[19,20] Therefore, in the first step, theoretical investigation of gravitational waves is very helpful to direct the detection of the gravitational waves. Non-radial oscillations of neutron stars, which can be excited by several mechanisms, such as gravitational collapse, glitch or phase transitions, are considered as one of the important potential candidates for producing detectable gravitational waves.[21−24] Axial $w$-modes are one of the most substantial non-radial oscillations, essentially a kind of space-time oscillations which exists in all relativistic stars.[21,22,25,26] As the crust has a significant

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effect on the global properties of neutron stars, especially the stellar radius,[10,27] it is expected that there may be also a distinct imprint of the crust on the gravitational waves emitted from the non-radial oscillations, especially on the axial $w$-modes. This is the major objective we will investigate in this work.

## 2. Equation of state of the neutron-star-crust matter and the neutron-star-core matter

The EOS of the neutron-star-crust matter is a crucial input in the study of the effect of the stellar curst on the neutron star properties. Theoretically, by minimizing the total energy density for a given baryon density under the assumption of $\beta$-equilibrium and electric charge neutrality, one can determine the EOS of the crust.

The EOS of the outer crust has been well studied by theories and experiments.[28] In this paper, two of the widely used EOS of the outer curst of neutron star will be adopted, they are BPS[11] and HP.[29] It is shown that only at a very low density (below $10^7$ kg·m$^{-3}$) is there a distinct difference between these two kinds of EOS; while at the greater density region, they are very similar. The EOS of the outer crust has a very faint effect on the global properties of the neutron star.

However, to the EOS of the inner crust, the case becomes more complicated. The matter in the inner crust of a neutron star is not easy to reproduce in the terrestrial laboratory because of the presence of the neutron gas. Consequently, the description of the EOS of the crust beyond neutron drip depends on the theoretical models.[9] In this work, two typical EOSs of the inner crust of neutron star: the BPS[11,30] and the PRL[13] will be adopted. The BPS EOS for the inner crust describes the nuclei by a compressible liquid-drop model for taking into account the effect of the free neutrons and the Coulomb interaction between nuclei.[11,30] The PRL EOS for the inner crust is assumed that the matter is composed of electrons, non-relativistic neutrons and protons in beta equilibrium and the interaction is described by the generalized Skyrme interaction.[13] The two chosen EOSs of the inner crust of neutron star are displayed in the inset of Fig. 1. It is shown that the significant differences between the BPS EOS and PRL EOS of the inner crust are restricted to the density interval $5 \times 10^{14}$ kg·m$^{-3}$–$4 \times 10^{15}$ kg·m$^{-3}$.

The EOS of the neutron-star-core matter is still very uncertain because of the lesser understanding of the symmetry energy of the nucleonic matter.[2,3] Therefore, the EOS of the core matter is also model-dependent. Nowadays, there are dozens of EOSs describing the matter with density beyond the normal nuclear density.[1−3] Although at supra-nuclear densities exotica may appear, the nucleons (especially the neutrons) are believed to dominate the core of the neutron star.[1] So in this work we only consider the EOSs of the core matter composed of neutrons, protons and electrons. Here we choose two representative EOSs of the core matter: EOS L,[31,32] MDIx0.[33,34] EOS L represents the stiffer EOS, where the nucleons are assumed to interact through the exchange of σ-, ω- and ρ-mesons. Hybrid methods are adopted in model L: the exchange of heavy particles (ω and ρ) is described according to non-relativistic potentials, while the σ-meson is taken into account by the relativistic field theory and the mean-field approximation.[31,32] As mentioned in the introduction, massive neutron stars are observed recently,[16−18] which need a stiff EOS to describe their matter. EOS L is such a stiff EOS to support the massive neutron star. The MDIx0 is calculated by the MDI (Momentum Dependent Interaction) and is obtained from the Hartree–Fock calculations by using a modified Gogny interaction[33] and constrained by the recent terrestrial nuclear laboratory data.[3,34,35] It is shown that the MDIx0 EOS is very similar to the widely used APR EOS.[3,36] The chosen EOSs of the neutron-star-core matter are displayed in Fig. 1.

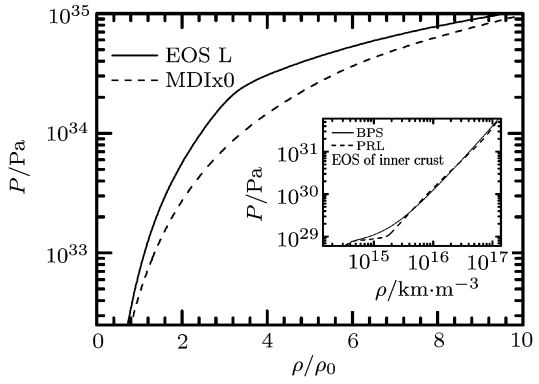



**Fig. 1.** Two representative equations of state (EOSs) of the neutron-star-core matter considered in this work, where the energy density $\rho$ is in units of saturation nuclear density $\rho_0$. The inset shows the two typical EOSs of the neutron-star-crust matter, for details see the text.

The mass–radius relationship of the corresponding static neutron stars is plotted in Fig. 2, where the models are constructed with the chosen EOSs (EOS L and MDIx0) of the core matter combining with the two typical EOSs of the crust matter: BPS and HP & PRL (where the latter one is a combination of the outer crust EOS HP and an inner crust EOS PRL). It is interesting to notice that, in Fig. 2, the EOS of the crust matter affects slightly the mass–radius relationship of the neutron star in the maximum mass region or in the minimum mass region supported by the corresponding EOSs. Except for the extremum region, there is a significant effect of the crust on the bulk properties of neutron stars, especially to the stiff core EOS (EOS L), which will consequently lead to a distinct effect on the axial oscillations of the stellar. By the way, the neutron stars constructed by the stiffest EOS have much smaller central densities.

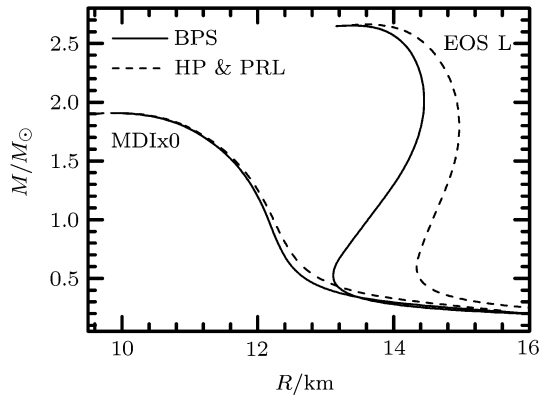



**Fig. 2.** Mass–radius relationship of the static neutron stars, where the models are constructed with the two EOSs of the core matter, in combination with the two chosen crust EOSs.

## 3. Perturbation equations for the axial *w*-mode oscillation

In this section, the equation of axial $w$-mode perturbation and the formalism for integrating the perturbed equation inside and outside the neutron star are briefly reviewed.

According to Chandrasekhar & Ferrari,[25,26] the axial perturbation equations for a static neutron star can be simplified to a second-order differential equation for a function $Z(r)$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}r_*^2} + [\omega^2 - V(r)]Z = 0, \tag{1}$$

where $\omega(= \omega_\mathrm{r} + \mathrm{i}\,\omega_\mathrm{i})$ is the complex eigen-frequency of the axial $w$-mode, the real part of which represents the actual frequency of the gravitational waves from the oscillation and the imaginary part represents the energy loss rate of the oscillation. Inside the star, the tortoise coordinate $r_*$ and the potential function $V$ are defined as

$$\frac{\mathrm{d}}{\mathrm{d}r_*} = \mathrm{e}^{\lambda-\nu}\frac{\mathrm{d}}{\mathrm{d}r} \tag{2}$$

and

$$V = \frac{\mathrm{e}^{2\nu}}{r^3}[l(l+1)r + 4\pi r^3(\rho - p) - 6m], \tag{3}$$

where $l$ is the spherical harmonic index (used in describing the perturbed metric and only the case $l = 2$ is considered in this work); $\rho$ and $p$ are the density and pressure, respectively; and $m$ is the mass inside radius $r$. The functions $\mathrm{e}^\nu$ and $\mathrm{e}^\lambda$ are given by the line element for a static neutron star

$$-\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + \mathrm{e}^{2\lambda}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{4}$$

Outside the neutron star, equations (2) and (3) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}r_*} = \frac{r-2M}{r}\frac{\mathrm{d}}{\mathrm{d}r} \tag{5}$$

and

$$V = \frac{r-2M}{r^4}[l(l+1)r - 6M], \tag{6}$$

where $M$ is the total mass of the star.

The solutions to this problem are subject to a set of boundary conditions constructed by Chandrasekhar & Ferrari:[25] regular at the neutron star centre, continuous at the surface and behaving as a purely outgoing wave at infinity.

As discussed in Refs. [27], [37], and [38], the $w$-modes have larger imaginary parts and thus introduce considerable difficulties to dealing with the boundary conditions at infinity. A practical way to overcome these difficulties is to use the continued fraction method.[21,37]

According to the continued fraction method, one important step to calculate the eigen-frequency of the axial $w$-mode is integrating Eq. (1) from the centre to a selected point outside the star. In the process of numerical integration, it is convenient to separate the real and the imaginary parts of Eq. (1).[26] Accordingly, by writing

$$Z = Z_\mathrm{r} + \mathrm{i}\,Z_\mathrm{i}, \tag{7}$$

equation (1) can be separated into the following pair of equations

$$\frac{\mathrm{d}^2 Z_\mathrm{r}}{\mathrm{d}r_*^2} + [\omega_\mathrm{r}^2 - \omega_\mathrm{i}^2 - V(r)]Z_\mathrm{r} - 2\omega_\mathrm{r}\omega_\mathrm{i}Z_\mathrm{i} = 0, \quad (8)$$

$$\frac{\mathrm{d}^2 Z_\mathrm{i}}{\mathrm{d}r_*^2} + [\omega_\mathrm{r}^2 - \omega_\mathrm{i}^2 - V(r)]Z_\mathrm{i} - 2\omega_\mathrm{r}\omega_\mathrm{i}Z_\mathrm{r} = 0. \quad (9)$$

Inside the star, according to Eqs. (2) and (3), the above pair of equations can be rewritten in a direct integrable form

$$\begin{aligned}\frac{\mathrm{d}^2 Z_\mathrm{r}}{\mathrm{d}r^2} &+ \frac{\mathrm{e}^{2\lambda}}{r^2}[2m - 4\pi r^3(\rho - p)]\frac{\mathrm{d}Z_\mathrm{r}}{\mathrm{d}r} \\ &- \frac{\mathrm{e}^{2\lambda}}{r^2}\left[4\pi r^2(\rho - p) - \frac{6m}{r} + l(l+1)\right. \\ &\left. - \frac{r^2}{\mathrm{e}^{2\nu}}(\omega_\mathrm{r}^2 - \omega_\mathrm{i}^2)\right]Z_\mathrm{r} - 2\omega_\mathrm{r}\omega_\mathrm{i}\frac{\mathrm{e}^{2\lambda}}{\mathrm{e}^{2\nu}}Z_\mathrm{i} = 0, \quad (10)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2 Z_\mathrm{i}}{\mathrm{d}r^2} &+ \frac{\mathrm{e}^{2\lambda}}{r^2}[2m - 4\pi r^3(\rho - p)]\frac{\mathrm{d}Z_\mathrm{i}}{\mathrm{d}r} \\ &- \frac{\mathrm{e}^{2\lambda}}{r^2}\left[4\pi r^2(\rho - p) - \frac{6m}{r} + l(l+1)\right. \\ &\left. - \frac{r^2}{\mathrm{e}^{2\nu}}(\omega_\mathrm{r}^2 - \omega_\mathrm{i}^2)\right]Z_\mathrm{i} + 2\omega_\mathrm{r}\omega_\mathrm{i}\frac{\mathrm{e}^{2\lambda}}{\mathrm{e}^{2\nu}}Z_\mathrm{r} = 0. \quad (11)\end{aligned}$$

From the requirement that the function $Z(r)$ is regular at $r = 0$, $Z_\mathrm{r}$ and $Z_\mathrm{i}$ can be expanded near the star centre as[26]

$$\begin{aligned}Z_\mathrm{r} &= r^{l+1} + \frac{1}{2(2l+3)}\left\{4\pi(l+2)\left[\frac{1}{3}(2l-1)\rho_0 - p_0\right]\right. \\ &\left. - \mathrm{e}^{-2\nu_0}(\omega_\mathrm{r}^2 - \omega_\mathrm{i}^2) + 2\mathrm{e}^{-2\nu_0}\omega_\mathrm{r}\omega_\mathrm{i}\right\}r^{l+3}, \quad (12)\end{aligned}$$

$$\begin{aligned}Z_\mathrm{i} &= r^{l+1} + \frac{1}{2(2l+3)}\left\{4\pi(l+2)\left[\frac{1}{3}(2l-1)\rho_0 - p_0\right]\right. \\ &\left. - \mathrm{e}^{-2\nu_0}(\omega_\mathrm{r}^2 - \omega_\mathrm{i}^2) - 2\mathrm{e}^{-2\nu_0}\omega_\mathrm{r}\omega_\mathrm{i}\right\}r^{l+3}, \quad (13)\end{aligned}$$

where $\rho_0$, $p_0$, and $\nu_0$ are values at the centre of the star.

Outside the star, according to Eqs. (5) and (6), equations (10) and (11) reduce to

$$\begin{aligned}&\frac{\mathrm{d}^2 Z_\mathrm{r}}{\mathrm{d}r^2} + \frac{2M}{r^2 - 2Mr}\frac{\mathrm{d}Z_\mathrm{r}}{\mathrm{d}r} - \left(1 - \frac{2M}{r}\right)^{-1}\left\{\frac{\omega_\mathrm{i}^2 - \omega_\mathrm{r}^2}{1 - 2M/r}\right. \\ &\left. + \frac{l(l+1)r - 6M}{r^3}\right\}Z_\mathrm{r} - \frac{2\omega_\mathrm{r}\omega_\mathrm{i}}{(1 - 2M/r)^2}Z_\mathrm{i} = 0, \quad (14)\end{aligned}$$

$$\begin{aligned}&\frac{\mathrm{d}^2 Z_\mathrm{i}}{\mathrm{d}r^2} + \frac{2M}{r^2 - 2Mr}\frac{\mathrm{d}Z_\mathrm{i}}{\mathrm{d}r} - \left(1 - \frac{2M}{r}\right)^{-1}\left\{\frac{\omega_\mathrm{i}^2 - \omega_\mathrm{r}^2}{1 - 2M/r}\right. \\ &\left. + \frac{l(l+1)r - 6M}{r^3}\right\}Z_\mathrm{i} + \frac{2\omega_\mathrm{r}\omega_\mathrm{i}}{(1 - 2M/r)^2}Z_\mathrm{r} = 0, \quad (15)\end{aligned}$$

In this work, we will employ the continued fraction method to calculate the eigen-frequencies of axial $w$-modes. The details of this method can be found in Refs. [21] and [38].

# 4. Imprint of the crust on gravitational waves from the axial $w$-mode oscillations

In what follows, the imprints of the neutron star crust EOS on the frequencies and damping times of the gravitational waves from the axial $w$-mode oscillations are presented. Shown in Fig. 3 is the frequency of the $w_\mathrm{I}$-mode as a function of the stellar mass $M$ for the static neutron stars with EOSs constructed by using the chosen EOSs of the core combining with the two EOSs of the crust where the filled circles denote the neutron stars with a BPS EOS for the crust matter and the filled squares denote the neutron stars with an HP & PRL EOS for the crust matter. It is shown that to the neutron star with the stiff core EOS (EOS L), there is a distinct imprint of the crust on the frequencies of the axial $w_\mathrm{I}$-modes, while to the relative softer core EOS (MDIx0), the effect is too weak to distinguish. By the way, our calculation shows that the effect of the crust on the damping rate of the $w_\mathrm{I}$-modes is very tiny, thus the results of the damping rate of the $w_\mathrm{I}$-mode are not presented here.

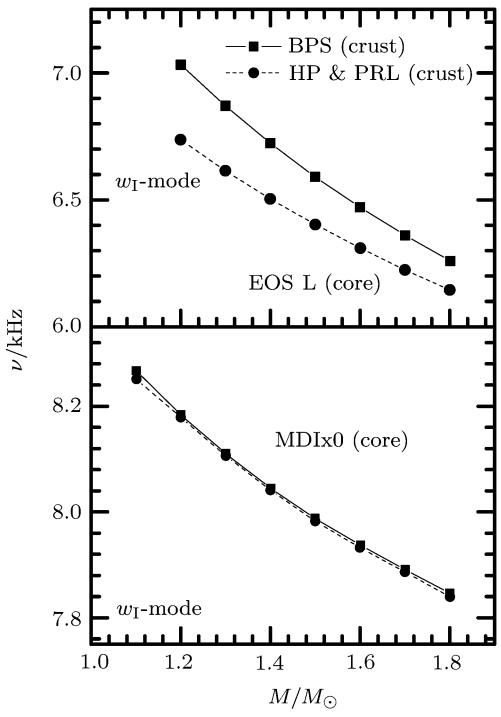



**Fig. 3.** Frequency of $w_\mathrm{I}$-mode as a function of the stellar mass $M$.

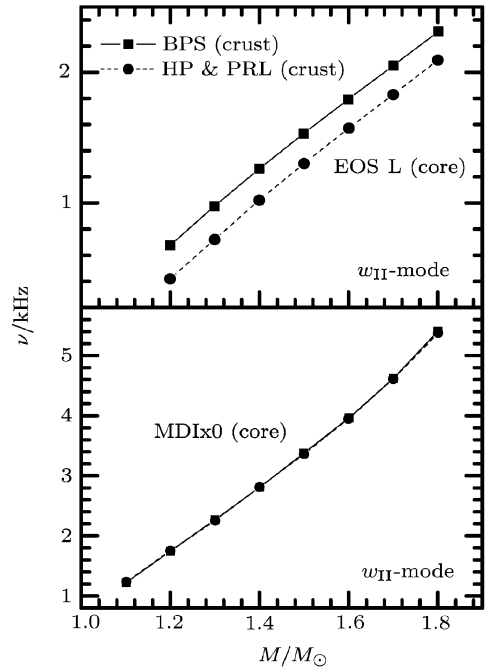



**Fig. 4.** Frequency of $w_{\rm II}$-mode as a function of the stellar mass $M$.

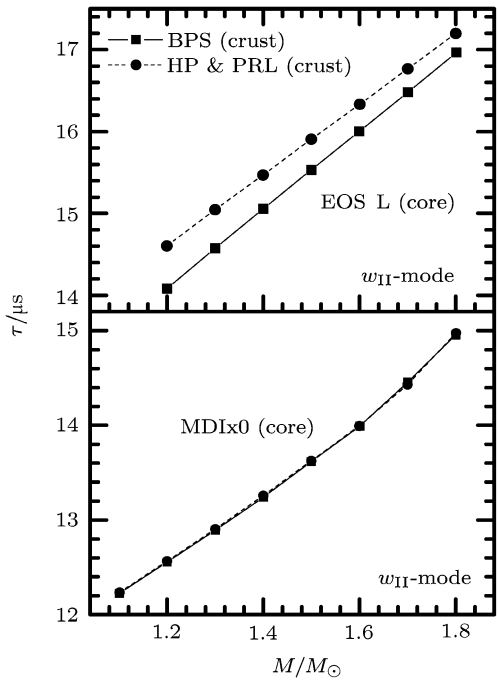



**Fig. 5.** Damping time of $w_{\rm II}$-mode as a function of the stellar mass $M$.

In Figs. 4 and 5, the effects of the crust on the frequency and damping time of $w_{\rm II}$-modes for the constructed models are displayed. It is shown that to the stiff EOS (EOS L) of the core, there is a distinct effect of the crust EOS on both the frequency and damping time of $w_{\rm II}$-mode, while to the softer EOS (MDIx0) of the core, the effect of the crust is indistinguishable. It is interesting to notice that, to the stiff EOS (EOS L) of the core, the frequency $\omega$ and the damping time $\tau$ of the $w_{\rm II}$-mode vary linearly as a function of the mass $M$. Therefore, once the stiff EOS of the core is approved by the observation in the near future, this linearly varying relation is very significant in the prediction of the gravitation radiation of the axial $w$-mode oscillation. Furthermore, at a fixed stellar mass, say 1.4 $M_\odot$, the star with a stiffer EOS of the core will have a smaller frequency and a longer damping time to both the axial $w_{\rm I}$-mode and $w_{\rm II}$-mode.

As shown above, to a stiff EOS of the core, which is needed for supporting the recent observations of the massive neutron star,[16−18] there is a distinct imprint of the crust EOS on the frequency of the axial $w_{\rm I}$-mode and both the frequency and damping time of the axial $w_{\rm II}$-mode, so its meaningfull or important to obtain a precise crust EOS in predicting the gravitational radiation from the axial $w$-mode.

## 5. Summary

In summary, we have examined the effect of the EOS of the neutron-star-crust matter on the gravitational waves from the axial $w$-modes of neutron star and found that to the stiff EOS of the neutron-star-core matter there is a significant imprint of the crust EOS on the frequencies of the $w_{\rm I}$-, $w_{\rm II}$-modes and the damping times of the $w_{\rm II}$-modes. These results indicate that it is very important to investigate the EOS of the neutron star crust matter for predicting the gravitational radiations of the axial $w$-mode oscillations.

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