The relation between Hardy's non-locality and violation of Bell inequality*

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We give an analytic quantitative relation between Hardy's non-locality and Bell operator. We find that Hardy's non-locality is a sufficient condition for the violation of Bell inequality, the upper bound of Hardy's non-locality allowed by information causality just corresponds to Tsirelson bound of Bell inequality and the upper bound of Hardy's non-locality allowed by the principle of no-signaling just corresponds to the algebraic maximum of Bell operator. Then we study the Cabello's argument of Hardy's non-locality (a generalization of Hardy's argument) and find a similar relation between it and violation of Bell inequality. Finally, we give a simple derivation of the bound of Hardy's non-locality under the constraint of information causality with the aid of the above derived relation between Hardy's non-locality and Bell operator.

Keywords: Hardy's non-locality, Bell inequality, information causality

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1. Introduction

Quantum non-locality is always a fundamental problem in physics research. The violation of Bell inequality claims that quantum non-locality cannot be reproduced with the hidden variable local model.[1,2] In 1992 Hardy proposed his theorem (Hardy's non-locality) which is a manifestation of quantum non-locality without using inequality.[3,4] However, the amount of quantum non-locality is limited by Tsirelson bound,[5] and the whole boundary of quantum non-locality of binary-input and binary-output model has also been studied in Refs. [6] and [7]. Tsirelson bound (including the boundary of quantum non-locality) does not originate solely from the principle of no-signaling, one class of no-signaling theory, initiated by Popescu and Rohrlich (PR) correlation,[8] allows the Bell operator to take its algebraic maximum which greatly exceeds Tsirelson bound. So identifying the physical principles under the limits of quantum non-locality is now an intriguing problem in foundational research of quantum mechanics (QM). In 2005, Dam[9] showed that the PR correlation makes communication complexity trivial (communication complexity is not trivial in QM), but which seems highly implausible in nature. Recently, Pawłowski *et al.*[10] introduced a new physical principle, which is named information causality. This new principle states that the communication of m classical bits causes information gain of at most m bits and they proved that this principle can distinguish physical theories from other no-signaling theories that are endowed with stronger correlations than quantum physics. New progress has been made in Refs. [11] and [12] recently, the authors have proved that the quantum non-locality originates from the combination of the principle of no-signaling and local quantum measurement assumption.

In order to investigate quantum correlation, Barret *et al.*[13] treated it in a general correlation theory—no-signaling theory. They found that no-signaling correlations formed a polytope in probability space and this no-signaling polytope contained the quantum correlations as a subset. In their work they provided a mathematical framework for the research of quantum non-locality. Recently, within this mathematical framework of no-signaling polytope, Ahanj *et al.*[14] gave a bound on Hardy's non-locality under the constraint of information causality. In the present paper we will study the relation between Hardy's non-locality and violation of Bell inequality with the help of no-signaling polytope. We find that the Hardy's non-locality is a sufficient condition for the violation of Bell inequality, the upper bound of Hardy's non-locality limited by the information causality just corresponds to Tsirelson bound of Bell inequality and the

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upper bound of Hardy's non-locality allowed by the principle of no-signaling just corresponds to the algebraic maximum of Bell operator. Then we study the Cabello's argument of Hardy's non-locality and find a similar relation between it and the violation of Bell inequality. Finally, we give a simple derivation of the bound of Hardy's non-locality under the constraint of information causality.

2. No-signaling polytope and Hardy's non-locality

In this section we give a brief introduction of no-signaling polytope and Hardy's non-locality.

2.1. Non-signaling polytope

We still consider the case in which Alice and Bob each are choosing from two input, each of them has two possible outputs. We denote $X(Y) \in \{0,1\}$ and $a(b) \in \{0,1\}$ as Alice's (Bob's) observable and outcome respectively. The joint probabilities $p_{ab|XY}$ form a table with 2^4 entries, although these are not all independent due to the constraints of normalization and the principle of no-signaling. These constraints lead the entire joint probabilities to a convex subset in the form of a polytope in 2^4-dimensional probability vector space, one calls this polytope as no-signaling polytope.[13] No-signaling polytope is eight-dimensional, has 24 vertices, 16 of which are local vertices and 8 of which are nonlocal vertices (they all correspond to PR correlations).

The local vertices can be expressed as

$$p_{ab|XY}^{\alpha\beta\gamma\delta} = \begin{cases} 1, & \text{if} \quad a = \alpha X \oplus \beta, \\ & \quad\ b = \gamma Y \oplus \delta; \\ 0, & \text{else}, \end{cases} \tag{1}$$

where $\alpha, \beta, \gamma, \delta \in \{0,1\}$ and $\oplus$ denotes addition modulo 2.

And the eight nonlocal vertices are

$$p_{ab|XY}^{\alpha\beta\gamma} = \begin{cases} \dfrac{1}{2}, & \text{if} \quad a \oplus b = XY \oplus \alpha X \oplus \beta Y \oplus \gamma, \\ 0, & \text{else}, \end{cases} \tag{2}$$

where $\alpha, \beta, \gamma \in \{0,1\}$.

For the case of two inputs and two outputs, there are eight nontrivial facets of the local correlations and they correspond to eight CHSH inequalities as well as eight nonlocal vertices respectively. Define $\langle ij \rangle$ as

$$\langle ij \rangle = \sum_{a,b=0}^{1} (-1)^{a+b} p_{ab|X=i,Y=j}, \tag{3}$$

then these eight CHSH inequalities can be expressed as the following inequalities:

$$\begin{aligned} B_{\alpha,\beta,\gamma} \equiv\ & (-1)^{\gamma}\langle 00 \rangle + (-1)^{\beta+\gamma}\langle 01 \rangle + (-1)^{\alpha+\gamma}\langle 10 \rangle \\ & + (-1)^{\alpha+\beta+\gamma+1}\langle 11 \rangle \\ \leq\ & 2, \end{aligned} \tag{4}$$

where $\alpha, \beta, \gamma \in \{0,1\}$. We can find that the algebraic maximum of Bell operator is $B_{\alpha,\beta,\gamma} = 4$, each choice of α, β, γ corresponds to one nonlocal vertex of Eq. (2), thus there is a one-to-one correspondence between the nonlocal vertices of no-signaling polytope and the CHSH inequalities.[2,15–17] It is easy to check that each nonlocal vertex returns to a value for the corresponding Bell operator of $B_{\alpha,\beta,\gamma} = 4$.

2.2. Hardy's non-locality

Consider that Alice and Bob share two spin-1/2 particles, denote $\{A, A'\}$ as the measurement set of Alice and $\{B, B'\}$ as the measurement set of Bob and all outcomes of measurement only take values of ± 1. Now consider the following joint probabilities:

$$P(A = +1, B = +1) = q_1, \tag{5}$$

$$P(A' = -1, B = -1) = 0, \tag{6}$$

$$P(A = -1, B' = -1) = 0, \tag{7}$$

$$P(A' = -1, B' = -1) = q_2, \tag{8}$$

when $q_1 = 0$ and $q_2 > 0$ the above four equations represent Hardy's argument of Hardy's non-locality. The general case of $0 \leq q_1 < q_2$ corresponds to the Cabello's argument which is a generalization of Hardy's argument. It can be proved that there is no local realistic theory that can reproduce the predictions of Eqs. (5)–(8).[14] To show this, we consider that there are some local realistic states for which $A' = -1$ and $B' = -1$, corresponding to the validity of Eq. (8). For these local realistic states, equations (6) and (7) show that the outcomes of A and B must be equal to $+1$, so according to local realistic theory, $P(A = +1, B = +1)$ should be at least equal to q_2 and this contradict $q_1 < q_2$. However quantum entangle state can reproduce Hardy's non-locality with suitable measurement setting, an example of the most general nonmaximally entangle state which can produce Hardy's non-locality can be seen in Ref. [18].

So Hardy's non-locality (theorem) manifest the contradiction without inequalities between local realism theory and quantum mechanics.

In Ref. [14], the expressions of Hardy's nonlocality (Hardy's argument and Cabello's argument) were given in terms of vertices of no-signalling polytope. Using the same correspondence as that in Ref. [14]: $(X=0)\leftrightarrow A, (X=1)\leftrightarrow A'$, $(Y=0)\leftrightarrow B, (Y=1)\leftrightarrow B'$ and $a,b=0(1)\leftrightarrow +1(-1)$, one can find that there are only five of the 16 local vertices and one of the eight nonlocal vertices satisfy Hardy's argument Eqs. (5)–(8) ($q_1=0$), these vertices are $p^{0001}_{ab|XY}$, $p^{0011}_{ab|XY}$, $p^{0100}_{ab|XY}$, $p^{1100}_{ab|XY}$, $p^{1111}_{ab|XY}$, and $p^{001}_{ab|XY}$. So Hardy's argument of Eqs. (5)–(8) can be written as a linear superposition of the above six vertices:[14]

$$\begin{aligned} p^{\mathcal{H}}_{ab|XY} &= c_1 p^{0001}_{ab|XY} + c_2 p^{0011}_{ab|XY} + c_3 p^{0100}_{ab|XY} \\ &\quad + c_4 p^{1100}_{ab|XY} + c_5 p^{1111}_{ab|XY} + c_6 p^{001}_{ab|XY}, \end{aligned} \quad (9)$$

where $\sum_{i=1}^{6} c_i = 1$.

It is easy to check from Eq. (9) that the success probability q_2 for Hardy's argument is given by $q_2 = p^{\mathcal{H}}_{11|11} = c_6/2$. We can find that under the no-signaling constraint, the maximum of q_2 is $1/2$ which is achieved when $c_6 = 1$ and the remaining c_i's$= 0$. This result has also been derived in Refs. [19] and [20]. Recently Ahanj *et al.*[14] gave $q_{2\max} = (\sqrt{2}-1)/2$ under the constraint of information causality.[10]

Cabello's argument can also be expressed as a linear superposition of the vertices which satisfy Eqs. (5)–(8) in the case of $0 \le q_1 < q_2$:[14]

$$\begin{aligned} & p^{\mathcal{C}}_{ab|XY} \\ &= p^{\mathcal{H}}_{ab|XY} + c_7 p^{0000}_{ab|XY} + c_8 p^{0010}_{ab|XY} + c_9 p^{1000}_{ab|XY} \\ &\quad + c_{10} p^{1010}_{ab|XY} + c_{11} p^{110}_{ab|XY}, \end{aligned} \quad (10)$$

where the Hardy's argument $p^{\mathcal{H}}_{ab|XY}$ is given in Eq. (9), and the coefficients c_i's satisfy the new normalization condition $\sum_{i=1}^{11} c_i = 1$. It is easy to calculate the success probability for Cabello's argument by using Eq. (10)

$$\begin{aligned} w &\equiv q_2 - q_1 \\ &= p^{\mathcal{C}}_{11|11} - p^{\mathcal{C}}_{00|00} \\ &= \frac{c_6 - c_{11}}{2} - c_7 - c_8 - c_9. \end{aligned} \quad (11)$$

Recently Ahanj *et al.*[14] gave $w_{\max} = (\sqrt{2}-1)/2$ under the constraint of information causality.[10]

3. Hardy's non-locality and Bell inequality

Now we study the relation between the Hardy's non-locality and violation of Bell inequality. We first discuss the relation between q_2 and Bell operator in the case of $q_1 = 0$ (Hardy's argument). We notice that the expression of Hardy's argument (Eq. (9)) has only one non-local vertex $p^{001}_{ab|XY}$, so it is natural to adopt the Bell inequality corresponding to this non-local vertex:

$$B_{0,0,1} \equiv -\langle 00\rangle - \langle 01\rangle - \langle 10\rangle + \langle 11\rangle \le 2. \quad (12)$$

By using Eqs. (3) and (9), we can obtain

$$\begin{aligned} \langle 00\rangle &= -c_1 - c_2 - c_3 - c_4 + c_5 - c_6, \\ \langle 01\rangle &= -c_1 + c_2 - c_3 - c_4 - c_5 - c_6, \\ \langle 10\rangle &= -c_1 - c_2 - c_3 + c_4 - c_5 - c_6, \\ \langle 11\rangle &= -c_1 + c_2 - c_3 + c_4 + c_5 + c_6. \end{aligned} \quad (13)$$

Then we obtain the value of Bell operator $B_{0,0,1}$ as

$$\begin{aligned} B_{0,0,1} &= 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5 + 4c_6 \\ &= 2c_6 + 2 \\ &= 4q_2 + 2. \end{aligned} \quad (14)$$

In the above calculation we have used the normalization condition of $\sum_{i=1}^{6} c_i = 1$ and the relation of $q_2 = c_6/2$. In 2000, Cereceda derived this result as shown in Ref. [20].

From Eq. (14) we find that if the success probability $q_2 > 0$ the violation of Bell inequality can be achieved and it can also be said that the Hardy's non-locality is a sufficient condition for the violation of Bell inequality. We also find that when $q_2 = (\sqrt{2}-1)/2$ the Bell operator $B_{0,0,1}$ reaches Tsirelson bound $2\sqrt{2}$ and this value ($q_2 = (\sqrt{2}-1)/2$) is just equal to the upper bound of Hardy's non-locality under the constraint of information causality.[14] However, it must be noticed that in quantum mechanics the maximum probability of success of the Hardy's non-locality is $q_2 = 0.09$,[21] the reason is that there are also non-quantum correlations which are under Tsirelson bound $2\sqrt{2}$. Under the no-signaling constraint the maximum of q_2 is $1/2$, substitute it into Eq. (14), then we will obtain just the algebraic maximum of Bell operator as $B_{0,0,1} = 4$.

Then we discuss the relation between w and Bell operator in the case of $0 < q_1 < q_2$ (Cabello's argument).

By using Eqs. (3) and (10), we can have

$$\langle 00\rangle = -c_1 - c_2 - c_3 - c_4 + c_5 - c_6$$

$$
\begin{aligned}
&+ c_7 + c_8 + c_9 + c_{10} + c_{11},\\
\langle 01\rangle &= -c_1 + c_2 - c_3 - c_4 - c_5 - c_6\\
&+ c_7 - c_8 + c_9 - c_{10} - c_{11},\\
\langle 10\rangle &= -c_1 - c_2 - c_3 + c_4 - c_5 - c_6\\
&+ c_7 + c_8 - c_9 - c_{10} - c_{11},\\
\langle 11\rangle &= -c_1 + c_2 - c_3 + c_4 + c_5 + c_6\\
&+ c_7 - c_8 - c_9 + c_{10} - c_{11}. \qquad (15)
\end{aligned}
$$

Here we still adopt the Bell inequality of Eq. (12) which corresponds to non-local vertex $p^{001}_{ab|XY}$, we obtain the value of Bell operator $B_{0,0,1}$ for Cabello's argument of Eq. (10) as

$$
\begin{aligned}
B_{0,0,1} &= -\langle 00\rangle - \langle 01\rangle - \langle 10\rangle + \langle 11\rangle\\
&= 2 + 2c_6 - 2c_{11} - 4(c_7 + c_8 + c_9)\\
&= 2 + 4w. \qquad (16)
\end{aligned}
$$

In the above calculation we have used the normalization condition of $\sum_{i=1}^{11} c_i = 1$ and the relation of Eq. (11).

From Eq. (16) we find a similar relation between w and violation of Bell inequality to that of q_2's. If the success probability for Cabello's argument $w = q_2 - q_1 > 0$ the violation of Bell inequality can be achieved. One can also find that when $w = (\sqrt{2}-1)/2$ the Bell operator $B_{0,0,1}$ reaches Tsirelson bound $2\sqrt{2}$ and this value ($w = (\sqrt{2}-1)/2$) is just equal to the upper bound of Cabello's non-locality under the constraint of information causality.[14] Under the no-signaling constraint, the maximum of w is $1/2$ when $c_6 = 1$ and the rest of the c_i's are equal to 0, substitute it into Eq. (16) we obtain just the algebraic maximum of Bell operator of $B_{0,0,1} = 4$.

4. Bound of Hardy's non-locality in the limit of information causality

In this section, we give a simple derivation of the bound of Hardy's non-locality under the constraint of information causality with the aid of the above derived relation between Hardy's non-locality and Bell operator.

The general Bell inequality of Eq. (4) can also be written as

$$
\begin{aligned}
&B'_{\alpha,\beta,\gamma}\\
&= \frac{1}{4}\sum_{X,Y=0}^{1} p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma | XY)\\
&\leq \frac{3}{4}, \qquad (17)
\end{aligned}
$$

where $\alpha, \beta, \gamma \in \{0, 1\}$. The choice of $\alpha = 0, \beta = 0$ and $\gamma = 0$ corresponds to the standard CHSH inequality[2] and is widely used in Refs. [11], [12], and [22]. The relation between $B'_{\alpha,\beta,\gamma}$ and $B_{\alpha,\beta,\gamma}$ is

$$
\begin{aligned}
B'_{\alpha,\beta,\gamma} &= \frac{B_{\alpha,\beta,\gamma} + 4}{8}\\
&= \frac{4q_2 + 6}{8} \quad \text{(Hardy's argument)}\\
&= \frac{4w + 6}{8} \quad \text{(Cabello's argument).} \qquad (18)
\end{aligned}
$$

Tsirelson bound of $B'_{\alpha,\beta,\gamma}$ is $(\sqrt{2}+2)/4$ when $q_2 = (\sqrt{2}-1)/2$ (or $w = (\sqrt{2}-1)/2$).

The general expression of information causality can be written as

$$
A \equiv (2P_1 - 1)^2 + (2P_2 - 1)^2 \leq 1, \qquad (19)
$$

where

$$
\begin{aligned}
P_1 &= \frac{1}{2}\big[p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma | 00)\\
&\quad + p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma | 10)\big],\\
P_2 &= \frac{1}{2}\big[p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma | 01)\\
&\quad + p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma | 11)\big]. \qquad (20)
\end{aligned}
$$

The expression of this principle in Ref. [10] corresponds to the choice of $\alpha = 0, \beta = 0$ and $\gamma = 0$.

The $B'_{\alpha,\beta,\gamma}$ can be written as $(P_1 + P_2)/2$ and then it can be expressed as a function of A, i.e.,

$$
\begin{aligned}
B'_{\alpha,\beta,\gamma} &= \frac{(\sqrt{A}\sin\theta + \sqrt{A}\cos\theta + 2)}{4}\\
&\leq \frac{(\sqrt{2A} + 2)}{4}. \qquad (21)
\end{aligned}
$$

We can find that under the constraint of information causality $A \leq 1$, the upper bound of $B'_{\alpha,\beta,\gamma}$ is $(\sqrt{2}+2)/4$, therefore from Eq. (18) we find that the upper bound of q_2 and w both are $(\sqrt{2}-1)/2$ under the constraint of information causality. This upper bound of q_2 and w can be achieved in the limits of information causality. For example we can take $P_1 = P_2 = (\sqrt{2}+2)/4$, substitute them into Eq. (19) we have $A = 1$, so the information causality is observed; at the same time we substitute them into Eq. (17) and obtain $B'_{\alpha,\beta,\gamma} = (1/2)(P_1 + P_2) = (\sqrt{2}+2)/4$, so from Eq. (18) we can find the upper bound of q_2 and w ($(\sqrt{2}-1)/2$) has been reached. Now we complete the proof that the bound of Hardy's non-locality under the constraint of information causality is $(\sqrt{2}-1)/2$, which is the same as the result in Ref. [14].

5. Conclusion

In this work, we gave the quantitative relations between Hardy's/Cabello's argument of non-locality and violation of Bell inequality. We obtained the analytic expressions of the relations between the success probabilities of the Hardy's/Cabello's argument and the value of Bell operator and then we found that if and only if the success probabilities of the Hardy's/Cabello's argument are greater than zero, the violations of Bell inequality can be achieved. The bound values of the success probabilities of the Hardy's and Cabello's arguments under the constraint of information causality both correspond to Tsirelson bound of Bell operator and the bound values of these two success probabilities under the no-signaling constraint both correspond to the algebraic maximum of Bell operator. Finally, we gave a simple derivation of the bound of Hardy's non-locality under the constraint of information causality with the aid of the above derived relation between Hardy's non-locality and Bell operator, this bound is the main result of Ref. [14].

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