# Global exponential synchronization between Lü system and Chen system with unknown parameters and channel time-delay*

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This paper proposes a nonlinear feedback control method to realize global exponential synchronization with channel time-delay between the Lü system and Chen system, which are regarded as the drive system and the response system respectively. Some effective observers are produced to identify the unknown parameters of the Lü system. Based on the Lyapunov stability theory and linear matrix inequality technique, some sufficient conditions of global exponential synchronization of the two chaotic systems are derived. Simulation results show the effectiveness and feasibility of the proposed controller.

**Keywords:** global exponential synchronization, unknown parameters, channel time-delay, Lyapunov stability theory

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## 1. Introduction

During the last two decades, synchronization of the chaotic systems has attracted considerable attention due to its great potential applications in secure communication, chemical reactions and biological systems.[1] The first idea of synchronization of two chaotic systems with different initial conditions was introduced by Pecora and Carroll.[2] Many different methods are applied theoretically and experimentally to synchronize the chaotic systems, such as linear and nonlinear feedback control,[3−6] backstepping control,[7−9] variable structure control,[10−12] adaptive control,[13−16] impulsive control[17−23] and active control.[24,25] However, most of the methods mentioned above are designed to synchronize two identical chaotic systems. In fact, in many real worlds such as biological systems, laser array and cognitive processes, it is hardly the case that the structure of drive and response chaotic systems can be assumed to be identical, especially when chaotic synchronization is applied to secure communication. In this regard, some works on synchronization between different chaotic systems have been performed.[26−29] Moreover, for some special applications, the parameters of some systems cannot be exactly known a priori and the effect of these uncertainties will destroy the synchronization and even break it. Therefore, it is essential to investigate synchronization of two different systems in the presence of unknown parameters.

On the other hand, as is well known, fast convergence of a system is essential for real-time computation. In the meantime, it is required for the chaotic systems that they are not only synchronized but also with a fast synchronizing rate. Thus, exponential synchronization for chaotic systems has also been investigated in very recent years.[30−32]

In engineering applications, time-delay always exists. Strictly speaking, in the practical environment with signal propagation delays, it is not reasonable to require that the response system synchronizes with the drive system at exactly the same time. For this reason, Ref. [33] re-defined the chaotic synchronization in such a way that the state of response system at time $t$ asymptotically synchronizes with the drive system at time $t-\tau$, namely

$$\lim_{t\to\infty} \|\tilde{\boldsymbol{x}}(t) - \boldsymbol{x}(t-\tau)\| = 0,$$

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where $\tilde{\boldsymbol{x}}$ and $\boldsymbol{x}$ are the state of the response system and the drive system, respectively and $\tau$ is a finite channel time-delay, which is often unknown.

Inspired by the above work, the exponential synchronization of Lü system[34] and Chen system[35] is considered in this paper in the situation of existing channel time-delay and unknown drive system parameters. To estimate the unknown parameters, one kind of observer is proposed, which can identify the full parameters of the drive system. It is worth pointing out that, in Refs. [25] and [26], the processes of parameter estimation and chaos synchronization are simultaneous. By comparison, the two processes in this paper can be done separately (working in parallel also does well) and need not the design of adaptive laws, which is more flexible and simple. In addition, The controller is designed by linear matrix inequality (LMI), which can guarantee the synchronization in an exponential rate. To the best of our knowledge, the topic of nonlinear feedback control for global exponential synchronization between different chaotic systems with unknown parameters and channel time-delay has not been investigated yet.

The rest of the paper is organized as follows. Section 2 describes the systems and their synchronization problem. In Section 3, the observers identifying the unknown parameters of Lü system are presented. Controller design method and simulation results are given in Section 4 and Section 5, respectively. Section 6 concludes the paper.

**Notations** Throughout the paper, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the real number, the $n$-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively. The $\boldsymbol{X}<0$ ($\boldsymbol{X}>0$) means that $\boldsymbol{X}$ is a real symmetric negative (positive) definitive matrix. The $\boldsymbol{I}$ denotes the identity matrix with appropriate dimensions. The $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix 2-norm. The $\lambda_M(\boldsymbol{A})(\lambda_m(\boldsymbol{A}))$ represents the maximum (minimum) eigenvalue of the matrix $\boldsymbol{A}$.

## 2. System description and synchronization problem

Consider the Lü chaotic system described by the following differential equations:

$$\begin{cases} \dot{x}_1(t) = a(x_2(t) - x_1(t)), \\ \dot{x}_2(t) = -x_1(t)x_3(t) + cx_2(t), \\ \dot{x}_3(t) = x_1(t)x_2(t) - bx_3(t), \end{cases} \tag{1}$$

where $a$, $b$ and $c$ are the unknown parameters needed to be identified in the next section and the identification results are $\hat{a}$, $\hat{b}$ and $\hat{c}$ respectively. Therefore, when identification is finished, we nearly have $a=\hat{a}$, $b=\hat{b}$ and $c=\hat{c}$.

To investigate a synchronization problem between the Lü system and Chen system, we assume that the Lü system is the drive system and the controlled Chen system is the response system,

$$\begin{cases} \dot{\tilde{x}}_1(t) = \theta_1(\tilde{x}_2(t) - \tilde{x}_1(t)) + u_1(t), \\ \dot{\tilde{x}}_2(t) = (\theta_2 - \theta_1)\tilde{x}_1(t) - \tilde{x}_1(t)\tilde{x}_3(t) \\ \qquad\qquad +\theta_2\tilde{x}_2(t) + u_2(t), \\ \dot{\tilde{x}}_3(t) = \tilde{x}_1(t)\tilde{x}_2(t) - \theta_3\tilde{x}_3(t) + u_3(t), \end{cases} \tag{2}$$

where $\boldsymbol{u}(t) = [u_1(t), u_2(t), u_3(t)]^{\mathrm{T}}$ is the main control input, $\theta_1$, $\theta_2$ and $\theta_3$ are positive real constants.

Define the synchronization error state vector as $\boldsymbol{e}(t) = \tilde{\boldsymbol{x}}(t) - \boldsymbol{x}(t-\tau) = [\tilde{x}_1(t) - x_1(t-\tau), \tilde{x}_2(t) - x_2(t-\tau),\ \tilde{x}_3(t) - x_3(t-\tau)]^{\mathrm{T}}$, which is the state error between the response system at time $t$ and the drive system at time $t-\tau$, where $\tau$ is an unknown constant.

Replacing $t$ with $t-\tau$ in Eq. (1) gives the drive system equation at time $t-\tau$ as follows:

$$\begin{cases} \dot{x}_1(t-\tau) = a(x_2(t-\tau) - x_1(t-\tau)), \\ \dot{x}_2(t-\tau) = -x_1(t-\tau)x_3(t-\tau) + cx_2(t-\tau), \\ \dot{x}_3(t-\tau) = x_1(t-\tau)x_2(t-\tau) - bx_3(t-\tau), \end{cases} \tag{3}$$

and from Eqs. (2) and (3), the following compact form of the error system is obtained as

$$\dot{\boldsymbol{e}}(t) = \tilde{\boldsymbol{A}}\boldsymbol{e}(t) + \boldsymbol{M}\boldsymbol{e}(t) + (\tilde{\boldsymbol{A}} - \hat{\boldsymbol{A}})\boldsymbol{x}(t-\tau) + \boldsymbol{u}(t), \tag{4}$$

where

$$\hat{\boldsymbol{A}} = \begin{bmatrix} -\hat{a} & \hat{a} & 0 \\ 0 & \hat{c} & 0 \\ 0 & 0 & -\hat{b} \end{bmatrix}, \quad \tilde{\boldsymbol{A}} = \begin{bmatrix} -\theta_1 & \theta_1 & 0 \\ \theta_2 - \theta_1 & \theta_2 & 0 \\ 0 & 0 & -\theta_3 \end{bmatrix},$$

$$\boldsymbol{M}\boldsymbol{e}(t) = \begin{bmatrix} 0 \\ -\tilde{x}_1(t)\tilde{x}_3(t) \\ \tilde{x}_1(t)\tilde{x}_2(t) \end{bmatrix} - \begin{bmatrix} 0 \\ -\tilde{x}_1(t-\tau)\tilde{x}_3(t-\tau) \\ \tilde{x}_1(t-\tau)\tilde{x}_2(t-\tau) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -x_3(t-\tau) & 0 & -\tilde{x}_1(t) \\ x_2(t-\tau) & \tilde{x}_1(t) & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix},$$

i.e.

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 0 \\ -x_3(t-\tau) & 0 & -\tilde{x}_1(t) \\ x_2(t-\tau) & \tilde{x}_1(t) & 0 \end{bmatrix}.$$

**Remark 1** Note that chaotic states are bounded, thus there exists a real constant $m > 0$ such that $\|\boldsymbol{M}\|_\infty = \max\{|x_3| + |\tilde{x}_1|, |x_2| + |\tilde{x}_1|\} \leq m < +\infty$.

Here the aim is to design nonlinear feedback controller $\boldsymbol{u}(t)$ such that the error $\boldsymbol{e}(t)$ converges exponentially to zero, i.e. the trajectory of the response system (2) with any initial condition $\tilde{\boldsymbol{x}}_0(t)$ can exponentially approach the drive system (3) with any initial condition $\boldsymbol{x}_0(t-\tau)$ and finally implements global exponential synchronization.

For the purpose of simplicity, in the following sections, we abbreviate $\tilde{x}_i(t)$ and $x_i\ (t-\tau)$ as $\tilde{x}_i$ and $x_i$, respectively, where $i = 1, 2, 3$.

## 3. Identification of the unknown parameters

In this section, the observers are provided to identify the unknown parameters $a$, $b$ and $c$ of Lü system.

Since the parameters are fully unknown, we do not know the relevant dynamical information about the parameters. However, we can attain the state vector $(x_1, x_2, x_3)$. First, we use the following observers to identify the unknown parameters $b$ and $c$

$$\begin{cases} \dot{\delta}_1 = -k_1\delta_1 + k_1^3 \ln x_3 + k_1 x_1 x_2/x_3, \\ \hat{b} = \delta_1 - k_1 \ln x_3, \end{cases} \tag{5}$$

$$\begin{cases} \dot{\delta}_2 = -k_2 x_2^2 \delta_2 + k_2 x_1 x_2 x_3 - k_2^2 x_2^4/2, \\ \hat{c} = \delta_2 + k_2 x_2^2/2, \end{cases} \tag{6}$$

where $k_1, k_2 > 0$, $\delta_1$ and $\delta_2$ are auxiliary variables. The two observers can converge to $b$ and $c$ exponentially, see Ref. [36] for details of identification, which we omit here.

Now, we will pay more attention to designing the observer which can identify the unknown parameter $a$, we use the following observer:

$$\dot{\hat{a}} = -\hat{a}(x_2 - x_1)F(x_1, x_2) + \dot{x}_1 F(x_1, x_2), \tag{7}$$

where $F(x_1, x_2) \in \mathbb{R}$ is a gain function. Let

$$h = a - \hat{a}, \tag{8}$$

then from Eqs. (1) and (7), we obtain

$$\begin{aligned} \dot{h}(t) &= \dot{a} - \dot{\hat{a}} = -\dot{\hat{a}} \\ &= \hat{a}(x_2 - x_1)F(x_1, x_2) - a(x_2 - x_1)F(x_1, x_2) \\ &= -h(t)(x_2 - x_1)F(x_1, x_2), \end{aligned} \tag{9}$$

note that $a$ is a constant, thus $\dot{a} = 0$.

If $F(x_1, x_2)$ is selected as

$$F(x_1, x_2) = k_3(x_2 - x_1), \tag{10}$$

then we rewrite Eq. (9) as follows:

$$\dot{h}(t) + k_3(x_2 - x_1)^2 h(t) = 0. \tag{11}$$

Obviously, system (11) is exponentially stable for all $x_1$ and $x_2$, that is, $\hat{a}$ converges to $a$ with exponential rate as $t \to \infty$. Where $k_3 > 0$ is the convergence rate.

In practice it is hard to observe $\dot{x}_1$, so the observer (7) is of no use. To overcome the limitation, we introduce the variable

$$\delta_3 = \hat{a} - P(x_1, x_2), \tag{12}$$

where $P(x_1, x_2) \in \mathbb{R}$ is a design function and satisfies

$$F(x_1, x_2) = \partial P(x_1, x_2)/\partial x_1. \tag{13}$$

Choose $P(x_1, x_2) = -k_3(x_2 - x_1)^2/2$ and from Eqs. (1), (7) and (12), we obtain the observer as follows:

$$\begin{cases} \dot{\delta}_3 = \dot{\hat{a}} - \Big[\dfrac{\partial P(x_1, x_2)}{\partial x_1}\dot{x}_1 + \dfrac{\partial P(x_1, x_2)}{\partial x_2}\dot{x}_2\Big] \\ \quad = -\hat{a}(x_2 - x_1)F(x_1, x_2) + \dot{x}_1 F(x_1, x_2) \\ \qquad -\dfrac{\partial P(x_1, x_2)}{\partial x_1}\dot{x}_1 - \dfrac{\partial P(x_1, x_2)}{\partial x_2}\dot{x}_2 \\ \quad = -(\delta_3 + P(x_1, x_2))(x_2 - x_1)F(x_1, x_2) \\ \qquad -\dfrac{\partial P(x_1, x_2)}{\partial x_2}(-x_1 x_3 + \hat{c} x_2) \\ \quad = -k_3(x_2 - x_1)^2\delta_3 + k_3^2(x_2 - x_1)^4/2 \\ \qquad -k_3(x_2 - x_1)(x_1 x_3 - \hat{c} x_2), \\ \hat{a} = \delta_3 - k_3(x_2 - x_1)^2/2. \end{cases} \tag{14}$$

**Remark 2** Note that the observer (14) relies on both the first equation of system (3) and $\hat{c}$ in the observer (6). Compared with the observer (14), the observer (6) only relies on the second equation of the system (3). That is, when the structure of the first and third equations of system (3) or the parameters $a$ and $b$ are changed, the observer (6) is not changed. Similarly, the observer (5) has the same property. Hence, the observers (5) and (6) have stronger robustness. Generally, the above result is determined by the structure of the original system (3).

## 4. Controller design

In this section, we discuss how to design a controller $\boldsymbol{u}(t)$ to realize global exponential synchronization between the Lü system and Chen system. First, the following fact and lemma are needed to derive the main result.

**Fact 1** (Schur complement) The LMI

$$\begin{bmatrix} \boldsymbol{S}_{11}(\boldsymbol{x}) & \boldsymbol{S}_{12}(\boldsymbol{x}) \\ \boldsymbol{S}_{12}^{\mathrm{T}}(\boldsymbol{x}) & \boldsymbol{S}_{22}(\boldsymbol{x}) \end{bmatrix} < 0$$

is equivalent to $\boldsymbol{S}_{22}(\boldsymbol{x}) < 0$ and $\boldsymbol{S}_{11}(\boldsymbol{x}) - \boldsymbol{S}_{12}(\boldsymbol{x})\boldsymbol{S}_{22}^{-1}(\boldsymbol{x})\boldsymbol{S}_{12}^{\mathrm{T}}(\boldsymbol{x}) < 0$, where $\boldsymbol{S}_{11}(\boldsymbol{x}) = \boldsymbol{S}_{11}^{\mathrm{T}}(\boldsymbol{x})$, $\boldsymbol{S}_{22}(\boldsymbol{x}) = \boldsymbol{S}_{22}^{\mathrm{T}}(\boldsymbol{x})$ and $\boldsymbol{S}_{12}(\boldsymbol{x})$ depend affinely on $\boldsymbol{x}$.

**Lemma 1**[37] Given a continuous nonlinear system

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{g}(\boldsymbol{e}(t), t), \tag{15}$$

where $\boldsymbol{e}(t)$ is an $n \times 1$ vector, let $V(\boldsymbol{e}, t)$ be the associated Lyapunov function with the following properties:

$$\lambda_1 \|\boldsymbol{e}\|^2 \le V(\boldsymbol{e}, t) \le \lambda_2 \|\boldsymbol{e}\|^2, \ \forall (\boldsymbol{e}, t) \in \mathbb{R}^n \times \mathbb{R}, \tag{16}$$

$$\dot{V}(\boldsymbol{e}, t) \le -\lambda_3 V(\boldsymbol{e}, t) + \lambda_4 \mathrm{e}^{-\alpha t}, \ \forall (\boldsymbol{e}, t) \in \mathbb{R}^n \times \mathbb{R}, \tag{17}$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ and $\alpha$ are positive scalar constants. Then the state $\boldsymbol{e}(t)$ is globally exponentially stable in the sense that

$$\|\boldsymbol{e}(t)\| \le \begin{cases} \left[\dfrac{\lambda_2}{\lambda_1}\|\boldsymbol{e}(0)\|^2 \mathrm{e}^{-\lambda_3 t} + \dfrac{\lambda_4}{\lambda_1} t \mathrm{e}^{-\lambda_3 t}\right]^{1/2}, & \text{if } \lambda_3 = \alpha, \\ \left[\dfrac{\lambda_2}{\lambda_1}\|\boldsymbol{e}(0)\|^2 \mathrm{e}^{-\lambda_3 t} + \dfrac{\lambda_4}{\lambda_1(\lambda_3 - \alpha)}[\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\lambda_3 t}]\right]^{1/2}, & \text{if } \lambda_3 \ne \alpha. \end{cases} \tag{18}$$

Now, the following theorem gives a sufficient criterion for global exponential stability of the error system (4).

**Theorem 1** If there exist matrix $\boldsymbol{X} = \boldsymbol{P}^{-1} > 0$ and positive scalars $\gamma$ and $\delta$ such that the following inequality holds:

$$\begin{bmatrix} \tilde{\boldsymbol{A}}\boldsymbol{X} + \boldsymbol{X}\tilde{\boldsymbol{A}}^{\mathrm{T}} - 2\gamma\boldsymbol{I} & 2\delta\boldsymbol{X} \\ 2\delta\boldsymbol{X} & -2\delta\boldsymbol{I} \end{bmatrix} < 0, \tag{19}$$

and the nonlinear feedback control is selected as

$$\begin{aligned} \boldsymbol{u}(t) = {} & (\hat{\boldsymbol{A}} - \tilde{\boldsymbol{A}})\boldsymbol{x}(t-\tau) - \gamma\boldsymbol{P}\boldsymbol{e}(t) \\ & - \frac{m^2 \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)\boldsymbol{P}\boldsymbol{e}(t)}{m\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \|\boldsymbol{e}(t)\|_\infty + \varepsilon \mathrm{e}^{-\alpha t}}, \end{aligned} \tag{20}$$

then the origin of the error system (4) is globally exponentially stable in sense of Eq. (18) with

$$\begin{aligned} & \lambda_1 = \frac{1}{2}\lambda_m(\boldsymbol{X}^{-1}), \ \lambda_2 = \frac{1}{2}\lambda_M(\boldsymbol{X}^{-1}), \\ & \lambda_3 = 2\delta\lambda_m(\boldsymbol{X}), \ \ \lambda_4 = \varepsilon, \end{aligned} \tag{21}$$

where $\varepsilon$, $\alpha$ and $m$ are positive scalar constants.

**Proof** Construct a Lyapunov function in the form of

$$V = \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t), \tag{22}$$

the derivative of the Lyapunov function along the trajectory of system (4) is

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} = {} & \frac{1}{2}(\dot{\boldsymbol{e}}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) + \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{e}}(t)) \\ = {} & \frac{1}{2}\Big(\big(\tilde{\boldsymbol{A}}\boldsymbol{e}(t) + \boldsymbol{M}\boldsymbol{e}(t) + (\tilde{\boldsymbol{A}} - \boldsymbol{A})\boldsymbol{x}(t-\tau) \\ & + \boldsymbol{u}(t)\big)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}(t) + \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\big(\tilde{\boldsymbol{A}}\boldsymbol{e}(t) \\ & + \boldsymbol{M}\boldsymbol{e}(t) + (\tilde{\boldsymbol{A}} - \boldsymbol{A})\boldsymbol{x}(t-\tau) + \boldsymbol{u}(t)\big)\Big) \\ = {} & \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{P}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P} - 2\gamma\boldsymbol{P}\boldsymbol{P})\boldsymbol{e}(t) \\ & + \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{M}\boldsymbol{e}(t) \\ & - \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\frac{m^2\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)\boldsymbol{P}\boldsymbol{e}(t)}{m\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \|\boldsymbol{e}(t)\|_\infty + \varepsilon \mathrm{e}^{-\alpha t}}. \end{aligned} \tag{23}$$

By using Fact 1 (Schur complement), we find that the LMI (19) is equivalent to

$$\boldsymbol{P}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P} - 2\gamma\boldsymbol{P}\boldsymbol{P} + 2\delta\boldsymbol{I} < 0, \tag{24}$$

then using the inequality (24) and norm operation to Eq. (23), we have

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} \le {} & -\delta\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) + m\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \|\boldsymbol{e}(t)\|_\infty \\ & - \frac{m^2\|\boldsymbol{P}\boldsymbol{e}(t)\|^2 \|\boldsymbol{e}(t)\|^2}{m\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \|\boldsymbol{e}(t)\|_\infty + \varepsilon \mathrm{e}^{-\alpha t}} \\ \le {} & -\frac{2\delta}{\lambda_M(\boldsymbol{P})}V + m\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \|\boldsymbol{e}(t)\|_\infty \\ & - \frac{m^2\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty^2 \|\boldsymbol{e}(t)\|_\infty^2}{m\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \|\boldsymbol{e}(t)\|_\infty + \varepsilon \mathrm{e}^{-\alpha t}} \\ = {} & -2\delta\lambda_m(\boldsymbol{X})V + \frac{m\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \|\boldsymbol{e}(t)\|_\infty \varepsilon \mathrm{e}^{-\alpha t}}{m\|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \|\boldsymbol{e}(t)\|_\infty + \varepsilon \mathrm{e}^{-\alpha t}} \\ \le {} & -2\delta\lambda_m(\boldsymbol{X})V + \varepsilon \mathrm{e}^{-\alpha t}, \end{aligned} \tag{25}$$

where the following inequalities are utilized:

(i) $\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) \leq \lambda_M(\boldsymbol{P})\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)$, for $\boldsymbol{P} > 0$;

(ii) $0 \leq \|\boldsymbol{e}(t)\|_\infty \leq \|\boldsymbol{e}(t)\|$, $0 \leq \|\boldsymbol{P}\boldsymbol{e}(t)\|_\infty \leq \|\boldsymbol{P}\boldsymbol{e}(t)\|$;

(iii) $0 \leq l_1 l_2/(l_1 + l_2) \leq l_1,\ \forall l_1 > 0, l_2 \geq 0$.

Applying Lemma 1 to inequality (25), the parameters in Eq. (21) can be obtained easily. Then the origin of Eq. (4) is globally exponentially stable, which implies that the drive system (3) and the response system (2) are globally exponentially synchronized. This completes the proof.

**Remark 3** In Eq. (20), the value of channel time-delay $\tau$ is not required to be known for executing the control, since we can obtain the time-delay signal $\boldsymbol{x}(t-\tau)$ easily and use it in the control blindly when it is received.

To realize the synchronization between the systems (3) and (2), in conclusion, the following steps are taken:

Step 1) Designing the observers (5) and (6) to identify the unknown parameters $b$ and $c$ respectively.

Step 2) When the identification in Step 1 is complete, the unknown parameter $a$ can be identified by the observer (14) and the estimate $\hat{c}$ in the observer (6).

Step 3) When the parameters $a$, $b$ and $c$ are identified completely, the controller Eq. (20) is applied to the response system (2) to realize the synchronization.

## 5. Simulation results

In this section, simulation results are given to verify and demonstrate the effectiveness of the observers (5), (6), (14) and the control applicability of the proposed controller Eq. (20).

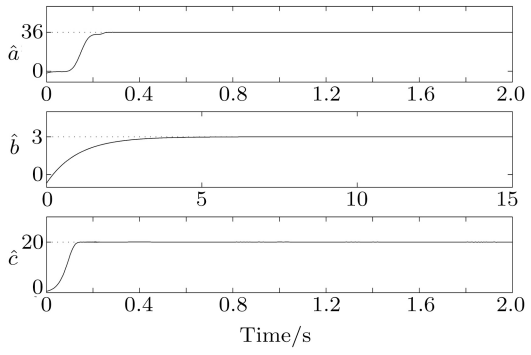



**Fig. 1.** Estimate values of parameters $a$, $b$, $c$ of Lü system with $k_1 = k_2 = k_3 = 1$.

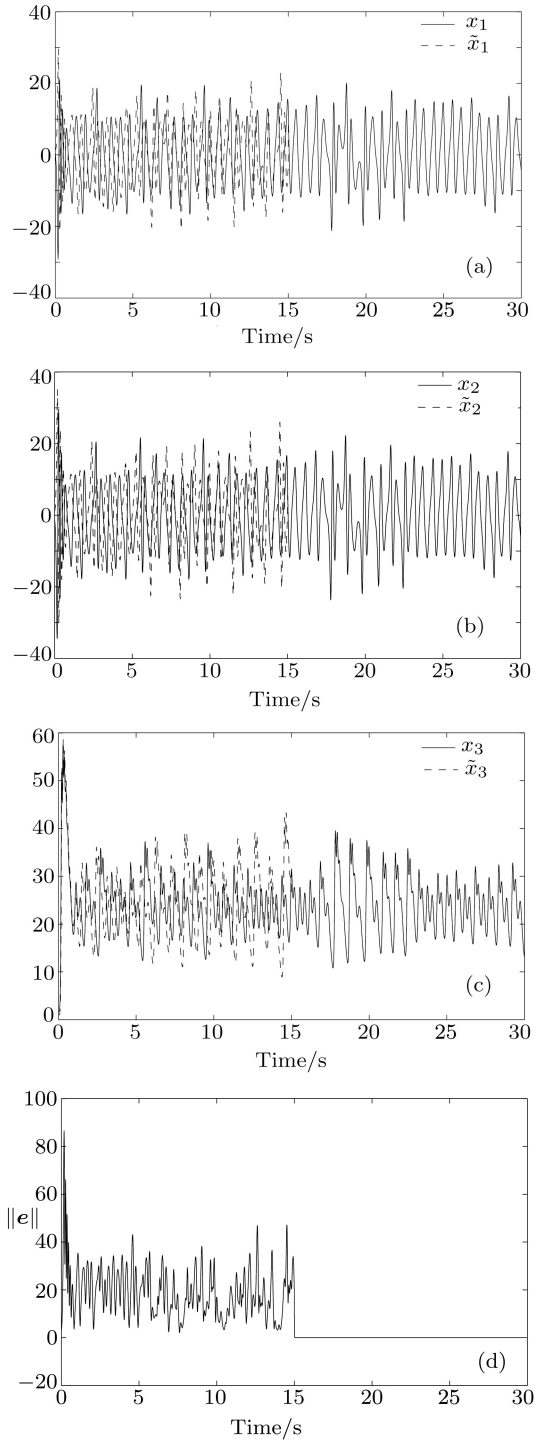



**Fig. 2.** (a)–(c) The time response of states $(x_1, x_2, x_3)$ of the drive system (3) and the states $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ of the response system (2), where the controller (20) is applied at $t = 15$ s; (d) the Euclidean norm of error state vector.

In simulation, the fourth order Runge–Kutta method is used to solve the systems of differential equations (2) and (3) with the controller Eq. (20). We select the 'unknown' parameters of Lü system as $a = 36$, $b = 3$, $c = 20$ and the parameters of Chen system as $\theta_1 = 35$, $\theta_2 = 28$, $\theta_3 = 3$, respectively and $k_1 = k_2 = k_3 = 1$. The initial values of the drive

and the response systems are $\boldsymbol{x}_0 = [1, -1, 2]^{\mathrm{T}}$ and $\tilde{\boldsymbol{x}}_0 = [2, 1, 1]^{\mathrm{T}}$, respectively. Figure 1 shows that the estimates of parameters converge to some constants, which demonstrates the effectiveness of the observers (5), (6) and (14). After 15 s, the parameters $a$, $b$ and $c$ have been identified completely. From now on, we apply the above nonlinear feedback controller (20) to the response system. Let $\varepsilon = 1$, $\alpha = 1$, $m = 75$ and $\delta = 10$. Solving the LMI (19), we obtain the possible solutions:

$$\gamma = 86.2491,$$
$$\boldsymbol{X} = \boldsymbol{P}^{-1} = \begin{bmatrix} 2.9696 & -0.1609 & 0 \\ -0.1609 & 1.7490 & 0 \\ 0 & 0 & 2.6497 \end{bmatrix}.$$

Hence, by Theorem 1, the origin of the error system (4) via the controller (20) is globally exponentially stable in sense of Lemma 1 with $\lambda_1 = 0.1672$, $\lambda_2 = 0.2893$, $\lambda_3 = 34.5623$, $\lambda_4 = 1$, and the simulation results are given in Fig. 2. Obviously, the system (4) is well stabilized.

## 6. Conclusion

In this paper, we investigate global exponential synchronization with channel time-delay between the Lü system and Chen system via a nonlinear feedback control. To identify the fully unknown parameters of the drive system, three effective observers are designed. Furthermore, based on the Lyapunov stability theory and LMI technique, some sufficient conditions of global exponential synchronization of the two chaotic systems are derived. Simulation results show the effectiveness of the proposed method. It should be pointed out that the methods mentioned above can also be applied to most typical chaotic systems, whose error system can be described by system (4), such as the Lorenz system, Rössler system, unified system, several variants of Chua's circuits. Due to the similarity and analogy, the details are not further discussed here.

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