Impulsive control of chaotic systems and its applications in synchronization*

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In this paper, some novel sufficient conditions for asymptotic stability of impulsive control systems are presented by comparison systems. The results are used to obtain the conditions under which the chaotic systems can be asymptotically controlled to the origin via impulsive control. Compared with some existing results, our results are more relaxed in the sense that the Lyapunov function is required to be nonincreasing only along a subsequence of switchings. Moreover, a larger upper bound of impulsive intervals for stabilization and synchronization is obtained.

Keywords: comparison systems, impulsive control, chaotic systems, synchronization

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1. Introduction

There exist many practical examples of impulsive control systems in the real world. Three typical examples are the insect population control system, the chemical reactor system and the financial system.[1] Many researchers have studied impulsive systems and impulsive control in recent years. In Refs. [1]–[4], the authors have considered the stability of impulsive systems by using Lyapunov functions and the Lyapunov functions are required to be nonincreasing along the whole sequence or subsequence of the switchings. The impulsive controller has a simple structure and the controller is discontinuous, which is useful for digital communication systems. Liu and Zhao[5] have presented a new approach to the impulsive stabilization of Liu's systems. Sun *et al.*[6] have obtained some less conservative conditions for the impulsive control of a class of nonlinear systems by using the results in Ref. [3]. Li *et al.*[7] have also given some less conservative conditions for asymptotic stability of impulsive systems with impulses at fixed time and the results have been used to design the impulsive control of a class of nonlinear systems.

On the other hand, a chaotic system has complex dynamical behaviours that possess some special features, such as being extremely sensitive to tiny variations of initial conditions, having bounded trajectories in the phase space with a positive leading Lyapunov. Chaos control and synchronization have been extensively studied over the last decade.[8−21] Lu and Cao[12] have studied the adaptive complete synchronization of chaotic and hyperchaotic systems with totally unknown parameters. In Ref. [13], a unified synchronization criterion for impulsive dynamical networks has been presented.

The concept of impulsive control and synchronization of chaotic systems was first reported in Refs. [4] and [22]. Since then, this problem has been closely connected to the theory of impulsive differential equations. A rigorous theory of the asymptotic stability of impulsive control and synchronization has been presented for autonomous chaotic systems in Refs. [23] and [24] and aroused the interest of many researchers for their various applications in information science, economic systems and autonomous control systems.[4,9,10,25−27] Some applications of impulsive chaotic synchronization to chaotic secure com-

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munication and to chaotic digital code-division multiple access (CDMA) systems have been presented in Refs. [4], [23] and [24] and in Ref. [28], respectively. So far, the continuous and the impulsive synchronization methods have been applied to several well-known chaotic circuits.[29−31] Furthermore, in Ref. [32], Li *et al.* have presented a general theory of impulsive control and analysis of stochastic systems.

In the present paper, based on the theory of comparison systems, we derive some new sufficient conditions for the stability of impulsive systems with impulses at fixed time and the obtained results are used to design impulsive control laws for chaotic systems. Moreover, our method is used to study the impulsive control and synchronization of Chua's circuits. The estimation of the upper bound of the impulsive interval, which is greatly improved by our results, is also presented. Compared with some existing results in Refs. [4], [6], [7], [16], [21] and [33], our results are less conservative.

The rest of the present paper is organized as follows. In Section 2, a new theorem on the stability of impulsive differential equations is given. In Section 3, a stability criterion for impulsive control of chaotic systems is derived. In Section 4, the theory of impulsive control and synchronization of Chua's circuits is presented. An example is given to illustrate the effectiveness and less conservativeness of our results in Section 5. Finally, we draw some conclusions from the present study in Section 6.

2. Impulsive control of nonlinear systems

An impulsive differential system with impulses at fixed time is described by

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)), & t \neq t_k, \\ \Delta \boldsymbol{x}(t) = \boldsymbol{x}(t^+) - \boldsymbol{x}(t^-) = \boldsymbol{U}(k, \boldsymbol{x}), & t = t_k, \end{cases} \tag{1}$$

for $k = 1, 2, \ldots$, where $\boldsymbol{f} : R^+ \times R^n \to R^n$ is continuous; $\boldsymbol{U} : R \times R^n \to R^n$ is continuous; $\boldsymbol{x} \in R^n$ is the state variable; $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$, $t_k \to \infty$ as $k \to \infty$.

Definition 1 Let $V : R^+ \times R^n \to R^+$, then V is said to belong to class V_0 if

(i) V is continuous in $(t_{k-1}, t_k] \times R^n$ and

$$\lim_{(t,\boldsymbol{y}) \to (t_k^+, \boldsymbol{x})} V(t, \boldsymbol{y}) = V(t_k^+, \boldsymbol{x})$$

exists for each $\boldsymbol{x} \in R^n$, $k = 1, 2, \ldots$;

(ii) V is locally Lipschitzian in $\boldsymbol{x}$.

Definition 2 For $(t, \boldsymbol{x}) \in (t_{i-1}, t_i] \times R^n$ we define

$$D^+V(t, \boldsymbol{x}) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, \boldsymbol{x} + h\boldsymbol{f}(t, \boldsymbol{x})) - V(t, \boldsymbol{x})].$$

Definition 3 Comparison system: let $V \in V_0$ and assume that

$$\begin{cases} D^+V(t, \boldsymbol{x}) \leq g(t, V(t, \boldsymbol{x})), \ t \neq t_k, \\ V(t, \boldsymbol{x} + \boldsymbol{U}(k, \boldsymbol{x})) \leq \psi_k(V(t, \boldsymbol{x})), \\ \quad t = t_k, \ k = 1, 2, \ldots, \end{cases} \tag{2}$$

where $\boldsymbol{g} : R^+ \times R^+ \to R$ is continuous and $\psi_k : R^+ \to R^+$ is non-decreasing, then the following system

$$\begin{cases} \dot{w}(t) = g(t, w(t)), & t \neq t_k, \\ w(t_k^+) = \psi_k(w(t_k)), & t = t_k, \ k = 1, 2, \ldots, \\ w(t_0^+) = w_0 \geq 0 \end{cases} \tag{3}$$

will be said to be the comparison system of system (1).

To support our analysis, the following result is presented.

Lemma 1[6] Assume the following three conditions are satisfied:

(i) $V : R^+ \times R^n \to R^+$, $V \in V_0$, $K'(t)V(t, \boldsymbol{x}) + K(t)D^+V(t, \boldsymbol{x}) \leq g(t, K(t)V(t, \boldsymbol{x}))$, $t \neq t_k$, where g is continuous in $(t_{k-1}, t_k] \times R^n$. For each $x \in R^n$, $k = 1, 2, \ldots$, $\lim_{(t,\boldsymbol{y}) \to (t_k^+, \boldsymbol{x})} g(t, \boldsymbol{y}) = g(t_k^+, \boldsymbol{x})$ exists. $K(t) \geq c > 0$, $\lim_{t \to t_k^-} K(t) = K(t_k)$, $\lim_{t \to t_k^+} K(t)$ exists, $k = 1, 2, \ldots$, $K'(t) = \lim_{h \to 0^+} \frac{1}{h}[K(t+h) - K(t)]$;

(ii) $K(t_k + 0)V(t_k + 0, \boldsymbol{x} + \boldsymbol{U}(k, \boldsymbol{x})) \leq \psi_k(K(t_k)V(t_k, \boldsymbol{x}))$, $k = 1, 2, \ldots$;

(iii) $V(t, 0) = 0$ and $a(||\boldsymbol{x}||) \leq V(t, \boldsymbol{x})$ on $R^+ \times R^n$, where $a(\cdot) \in \Re$ (class of continuous strictly increasing functions $a : R^+ \to R^+$ such that $a(0) = 0$), then the globally asymptotic stability of the trivial solution $w = 0$ of comparison system (3) will imply the globally asymptotic stability of the trivial solution of impulsive system (1).

Employing the lemma, we can obtain a new impulsive control result of system (1). Let $m > 1$ be an integer, let $g(t, w) = \dot{\lambda}(t)w$, where $\lambda(t)$ is non-decreasing, $\lambda(t)$ is left continuous at t_k, i.e., $\lim_{t \to t_k^-} \lambda(t) = \lambda(t_k)$ and $\lim_{t \to t_k^+} \lambda(t)$ exists and let $\psi_k(w) = d_k w$, where $d_k > 0$ for all $k = 1, 2, \ldots$, then the origin of system (1) will be globally asymptotically

stable if the conditions of Lemma 1 and the following conditions hold

$$\text{(i)} \qquad d_0 = \inf_k\{d_k\} > 0; \tag{4}$$

(ii) there exists an $r > 1$ such that

$$\sum_{j=1}^{m} \lambda(t_{m(k-1)+j+1}) + \ln\left(r\prod_{j=1}^{m} d_{m(k-1)+j}\right) \leq \sum_{j=1}^{m} \lambda(t^+_{m(k-1)+j}),\ k = 1, 2, \ldots. \tag{5}$$

Especially, when $\lambda(t) \in C^1[R^+, R^+]$, $\dot{\lambda}(t) \geq 0$, then condition (ii) can be simplified to 'there exists an $r > 1$ such that $\lambda(t_{mk+1}) + \ln\left(r\prod_{j=1}^{m} d_{m(k-1)+j}\right) \leq \lambda(t_{m(k-1)+1})$, $k = 1, 2, \ldots$'.

The selection of m depends on the actual system considered. When $m = 2$, then condition (ii) can be simplified to $\lambda(t_{2k}) + \lambda(t_{2k+1}) + \ln(rd_{2k-1}d_{2k}) \leq \lambda(t^+_{2k-1}) + \lambda(t^+_{2k})$, $k = 1, 2, \ldots$. When $m = 1$, letting $g(t, w) = \dot{\lambda}(t)w$, $\lambda \in C^1[R^+, R^+]$, $\dot{\lambda}(t) \geq 0$, $\psi_k(w) = d_k w$, $d_k \geq 0$ for all k, we can obtain that the origin of system (1) is globally asymptotically stable if the conditions of Lemma 1 hold and there exists $r > 1$ such that $\lambda(t_{k+1}) + \ln(rd_k) \leq \lambda(t_k)$ (see Ref. [2]).

3. Impulsive control of chaotic systems

In this section, we study the impulsive control of chaotic systems by using the obtained results in the previous section. The following chaotic system is first considered:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{\phi}(\boldsymbol{x}(t)), \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), \end{cases} \tag{6}$$

where $\boldsymbol{x}(t) \in R^n$ is the state vector, $\boldsymbol{A}$ is the $n \times n$ constant matrix, $\boldsymbol{y} \in R^r$ is the output vector, $\boldsymbol{C}$ is the $r \times n$ constant matrix, $\boldsymbol{\phi} : R^n \to R^n$ is a nonlinear function that gives rise to chaotic dynamics and satisfies

$$2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\phi}(\boldsymbol{x}(t)) \leq \boldsymbol{x}^{\mathrm{T}}(t)(\boldsymbol{H} + \boldsymbol{H}^{\mathrm{T}})\boldsymbol{x}(t), \tag{7}$$

where $\boldsymbol{H}$ is the matrix that is known in advance depends on chaotic system.

The impulsive control of chaotic system is then given as follows:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{\phi}(\boldsymbol{x}(t)), \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), & t \neq t_k, \\ \boldsymbol{x}(t_k^+) = \boldsymbol{x}(t_k) + \boldsymbol{U}(t_k, \boldsymbol{x}(t_k)), \\ \boldsymbol{U}(t_k, \boldsymbol{x}(t_k)) = \boldsymbol{B}_k\boldsymbol{C}\boldsymbol{x}(\boldsymbol{x}_k), & k = 1, 2, \ldots, \end{cases} \tag{8}$$

where $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$; $t_k \to \infty$ as $k \to \infty$. For the integer $m > 1$, we denote

$$\begin{aligned} \Delta_1 &= \sup_{k=1,2,\ldots}\{t_k - t_{k-1}\}, \\ &\vdots \\ \Delta_m &= \sup_{k=1,2,\ldots}\{t_{m(k-1)+1} - t_{m(k-2)+1}\}. \end{aligned} \tag{9}$$

To use the results obtained in the previous section, system (8) can be rewritten as

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{\phi}(\boldsymbol{x}(t)), & t \neq t_k, \\ \Delta\boldsymbol{x}(t_k) = \boldsymbol{B}_k\boldsymbol{C}\boldsymbol{x}(t_k), & k = 1, 2, \ldots, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0. \end{cases} \tag{10}$$

If $||\boldsymbol{\phi}(\boldsymbol{x})|| \leq L||\boldsymbol{x}||$, where $L > 0$ is a constant, let the $n \times n$ matrix $\boldsymbol{P}$ be symmetric and positive definite, $\lambda_{\max}$ and $\lambda_{\min}$ be the largest and the smallest eigenvalues of $\boldsymbol{P}$, respectively, let $\boldsymbol{Q} = \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}$ and $\boldsymbol{Q} \leq \gamma_1\boldsymbol{P}$ with γ_1 being a constant and let m be an integer and $m > 1$, then the origin of impulsive system (10) is asymptotically stable if the following conditions hold:

(i) $(\boldsymbol{I} + \boldsymbol{B}_k\boldsymbol{C})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{I} + \boldsymbol{B}_k\boldsymbol{C}) \leq \gamma_2(k)\boldsymbol{P}$, where $\boldsymbol{I}$ is the identity matrix, $\gamma_2(k)$, $(k = 1, 2, \ldots)$ are positive constants and $\inf_k\{\gamma_2(k)\} > 0$;

(ii) there exist $a > 1$ and a differentiable (at $t \neq t_i$, $i = 1, 2, \ldots$) and non-increasing function $K(t) \geq c > 0$ which satisfies

$$-\frac{K'(t)}{K(t)} \leq \gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{1}{t_{mk+1} - t_{m(k-1)+1}} \times \ln\left[\frac{1}{a}\prod_{j=1}^{m}\frac{K(t^+_{m(k-1)+j})}{\gamma_2(t_{m(k-1)+j})K(t_{m(k-1)+j+1})}\right], \tag{11}$$

for $k = 1, 2, \ldots$.

In the following, we show the importance of above two conditions in the stability analysis. Let $V(t, \boldsymbol{x}(t)) = \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t)$, then we will have

$$\begin{aligned} &K(t)D^+V(\boldsymbol{x}(t)) + K'(t)V(\boldsymbol{x}(t)) \\ &= K(t)((\boldsymbol{A}\boldsymbol{x} + \boldsymbol{\phi}(\boldsymbol{x}))^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{\phi}(\boldsymbol{x}))) \\ &\quad + K'(t)V(\boldsymbol{x}(t)) \end{aligned}$$

$$\begin{aligned}
&\leq K(t)(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}})V(\boldsymbol{x}(t)) + K'(t)V(\boldsymbol{x}(t)) \\
&= \left[\left(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}}\right) + \frac{K'(t)}{K(t)}\right]K(t)V(\boldsymbol{x}(t)), \\
&\quad t \neq t_k. \qquad (12)
\end{aligned}$$

Hence condition (i) of Lemma 1 is satisfied with

$$g(t, w(t)) = \left[\left(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}}\right) + \frac{K'(t)}{K(t)}\right]w(t).$$

When $t = t_k$, $k = 1, 2, \ldots$, from (i) we have

$$\begin{aligned}
&K(t^+)V(t^+, \boldsymbol{x}(t) + \boldsymbol{U}(t_k, \boldsymbol{x}(t_k))) \\
&\leq K(t)V((\boldsymbol{I} + \boldsymbol{B}_k\boldsymbol{C})\boldsymbol{x}(t)) \\
&= K(t)\boldsymbol{x}^{\mathrm{T}}(t)(\boldsymbol{I} + \boldsymbol{B}_k\boldsymbol{C})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{I} + \boldsymbol{B}_k\boldsymbol{C})\boldsymbol{x}(t) \\
&\leq K(t)\gamma_2(k)V(t, \boldsymbol{x}(t)). \qquad (13)
\end{aligned}$$

Then condition (ii) of Lemma 1 is satisfied with $\psi_k(w(t)) = \gamma_2(k)w(t)$, $k = 1, 2, \ldots$. It is clear that the asymptotic stability of impulsively control system (10) can be guaranteed by that of the following comparison system:

$$\begin{cases}
\dot{w}(t) = \left[\left(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}}\right) + \frac{K'(t)}{K(t)}\right]w(t), \ t \neq t_k, \\
w(t_k^+) = \gamma_2(k)w(t_k), \ t = t_k, \ k = 1, 2, \ldots, \\
w(t_0^+) = w_0 \geq 0.
\end{cases} \qquad (14)$$

Since $K(t)$ is non-increasing, for any $i = 0, 1, 2, \ldots$,

$$\begin{aligned}
&\lambda(t_{i+1}) - \lambda(t_i^+) \\
&= \int_{t_i^+}^{t_{i+1}} \left(\left(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}}\right) + \frac{K'(t)}{K(t)}\right)\mathrm{d}t \\
&= \left(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}}\right)(t_{i+1} - t_i) + \ln\frac{K(t_{i+1})}{K(t_i^+)}. \qquad (15)
\end{aligned}$$

Furthermore, from condition (ii) we obtain

$$\begin{aligned}
&\sum_{j=1}^{m}\lambda(t_{m(k-1)+j+1}) - \sum_{j=1}^{m}\lambda(t_{m(k-1)+j}^+) \\
&= \sum_{j=1}^{m}\left[\left(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}}\right)(t_{m(k-1)+j+1} - t_{m(k-1)+j})\right. \\
&\quad \left. + \ln\frac{K(t_{m(k-1)+j+1})}{K(t_{m(k-1)+j}^+)}\right] \\
&\leq \left(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}}\right)(t_{mk+1} - t_{m(k-1)+1}) \\
&\quad + \sum_{j=1}^{m}\left[\ln\frac{K(t_{m(k-1)+j+1})}{K(t_{m(k-1)+j}^+)}\right] \\
&\leq -\ln a\prod_{j=1}^{m}\gamma_2(t_{m(k-1)+j}), \qquad (16)
\end{aligned}$$

where the last inequality holds by inequality (11). Thus, inequality (5) is satisfied. Moreover, since $d_k = \gamma_2(k)$, $\inf_k\{d_k\} > 0$, condition (4) is also satisfied. Hence, system (10) is asymptotically stable.

When $m = 1$, replace inequality (11) with

$$\begin{aligned}
-\frac{K'(t)}{K(t)} &\leq \gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}} \\
&\leq \frac{1}{t_{k+1} - t_k}\ln\left[\frac{1}{a\gamma_2(k)}\frac{K(t_k^+)}{K(t_{k+1})}\right] \qquad (17)
\end{aligned}$$

for $k = 1, 2, \ldots$, then the origin of impulsive system (10) is also asymptotically stable.

Remark 1 One can obtain Lemma 1 in Ref. [4] by assuming $K(t)$ to be a constant, $\boldsymbol{B}_k = \boldsymbol{B}_{k+1}$, $k = 1, 2, \ldots$, $\boldsymbol{B}_k\boldsymbol{C}$ to be symmetric and $\rho(\boldsymbol{I} + \boldsymbol{B}_k) \leq 1$, where $\rho(\boldsymbol{I} + \boldsymbol{B}_k)$ denotes the spectral radius of matrix $\boldsymbol{I} + \boldsymbol{B}_k$, $k = 1, 2, \ldots$. Thus our results can be applied to a much wider class of nonlinear systems.

Remark 2 Especially, when $K(t)$ is a constant and $m = 2$, the condition $\inf_k\{\gamma_2(k)\} > 0$ depends only on $\boldsymbol{B}_k(k = 1, 2, \ldots)$, while the condition

$$\begin{aligned}
&\sup_i\left\{\gamma_2(i)\exp\left[\left(\gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}}\right)(t_{i+1} - t_i)\right.\right. \\
&\left.\left. + \ln\frac{K(t_{i+1})}{K(t_i^+)}\right]\right\} < \infty
\end{aligned}$$

in Theorem 3 of Ref. [7] depends on $\boldsymbol{B}_k(k = 1, 2, \ldots)$, $K(t)$, eigenvalues of $\boldsymbol{P}$ and so on. Hence, the conditions in the proposed result are much simpler and easier to verify than Theorem 3 of Ref. [7] and Theorem 3 in Ref. [6].

Remark 3 In our obtained results, $K(t)$ is non-increasing, then $\frac{K(t_{i+1})}{K(t_i^+)} \leq 1$ for any $i = 1, 2, \ldots$. Hence conditions (5.22) of Ref. [21], (21) of Ref. [4] and (22) of Ref. [6] cannot be derived from condition (11).

Let $K(t)$ be a constant and $\gamma_2(k) = \gamma_2$, $k = 1, 2, \ldots$, and replace inequality (11) with

$$0 \leq \gamma_1 + 2L\frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{1}{t_{mk+1} - t_{m(k-1)+1}}\ln\frac{1}{a\gamma_2^m} \qquad (18)$$

for $k = 1, 2, \ldots$, then the origin of impulsive system (10) will be asymptotically stable.

Remark 4 Condition (18) implies that the Lyapunov function $V(t, \boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$ is only required to be non-increasing along a subsequence $\{t_{m(k-1)+1}\}$, $k = 1, 2, \ldots$ of switchings rather than all the switchings $\{t_k\}$, $k = 1, 2, \ldots$ as in Refs. [1] and [4]. Hence, our result is less conservative.

Remark 5 When $m = 2$, from inequality (18), for $\sigma > 0$, let $(\sigma + 1)\Delta_1 = \Delta_2$, then

$$\Delta_1 \leq -\frac{\ln a\gamma_2^2}{(\gamma_1 + 2L(\lambda_{\max}/\lambda_{\min}))(1+\sigma)}.$$

Moreover, when $\sigma = 1$ as in Ref. [4], then

$$\Delta_1 \leq -\frac{\ln a\gamma_2^2}{2(\gamma_1 + 2L(\lambda_{\max}/\lambda_{\min}))},$$

which means that the upper bound of impulsive interval here is bigger than the upper bound obtained

$$\Delta_1 \leq -\frac{\ln a\gamma_2}{\gamma_1 + 2L(\lambda_{\max}/\lambda_{\min})}$$

in inequality (21) of Ref. [4] and inequality (25) of Ref. [1]. Their difference is $[\gamma_1 + 2L(\lambda_{\max}/\lambda_{\min})]^{-1} \times \ln\sqrt{a}$. It should be noted that the difference cannot be arbitrarily big, since a should satisfies $\ln(1/a\gamma_2^2) > 0$ obtained from inequality (18), where γ_2 is determined by the matrix $\boldsymbol{I} + \boldsymbol{B}_k\boldsymbol{C}$, $k = 1, 2, \ldots, r$. Hence, the choice of matrix $\boldsymbol{B}_k$ is important for the influence on the proposed results. The smaller the value of $\gamma_2(> 0)$, the bigger a can be chosen, the bigger the value of $\ln\sqrt{a}/[\gamma_1 + 2L(\lambda_{\max}/\lambda_{\min})]$ is, as a result, the bigger the upper bound of impulsive interval is. It should also be noticed that the bigger the value of m is, the bigger their difference is. The choice of m depends on the actual system considered.

We could use inequality (18) in order to find the control matrix $\boldsymbol{B}_k$, $k = 1, 2, \ldots$. For example, we may choose diagonal matrix $\boldsymbol{B}_k$, $k = 1, 2, \ldots$, such that inequality (18) holds.

In the following section, we will use the results obtained in this section to consider Chua's circuits and obtain some less conservative results of stabilization and synchronization.

4. Impulsive control on synchronization of Chua's circuits

Chua's circuit is a simple electronic circuit exhibiting a wide variety of bifurcation and chaotic phenomena. Because of its simplicity and universality, Chua's circuit has attracted much attention and has been studied via numerical, mathematical and experimental approaches. It is universal because Chua's circuit has proved mathematically to be chaotic in the sense of Shil'nikov's theorem.[34] It is simple because it contains only one simple nonlinear element and four linear elements. The dimensionless form of the Chua's circuit is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{\phi}(\boldsymbol{x}(t)), \tag{19}$$

where

$$\boldsymbol{A} = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}, \quad \boldsymbol{\phi}(\boldsymbol{x}) = \begin{bmatrix} -\alpha \boldsymbol{f}(\boldsymbol{x}_1) \\ 0 \\ 0 \end{bmatrix},$$

$$\boldsymbol{H} = \begin{bmatrix} |v_2\alpha| & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{20}$$

and $\boldsymbol{f}(\boldsymbol{x})$ is the piecewise-linear characteristic of the Chua's diode and is given as $\boldsymbol{f}(\boldsymbol{x}(t)) = v_1\boldsymbol{x}(t) = (1/2)(v_2 - v_1)(|\boldsymbol{x}(t) + 1| - |\boldsymbol{x}(t) - 1|)$, $v_2 < v_1 < 0$.

Given the impulsive control as follows:

$$\Delta\boldsymbol{x}(t_k) = \boldsymbol{B}\boldsymbol{x}(t_k), \quad k = 1, 2, \ldots, \tag{21}$$

where $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$; $t_k \to \infty$ as $k \to \infty$. Denote $\Delta_1, \ldots, \Delta_m$ as those in expression (9) for the integer $m > 1$.

Let $\chi(\boldsymbol{A}, \boldsymbol{H}) = \lambda_{\max}((\boldsymbol{A}+\boldsymbol{H})^{\mathrm{T}} + \boldsymbol{A} + \boldsymbol{H})$, where H satisfies inequality (7) and construct $V(t, \boldsymbol{x}(t)) = \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{x}(t)$. If there exist $a > 1$ and a differentiable at $t \neq t_i$, $i = 1, 2, \ldots$ and non-increasing function $K(t) \geq c > 0$ which satisfies

$$-\frac{K'(t)}{K(t)} \leq \chi(\boldsymbol{A}, \boldsymbol{H}) \leq \frac{1}{t_{2k+1} - t_{2k-1}} \times \ln\left[\frac{1}{ad_2^2}\frac{K(t_{2k-1}^+)K(t_{2k}^+)}{K(t_{2k})K(t_{2k+1})}\right] \tag{22}$$

for $k = 1, 2, \ldots$, where $d_2 = \rho^2(\boldsymbol{I} + \boldsymbol{B}) > 0$, then impulsive system (19) is also asymptotically stable.

Unlike the impulsive control Chua's circuit in Refs. [4] and [21], there is no need to assume $\boldsymbol{B}$ to be a symmetric matrix and satisfy $\rho(\boldsymbol{I} + \boldsymbol{B}) < 1$ here. Let $K(t)$ be a constant, $\boldsymbol{B}$ be a symmetric matrix, $\rho(\boldsymbol{I} + \boldsymbol{B}) < 1$ and $\Delta_2 = (1 + \epsilon_1)\Delta_1$, then we will obtain Corollary 5.4.1 in Ref. [21].

Now we study the impulsive synchronization of two Chua's circuits, which are called the drive system and the response system respectively. The drive system is given by Eq. (19) and the response system is given as follows:

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{\phi}(\tilde{\boldsymbol{x}}(t)), \tag{23}$$

where $\tilde{\boldsymbol{x}}^{\mathrm{T}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is the state variable of the system. Let $\boldsymbol{e}^{\mathrm{T}} = (e_1, e_2, e_3) = (x_1 - \tilde{x}_1, x_1 - \tilde{x}_1, x_1 - \tilde{x}_1)$

be the synchronization error,

$$\boldsymbol{\phi}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{\phi}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} -\alpha \boldsymbol{f}(\boldsymbol{x}) + \alpha \boldsymbol{f}(\tilde{\boldsymbol{x}}) \\ 0 \\ 0 \end{bmatrix}, \quad (24)$$

then error system of the impulsive synchronization under impulsive control (21) can be given by

$$\begin{cases} \dot{\boldsymbol{e}}(t) = \boldsymbol{A}\boldsymbol{e}(t) + \boldsymbol{\phi}(\boldsymbol{x},\tilde{\boldsymbol{x}}), & t \neq t_k, \\ \Delta \boldsymbol{e}(t_k) = \boldsymbol{B}\boldsymbol{e}(t_k), & k = 1,2,\ldots. \end{cases} \quad (25)$$

Similar to the case of stabilizing Chua's circuit, if there exist $a > 1$ and a differentiable (at $t \neq t_i,\ i = 1,2,\ldots$) and non-increasing function $K(t) \geq c > 0$ which satisfies inequality (22), then error system (25) is asymptotically stable.

In general, assume that the $n \times n$ matrix $\boldsymbol{P}$ is symmetric and positive definite, given by

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p_1 & p_2 \\ 0 & p_2 & p_3 \end{bmatrix}, \quad (26)$$

$(\boldsymbol{A}+\boldsymbol{H})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{H}) \leq b\boldsymbol{P}$, $b_0 = \min_p\{b\}$, m is an integer and $m > 1$, then the origin of impulsive system (25) is asymptotically stable if the following conditions hold:

(i) $(\boldsymbol{I}+\boldsymbol{B})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{I}+\boldsymbol{B}) \leq d\boldsymbol{P}$,

(ii) There exist $a > 1$ and a differentiable (at $t \neq t_i,\ i = 1,2,\ldots$) and non-increasing function $K(t) \geq c > 0$ which satisfies

$$-\frac{K'(t)}{K(t)} \leq b_0 \leq \frac{1}{t_{mk+1} - t_{m(k-1)+1}} \times \ln\left[\frac{1}{ad^m}\prod_{j=1}^{m}\frac{K(t^+_{m(k-1)+j})}{K(t_{m(k-1)+j+1})}\right] \quad (27)$$

for $k = 1,2,\ldots$.

In fact, consider the Lyapunov function as $V(\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}$, then we will have

$$2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\phi}(\boldsymbol{x},\tilde{\boldsymbol{x}}) \leq \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{H} + \boldsymbol{H}^{\mathrm{T}}\boldsymbol{P})\boldsymbol{e}, \quad t \neq t_k, \quad (28)$$

$$V(\boldsymbol{e}(t_k^+)) \leq dV(\boldsymbol{e}(t_k)). \quad (29)$$

Calculation gives that

$$\begin{aligned} & K(t)D^+V(\boldsymbol{e}(t)) + K'(t)V(\boldsymbol{e}(t)) \\ =\ & K(t)(\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{e} + \phi^{\mathrm{T}}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{P}e \\ & + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\phi}(\boldsymbol{x},\tilde{\boldsymbol{x}})) + K'(t)V(\boldsymbol{e}(t)) \\ =\ & K(t)(\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{e} \\ & + 2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\phi}(\boldsymbol{x},\tilde{\boldsymbol{x}})) + K'(t)V(\boldsymbol{e}(t)) \\ \leq\ & K(t)(\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{e} \\ & + \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{H} + \boldsymbol{H}^{\mathrm{T}}\boldsymbol{P})\boldsymbol{e}) + K'(t)V(\boldsymbol{e}(t)) \\ =\ & K(t)\boldsymbol{e}^{\mathrm{T}}((\boldsymbol{A}+\boldsymbol{H})^{\mathrm{T}}\boldsymbol{P} \\ & + \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{H}))\boldsymbol{e} + K'(t)V(\boldsymbol{e}(t)) \\ \leq\ & \left(b_0 + \frac{K'(t)}{K(t)}\right)K(t)V(\boldsymbol{e}(t)),\ t \neq t_k. \end{aligned} \quad (30)$$

Then the error system given by expression (25) can be shown to be asymptotically stable.

5. Example

To illustrate the effectiveness and the less conservativeness of our results, we give the following example of Chua's circuits.

The Chua's circuit is described by Eq. (19). Let the parameters of Chua's circuit be $v_1 = -0.722451121$, $v_2 = -1.138411196$, $\alpha = -9.35159085$, $\beta = -14.790313805$ and $\gamma = -0.016073965$ as in Ref. [21], then $\lambda_{\max}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}) = 13.4069$, $\lambda_{\max}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}) + 2|v_2\alpha| = 34.6988$ and $\chi(\boldsymbol{A},\boldsymbol{H}) = 16.8385$. Choose the impulsive matrix $\boldsymbol{B}$ as

$$\boldsymbol{B} = \begin{bmatrix} -1.04 & 0 & 0 \\ 0.02 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (31)$$

and let $\boldsymbol{P} = \boldsymbol{I}$ and $m = 3$, then $b_0 = \chi(\boldsymbol{A},\boldsymbol{H}) = 16.8385$.

Let $d = 0.0025$, $a = 300$, $t_{k+1} - t_k = \Delta_1$, $k = 1,2,\ldots$, then $\Delta_3 = 3\Delta_1$. For simplicity, let $K(t)$ be a constant. From inequality (27), we can choose Δ_1 such that

$$b_0 \leq \frac{1}{3\Delta_1}\ln\frac{1}{ad^3}, \quad (32)$$

i.e.,

$$0 < \Delta_1 \leq \frac{1}{3b_0}\ln\frac{1}{ad^3}. \quad (33)$$

By computation, $\Delta_1 \leq 0.2429$. Thus, the error system given by expression (25) has been shown to be asymptotically stable based on impulsive matrix (31) we have designed.

Table 1. Comparison among corresponding upper bound values of Δ_1.

Theorem 4.4	Ref. [21]	Ref. [16]	Ref. [33]
0.2429	0.0171	0.0080	0.0215

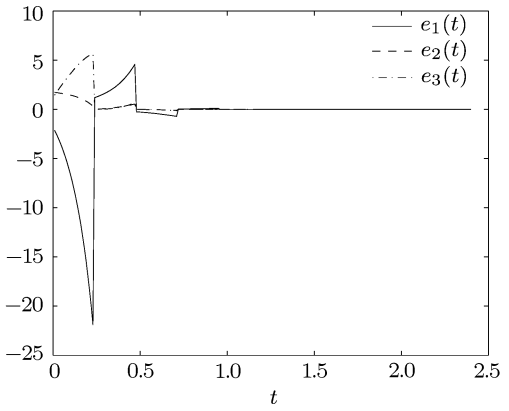


Fig. 1. Synchronization errors of Chua's circuits (25) with $\Delta_1 = 0.24$.

It should be noticed that our impulsive matrix $\boldsymbol{B}$ is unsymmetric, which means that the approaches to the impulsive control Chua's circuit in Refs. [4] and [21] are useless for our example. By choosing the integer $m = 3$, the upper bound value of Δ_1 can be calculated from inequality (33). In fact, we can choose a bigger integer m as shown in Remark 3.8, since the bigger m we choose, the bigger Δ_1 we may obtain from inequality (33). Using the proposed methods in Refs. [21], [33] and [16], the corresponding upper bound value of Δ_1 is listed in Table 1. It is obviously shown that the upper bound of impulsive interval is greatly improved by our results. Furthermore, the simulation result is shown in Fig. 1 with $\Delta_1 = 0.24$.

6. Conclusion

In this work, we have given a new general asymptotical stability criterion for a nonlinear impulsive system. The result is then used to design an impulsive control for chaotic systems. Less conservative conditions on stabilization and synchronization of Chua's circuits have been presented. An example is given to demonstrate the effectiveness of the theoretic results. Furthermore, a larger upper bound of the impulsive interval can be obtained through our approach, which is greater than some existing results. The obtained results can be easily used to control many systems, especially to stabilize and synchronize chaotic systems.

References

[1] Yang T 1999 *IEEE Trans. Automat. Contr.* **44** 1081

[2] Lakshmikantham V, Bainov D D and Simeonov P S 1989 *Theory of Impulsive Differential Equations* (Singapore: World Scientific)

[3] Kulev G V and Bainov D D 1990 *Comput. Math. Appl.* **19** 17

[4] Yang T and Chua L O 1997 *IEEE Trans. Circuits Syst.* I **44** 976

[5] Liu Y and Zhao S W 2010 *Commun. Theor. Phys.* **53** 994

[6] Sun J T, Zhang Y P and Wu Q D 2003 *IEEE Trans. Automat. Contr.* **48** 829

[7] Li Z G, Wen C Y and Soh Y C 2001 *IEEE Trans. Automat. Contr.* **46** 894

[8] Ott E, Grebogi C and Yorke J A 1990 *Phys. Rev. Lett.* **64** 1196

[9] Liu X Z 2001 *Nonl. Anal. TMA.* **47** 1081

[10] Li C D and Liao X F 2004 *Chaos, Solitons and Fractals* **22** 857

[11] Yang T, Yang L B and Yang C M 1997 *Physica* D **110** 18

[12] Lu J Q and Cao J D 2005 *Chaos* **15** 043901

[13] Lu J Q, Ho D W C and Cao J D 2010 *Automatica* **46** 1215

[14] Lu J Q, Cao J D and Ho D W C 2008 *IEEE Trans. Circuits Syst.* I **55** 1347

[15] Cao J D and Lu J Q 2006 *Chaos* **16** 013133

[16] Li Z G, Wen C Y, Soh Y C and Xie W X 2001 *IEEE Trans. Circuits Syst.* I **40** 1351

[17] Itoh M, Yang T and Chua L O 2001 *Int. J. Bifurcat. Chaos* **11** 551

[18] Wang D F and Han P 2008 *Chin. Phys.* B **17** 3603

[19] Zhang H G, Ma T D, Fu J and Tong S C 2009 *Chin. Phys.* B **18** 3751

[20] Zhang H G, Ma T D, Fu J and Tong S C 2009 *Chin. Phys.* B **18** 3742

[21] Li Z G, Soh Y C and Wen C Y 2005 *Switched and Impulsive Systems: Analysis, Design and Applications* (Berlin: Springer-Verlag)

[22] Yang T and Chua L O 1997 *Int. J. Bifurcat. Chaos* **7** 645

[23] Amritkar R E and Gupte N 1993 *Phys. Rev.* E **47** 3889

[24] Stojanovski T, Kocarev L and Parlitz U 1996 *Phys. Rev.* E **54** 2128

[25] Li Y, Wong K W, Liao X F and Li C D 2009 *Nonl. Anal. RWA* **10** 1712

[26] Wu X Q, Lu J A, Tse C K, Wang J J and Liu J 2007 *Chaos, Solitons and Fractals* **31** 631

[27] Ji Y, Wen C Y and Li Z G 2006 *Int. J. Bifurcat. Chaos* **16** 221

[28] Yang T and Chua L O 1997 *Int. J. Bifurcat. Chaos* **7** 2789

[29] Fortuna L, Frasca M and Rizzo A 2003 *Chaos, Solitons and Fractals* **17** 355

[30] Panas A I, Yang T and Chua L O 1998 *Int. J. Bifurcat. Chaos* **8** 639

[31] Itoh M, Yang T and Chua L O 1999 *Int. J. Bifurcat. Chaos* **9** 1393

[32] Li C G, Chen L N and Aihara K 2008 *Chaos* **18** 023132

[33] Wang Y W, Guan Z H and Xiao J W 2004 *Chaos, Solitons and Fractals* **14** 199

[34] Sliva C P 1993 *IEEE Trans. Circuits Syst.* I **40** 675