New results on stability criteria for neural networks with time-varying delays*

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In this paper, the problem of stability analysis for neural networks with time-varying delays is considered. By constructing a new augmented Lyapunov–Krasovskii's functional and some novel analysis techniques, improved delay-dependent criteria for checking the stability of the neural networks are established. The proposed criteria are presented in terms of linear matrix inequalities (LMIs) which can be easily solved and checked by various convex optimization algorithms. Two numerical examples are included to show the superiority of our results.

Keywords: neural networks, time-varying delays, stability, Lyapunov method

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1. Introduction

Neural Networks can be applied to the study of problems in physics such as chemical and material physics, high energy physics, reactor physics, optical physics, mathematical physics.[1] Eckmann *et al.*[2] mentioned that the physical quantities that emerge as essential characteristics of the network *in vitro* are propagation speeds, synaptic transmission, information creation and capacity. In Ref. [3], it was verified that neural networks are suitable for complex problems in computational statical physics. From Refs. [1]–[3], neural networks provide new opportunities for using neural networks in physics. With this regard in recent years, neural networks have been successfully applied to various fields such as pattern recognition, associative memories, image and signal processing, fixed-point computations and other scientific areas. Therefore, considerable attention[4−25] has been paid to the stability analysis of neural networks since the stability issue is a prerequisite to the applications of neural networks. On the other hand, in the implementation of neural networks, time-delays are frequently encountered in neural networks due to the finite switching speed of amplifiers and the inherent communication of neurons.[4−6] It is well recognized that a time-delay often makes the system oscillatory, or even unstable. Therefore, a lot of results of the delay-dependent stability analysis of neural networks with time-invariant or time-varying delays have been reported[14−25] because in general delay-dependent stability criteria are less conservative than delay-independent ones when the sizes of time-delays are small.[26]

The main aim in delay-dependent stability criteria of a dynamic system with time-delays is to increase the feasible region or to obtain maximum delay bounds of stability criteria such that the system under consideration is asymptotically stable for any time-delays less than maximum delay bounds. Therefore, how to choose Lyapunov–Krasovskii's functionals and derive the stability condition by calculating upper bounds of time derivative of Lyapunov–Krasovskii's functionals play important roles to increase the maximum bounds of time-delays.

In this regard, an augmented Lyapunov–Krasovskii's functional approach to delay-dependent stability analysis of uncertain neural networks with

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time-varying delays has been investigated.[18] In Ref. [20], by dividing delay intervals into two subintervals and utilizing free-weighting matrix techniques, improved delay-dependent stability criteria for delayed neural networks were proposed. Very recently, in Ref. [25], the triple integral form of Lyapunov–Krasovskii's functional and the methods of utilizing free-weighting matrices were proposed to enhance the feasible region of stability criteria. However, there in much room for further improvement to reduce the conservatism of stability criteria for neural networks with time-varying delays.

In this paper, we revisit the problem of delay-dependent stability analysis of neural networks with time-varying delays. Inspired by the work of Refs. [25] and [27], a triple integral form of Lyapunov–Krasovskii's functional is proposed. Then, unlike the method of Ref. [25], by taking more information about time-varying delays and states as augmented vectors, an improved stability criterion such that the considered neural networks are asymptotically stable is derived in terms of linear matrix inequalities (LMIs), which will be presented in Theorem 1. By utilizing information about time-varying delays, a new Lyapunov–Krasovskii's functional which is not considered in other literature will be proposed in Theorem 2. Through two numerical examples, it will be shown that the proposed stability criteria can provide larger delay bounds than the results in other literature.

Notation $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ denotes the set of $m\times n$ real matrix. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the notation $\boldsymbol{X} > \boldsymbol{Y}$ ($\boldsymbol{X} \geq \boldsymbol{Y}$) means that the matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive definite, (nonnegative). diag$\{\cdots\}$ denotes the block diagonal matrix. Symbol $\star$ represents the elements below the main diagonal of a symmetric matrix. $\boldsymbol{X}_{[f(t)]} \in \mathbb{R}^{m\times n}$ means that the elements of matrix $\boldsymbol{X}_{[f(t)]}$ include the scalar value of $f(t)$.

2. Problem statements

Consider the following neural networks with discrete time-varying delays:

$$\begin{aligned}\dot{\boldsymbol{y}}(t) =& -\boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{W}_0\boldsymbol{g}(\boldsymbol{y}(t)) \\ &+ \boldsymbol{W}_1\boldsymbol{g}(\boldsymbol{y}(t-h(t))) + \boldsymbol{b}, \end{aligned} \tag{1}$$

where $\boldsymbol{y}(t) = [y_1(t),\ldots,y_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the neuron state vector, n denotes the number of neurons in a neural network, $\boldsymbol{g}(\boldsymbol{y}(t)) = [g_1(y_1(t)),\ldots,g_n(y_n(t))]^{\mathrm{T}} \in \mathbb{R}^n$ means the neuron activation function, $\boldsymbol{g}(\boldsymbol{y}(t-h(t))) = [g_1(y_1(t-h(t))),\ldots,g_n(y_n(t-h(t)))]^{\mathrm{T}} \in \mathbb{R}^n$, $\boldsymbol{A} = \mathrm{diag}\{a_i\} \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, $\boldsymbol{W}_0 = (w_{ij}^0)_{n\times n} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{W}_1 = (w_{ij}^1)_{n\times n} \in \mathbb{R}^{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons and $\boldsymbol{b} = [b_1, b_2,\ldots,b_n]^{\mathrm{T}} \in \mathbb{R}^n$ represents a constant input vector.

The delay, $h(t)$, is a time-varying continuous function that satisfies

$$0 \leq h(t) \leq h_{\mathrm{U}}, \quad h_{\mathrm{Dl}} \leq \dot{h}(t) \leq h_{\mathrm{Du}}, \tag{2}$$

where h_{U} is a positive scalar, h_{Dl} and h_{Du} are any constant ones.

The activation functions, $g_i(y_i(t)), i = 1,\ldots,n$, are assumed to be nondecreasing, bounded and globally Lipschitz; that is,

$$\begin{aligned} &k_i^- \leq \frac{g_i(\xi_i)-g_j(\xi_j)}{\xi_i-\xi_j} \leq k_i^+, \ \ \xi_i,\ \xi_j \in \mathbb{R},\ \xi_i \neq \xi_j, \\ &(i = 1,\ldots,n,\ j = 1,\ldots,n), \end{aligned} \tag{3}$$

where k_i^+ and k_i^- are constants.

For simplicity, in stability analysis of system (1), the equilibrium point $\boldsymbol{y}^* = [y_1^*,\ldots,y_n^*]^{\mathrm{T}}$ whose uniqueness has been reported[13] is shifted to the origin by utilizing the transformation $\boldsymbol{x}(\cdot) = \boldsymbol{y}(\cdot) - \boldsymbol{y}^*$, which leads system (1) to the following form:

$$\dot{\boldsymbol{x}}(t) = -A\boldsymbol{x}(t) + \boldsymbol{W}_0\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}(t-h(t))), \tag{4}$$

where $\boldsymbol{x}(t) = [x_1(t),\ldots,x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector of the transformed system,

$$\boldsymbol{f}(\boldsymbol{x}(t)) = [f_1(x(t)),\ldots,f_n(x(t))]^{\mathrm{T}}$$

and

$$f_j(x_j(t)) = g_j(x_j(t)+y_j^*) - g_j(y_j^*)$$

with $f_j(0) = 0$, $(j = 1,\ldots,n)$.

From Eq. (3), $f_j(\cdot)$ satisfies the following condition:

$$k_j^- \leq \frac{f_j(\xi_j)}{\xi_j} \leq k_j^+,\ \forall\, \xi_j \neq 0,\ j = 1,\ldots,n, \tag{5}$$

which is equivalent to

$$\begin{aligned} &\left[f_j(\xi_j) - k_j^-\xi_j\right]\left[f_j(\xi_j) - k_j^+\xi_j\right] \leq 0, \\ &f_j(0) = 0,\ j = 1,\ldots,n. \end{aligned} \tag{6}$$

The objective of this paper is to investigate the delay dependent stability analysis of system (4), which will be conducted in Section 3.

Before deriving our main results, we state the following fact and lemma.

Fact 1 Given constant matrices $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_2$, $\boldsymbol{\Sigma}_3$ where $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_1^{\mathrm{T}}$ and $0 < \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^{\mathrm{T}}$, then $\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_3^{\mathrm{T}}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_3 < 0$ if and only if

$$\begin{bmatrix} \boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_3^{\mathrm{T}} \\ \boldsymbol{\Sigma}_3 & -\boldsymbol{\Sigma}_2 \end{bmatrix} < \boldsymbol{0}, \quad \text{or} \quad \begin{bmatrix} -\boldsymbol{\Sigma}_2 & \boldsymbol{\Sigma}_3 \\ \boldsymbol{\Sigma}_3^{\mathrm{T}} & \boldsymbol{\Sigma}_1 \end{bmatrix} < \boldsymbol{0}.$$

Lemma 1 For a positive matrix $\boldsymbol{M}$, the following inequality holds:

$$\begin{aligned} &-(\alpha-\beta)\int_\beta^\alpha \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{x}}(s)\mathrm{d}s \\ &\le \begin{bmatrix} \boldsymbol{x}(\alpha) \\ \boldsymbol{x}(\beta) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\boldsymbol{M} & \boldsymbol{M} \\ \star & -\boldsymbol{M} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\alpha) \\ \boldsymbol{x}(\beta) \end{bmatrix}. \end{aligned} \tag{7}$$

Proof According to Lemma 1,[28] one can see

$$\begin{aligned} &-(\alpha-\beta)\int_\beta^\alpha \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{x}}(s)\mathrm{d}s \\ &\le -\left(\int_\beta^\alpha \dot{\boldsymbol{x}}(s)\mathrm{d}s\right)^{\mathrm{T}} \boldsymbol{M}\left(\int_\beta^\alpha \dot{\boldsymbol{x}}(s)\mathrm{d}s\right). \end{aligned} \tag{8}$$

Then, inequality (7) can be obtained.

Lemma 2 For a positive matrix $\boldsymbol{M}$, the following inequality holds:

$$\begin{aligned} &-\frac{(\alpha-\beta)^2}{2}\int_\beta^\alpha\int_s^\alpha \boldsymbol{x}^{\mathrm{T}}(u)\boldsymbol{M}\boldsymbol{x}(u)\mathrm{d}u\mathrm{d}s \\ &\le -\left(\int_\beta^\alpha\int_s^\alpha \boldsymbol{x}(u)\mathrm{d}u\mathrm{d}s\right)^{\mathrm{T}} \\ &\quad\times \boldsymbol{M}\left(\int_\beta^\alpha\int_s^\alpha \boldsymbol{x}(u)\mathrm{d}u\mathrm{d}s\right). \end{aligned} \tag{9}$$

Proof From Lemma 1, the following inequality holds:

$$\begin{aligned} &-(\alpha-s)\int_s^\alpha \boldsymbol{x}^{\mathrm{T}}(u)\boldsymbol{M}\boldsymbol{x}(u)\mathrm{d}u \\ &\le -\left(\int_s^\alpha \boldsymbol{x}(u)\mathrm{d}u\right)^{\mathrm{T}} \boldsymbol{M}\left(\int_s^\alpha \boldsymbol{x}(u)\mathrm{d}u\right). \end{aligned} \tag{10}$$

By using Fact 1, inequality (10) is equivalent to

$$\begin{bmatrix} -\int_s^\alpha \boldsymbol{x}^{\mathrm{T}}(u)\boldsymbol{M}\boldsymbol{x}(u)\mathrm{d}u & \left(\int_s^\alpha \boldsymbol{x}(u)\mathrm{d}u\right)^{\mathrm{T}} \\ \star & -(\alpha-s)\boldsymbol{M}^{-1} \end{bmatrix} \le 0. \tag{11}$$

By integrating inequality (11) with respect to s from β to α, we have

$$\begin{bmatrix} -\int_\beta^\alpha\int_s^\alpha \boldsymbol{x}^{\mathrm{T}}(u)\boldsymbol{M}\boldsymbol{x}(u)\mathrm{d}u\mathrm{d}s & \left(\int_\beta^\alpha\int_s^\alpha \boldsymbol{x}(u)\mathrm{d}u\mathrm{d}s\right)^{\mathrm{T}} \\ \star & -\int_\beta^\alpha(\alpha-s)\boldsymbol{M}^{-1}\mathrm{d}s \end{bmatrix} \le 0. \tag{12}$$

From Fact 1, inequality (12) is equivalent to inequality (9). This completes the proof of Lemma 2.

Lemma 3[29] Let $\boldsymbol{\zeta} \in \mathbb{R}^n$, $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ such that $\mathrm{rank}(\boldsymbol{B}) < n$. Then, the following statements are equivalent:

(1) $\boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{\zeta} < \boldsymbol{0}$, $\boldsymbol{B}\boldsymbol{\zeta} = \boldsymbol{0}$, $\boldsymbol{\zeta} \ne \boldsymbol{0}$,

(2) $(\boldsymbol{B}^{\perp})^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{B}^{\perp} < \boldsymbol{0}$, where $\boldsymbol{B}^{\perp}$ is a right orthogonal complement of $\boldsymbol{B}$.

3. Main results

In this section, new delay-dependent stability criteria for neural networks with time-varying delays (2) will be derived. For simplicity of matrix representation, $\boldsymbol{e}_i$ $(i = 1,\ldots,11) \in \mathbb{R}^{11n\times n}$ are defined as block entry matrices. (For example, $\boldsymbol{e}_3^{\mathrm{T}} = [\boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{\mathrm{I}}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}]$). The notations for some matrices are defined as

$$\begin{aligned} \boldsymbol{\zeta}^{\mathrm{T}}(t) = \Big[&\boldsymbol{x}^{\mathrm{T}}(t) \quad \boldsymbol{x}^{\mathrm{T}}(t-h(t)) \quad \boldsymbol{x}^{\mathrm{T}}(t-h_{\mathrm{U}}) \quad \dot{\boldsymbol{x}}^{\mathrm{T}}(t) \quad \dot{\boldsymbol{x}}^{\mathrm{T}}(t-h_{\mathrm{U}}) \quad \frac{1}{h(t)}\int_{t-h(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)\mathrm{d}s \\ &\frac{1}{h_{\mathrm{U}}-h(t)}\int_{t-h_{\mathrm{U}}}^{t-h(t)} \boldsymbol{x}^{\mathrm{T}}(s)\mathrm{d}s \quad (1-\dot{h}(t))\dot{\boldsymbol{x}}^{\mathrm{T}}(t-h(t)) \quad \boldsymbol{f}^{\mathrm{T}}(x(t)) \quad \boldsymbol{f}^{\mathrm{T}}(x(t-h(t))) \quad \boldsymbol{f}^{\mathrm{T}}(x(t-h_{\mathrm{U}}))\Big], \\ \boldsymbol{\alpha}^{\mathrm{T}}(t) = \Big[&\boldsymbol{x}^{\mathrm{T}}(t) \quad \boldsymbol{x}^{\mathrm{T}}(t-h_{\mathrm{U}}) \int_{t-h_{\mathrm{U}}}^{t} \boldsymbol{x}^{\mathrm{T}}(s)\mathrm{d}s \quad \boldsymbol{x}^{\mathrm{T}}(t-h(t))\Big], \\ \boldsymbol{\beta}^{\mathrm{T}}(t) = \big[&\boldsymbol{x}^{\mathrm{T}}(t) \quad \dot{\boldsymbol{x}}^{\mathrm{T}}(t) \quad \boldsymbol{f}^{\mathrm{T}}(x(t))\big], \end{aligned}$$

$$\boldsymbol{\Pi}_1 = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_3 & \boldsymbol{e}_6 & \boldsymbol{e}_7 & \boldsymbol{e}_2 \end{bmatrix},$$
$$\boldsymbol{\Pi}_2 = \begin{bmatrix} \boldsymbol{e}_4 & \boldsymbol{e}_5 & \boldsymbol{e}_1 - \boldsymbol{e}_3 & \boldsymbol{e}_8 \end{bmatrix},$$
$$\boldsymbol{\Pi}_3 = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_4 & \boldsymbol{e}_9 \end{bmatrix},$$
$$\boldsymbol{\Pi}_4 = \begin{bmatrix} \boldsymbol{e}_3 & \boldsymbol{e}_5 & \boldsymbol{e}_{11} \end{bmatrix},$$
$$\boldsymbol{\Pi}_5 = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_9 \end{bmatrix},$$
$$\boldsymbol{\Pi}_6 = \begin{bmatrix} \boldsymbol{e}_2 & \boldsymbol{e}_8 & \boldsymbol{e}_{10} \end{bmatrix},$$

$$\boldsymbol{\Psi}_{[h(t)]} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & h(t)\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & (h_{\mathrm{U}} - h(t))\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix},$$

$$\widetilde{\mathscr{G}} = \begin{bmatrix} (1-h_{\mathrm{Du}})\boldsymbol{G}_{11} & \boldsymbol{G}_{12} & (1-h_{\mathrm{Du}})\boldsymbol{G}_{13} \\ \star & \dfrac{1}{1-h_{\mathrm{Dl}}}\boldsymbol{G}_{22} & \boldsymbol{G}_{23} \\ \star & \star & (1-h_{\mathrm{Du}})\boldsymbol{G}_{33} \end{bmatrix},$$

$$\mathscr{X}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{X}_1^{\mathrm{T}} & \boldsymbol{X}_2^{\mathrm{T}} & \boldsymbol{X}_3^{\mathrm{T}} & \boldsymbol{X}_4^{\mathrm{T}} & \boldsymbol{X}_5^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$
$$\mathscr{Y}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{Y}_1^{\mathrm{T}} & \boldsymbol{Y}_2^{\mathrm{T}} & \boldsymbol{Y}_3^{\mathrm{T}} & \boldsymbol{Y}_4^{\mathrm{T}} & \boldsymbol{Y}_5^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$
$$\mathscr{Z}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{Z}_1^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Z}_2^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$
$$\mathscr{W}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{W}\mathrm{T}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W}\mathrm{T}_2 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$
$$\boldsymbol{\Gamma} = \begin{bmatrix} -\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W_0} & \boldsymbol{W}_1 & \boldsymbol{0} \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{\Sigma}_{[h(t)]} = {} & \boldsymbol{\Pi}_1\boldsymbol{\Psi}_{[h(t)]}\mathscr{R}\boldsymbol{\Pi}_2^{\mathrm{T}} + \boldsymbol{\Pi}_2\mathscr{R}\boldsymbol{\Psi}_{[h(t)]}^{\mathrm{T}}\boldsymbol{\Pi}_1^{\mathrm{T}} \\ & + \boldsymbol{\Pi}_3\mathscr{N}\boldsymbol{\Pi}_3^{\mathrm{T}} - \boldsymbol{\Pi}_4\mathscr{N}\boldsymbol{\Pi}_4^{\mathrm{T}} + \boldsymbol{e}_9\boldsymbol{\Lambda}\boldsymbol{e}_4^{\mathrm{T}} \\ & + \boldsymbol{e}_4\boldsymbol{\Lambda}\boldsymbol{e}_9^{\mathrm{T}} - \boldsymbol{e}_1\boldsymbol{K}_{\mathrm{m}}\boldsymbol{\Lambda}\boldsymbol{e}_4^{\mathrm{T}} \\ & - \boldsymbol{e}_4\boldsymbol{\Lambda}\boldsymbol{K}_{\mathrm{m}}\boldsymbol{e}_1^{\mathrm{T}} + \boldsymbol{e}_1\boldsymbol{K}_{\mathrm{p}}\boldsymbol{\Delta}\boldsymbol{e}_4^{\mathrm{T}} + \boldsymbol{e}_4\boldsymbol{\Delta}\boldsymbol{K}_{\mathrm{p}}\boldsymbol{e}_1^{\mathrm{T}} \\ & - \boldsymbol{e}_9\boldsymbol{\Delta}\boldsymbol{e}_4^{\mathrm{T}} - \boldsymbol{e}_4\boldsymbol{\Delta}\boldsymbol{e}_9^{\mathrm{T}} + \boldsymbol{\Pi}_5\mathscr{G}\boldsymbol{\Pi}_5^{\mathrm{T}} - \boldsymbol{\Pi}_6\widetilde{\mathscr{G}}\boldsymbol{\Pi}_6^{\mathrm{T}} \\ & + h_{\mathrm{U}}\boldsymbol{e}_4\boldsymbol{Q}_1\boldsymbol{e}_4^{\mathrm{T}} + \mathscr{X}(\boldsymbol{e}_1^{\mathrm{T}} - \boldsymbol{e}_2^{\mathrm{T}}) + (\boldsymbol{e}_1 - \boldsymbol{e}_2)\mathscr{X}^{\mathrm{T}} \\ & + \mathscr{Y}(\boldsymbol{e}_2^{\mathrm{T}} - \boldsymbol{e}_3^{\mathrm{T}}) + (\boldsymbol{e}_2 - \boldsymbol{e}_3)\mathscr{Y}^{\mathrm{T}} \\ & + h_{\mathrm{U}}\boldsymbol{e}_1\boldsymbol{Q}_2\boldsymbol{e}_1^{\mathrm{T}} - \boldsymbol{e}_6(h(t)\boldsymbol{Q}_2)\boldsymbol{e}_6^{\mathrm{T}} \\ & - \boldsymbol{e}_7((h_{\mathrm{U}} - h(t))\boldsymbol{Q}_2)\boldsymbol{e}_7^{\mathrm{T}} \\ & + (h_{\mathrm{U}}^2/2)\boldsymbol{e}_4(\boldsymbol{Q}_3 + \boldsymbol{Q}_4)\boldsymbol{e}_4^{\mathrm{T}} \\ & - 2(\boldsymbol{e}_1 - \boldsymbol{e}_6)\boldsymbol{Q}_3(\boldsymbol{e}_1^{\mathrm{T}} - \boldsymbol{e}_6^{\mathrm{T}}) - 2(\boldsymbol{e}_2 - \boldsymbol{e}_7) \\ & \times \boldsymbol{Q}_3(\boldsymbol{e}_2^{\mathrm{T}} - \boldsymbol{e}_7^{\mathrm{T}}) + (h(t)\mathscr{Z})(\boldsymbol{e}_1^{\mathrm{T}} - \boldsymbol{e}_6^{\mathrm{T}}) \\ & + (\boldsymbol{e}_1 - \boldsymbol{e}_6)(h(t)\mathscr{Z}^{\mathrm{T}}) + ((h_{\mathrm{U}} - h(t))\mathscr{W}) \\ & \times (\boldsymbol{e}_2^{\mathrm{T}} - \boldsymbol{e}_7^{\mathrm{T}}) + (\boldsymbol{e}_2 - \boldsymbol{e}_7)((h_{\mathrm{U}} - h(t))\mathscr{W}^{\mathrm{T}}) \\ & - 2\boldsymbol{e}_1\boldsymbol{K}_{\mathrm{m}}\boldsymbol{H}_1\boldsymbol{K}_{\mathrm{p}}\boldsymbol{e}_1^{\mathrm{T}} + \boldsymbol{e}_1(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{H}_1\boldsymbol{e}_9^{\mathrm{T}} \\ & + \boldsymbol{e}_9\boldsymbol{H}_1(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{e}_1^{\mathrm{T}} - 2\boldsymbol{e}_9\boldsymbol{H}_1\boldsymbol{e}_9^{\mathrm{T}} \\ & - 2\boldsymbol{e}_2\boldsymbol{K}_{\mathrm{m}}\boldsymbol{H}_2\boldsymbol{K}_{\mathrm{p}}\boldsymbol{e}_2^{\mathrm{T}} + \boldsymbol{e}_2(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{H}_2\boldsymbol{e}_{10}^{\mathrm{T}} \\ & + \boldsymbol{e}_{10}\boldsymbol{H}_2(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{e}_2^{\mathrm{T}} - 2\boldsymbol{e}_{10}\boldsymbol{H}_2\boldsymbol{e}_{10}^{\mathrm{T}} \\ & - 2\boldsymbol{e}_3\boldsymbol{K}_{\mathrm{m}}\boldsymbol{H}_3\boldsymbol{K}_{\mathrm{p}}\boldsymbol{e}_3^{\mathrm{T}} + \boldsymbol{e}_3(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{H}_3\boldsymbol{e}_{11}^{\mathrm{T}} \\ & + \boldsymbol{e}_{11}\boldsymbol{H}_3(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{e}_3^{\mathrm{T}} - 2\boldsymbol{e}_{11}\boldsymbol{H}_3\boldsymbol{e}_{11}^{\mathrm{T}}. \end{aligned} \tag{13}$$

Now, we have the following theorem.

Theorem 1 For given positive scalars h_{U}, any ones h_{Dl} and h_{Du} which satisfy $h_{\mathrm{Dl}} < h_{\mathrm{Du}} < 1$, diagonal matrices $\boldsymbol{K}_{\mathrm{p}} = \mathrm{diag}\{k_1^+, \ldots, k_n^+\}$ and $\boldsymbol{K}_{\mathrm{m}} = \mathrm{diag}\{k_1^-, \ldots, k_n^-\}$, system (1) is asymptotically stable for $0 \leq h(t) \leq h_{\mathrm{U}}$ and $h_{\mathrm{Dl}} \leq \dot{h}(t) \leq h_{\mathrm{Du}} < 1$ if there exist positive diagonal matrices $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$, $\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1, \ldots, \delta_n\}$, $\boldsymbol{H}_j = \mathrm{diag}\{h_{1i}, \ldots, h_{1n}\}(j = 1, 2, 3)$, positive definite matrices $\mathscr{R} = [R_{ij}]_{4\times4}$, $\mathscr{N} = [N_{ij}]_{3\times3}$, $\mathscr{G} = [G_{ij}]_{3\times3}$, $\boldsymbol{Q}_i (i = 1, \ldots, 4)$, and any matrices $\boldsymbol{X}_i$, $\boldsymbol{Y}_i$ $(i = 1, \ldots, 5)$, $\boldsymbol{Z}_j$, $\boldsymbol{W}_j$ $(j = 1, 2)$ satisfying the following LMIs:

$$\begin{bmatrix} \left(\boldsymbol{\Gamma}^{\perp}\right)^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{[h(t)=0]}\right)\boldsymbol{\Gamma}^{\perp} & \boldsymbol{0} & \left(\boldsymbol{\Gamma}^{\perp}\right)^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{W}) & \left(\boldsymbol{\Gamma}^{\perp}\right)^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{Y}) \\ \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} \\ \star & \star & \star & -\boldsymbol{Q}_1 \end{bmatrix} < \boldsymbol{0}, \tag{14}$$

$$\begin{bmatrix} \left(\boldsymbol{\Gamma}^{\perp}\right)^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{[h(t)=h_{\mathrm{U}}]}\right)\boldsymbol{\Gamma}^{\perp} & \left(\boldsymbol{\Gamma}^{\perp}\right)^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{Z}) & \boldsymbol{0} & \left(\boldsymbol{\Gamma}^{\perp}\right)^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{X}) \\ \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} \\ \star & \star & \star & -\boldsymbol{Q}_1 \end{bmatrix} < \boldsymbol{0}, \tag{15}$$

where $\boldsymbol{\Sigma}_{[h(t)]}$, $\boldsymbol{\Gamma}$, $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$, and $\mathscr{W}$ are defined in (13) and $\boldsymbol{\Gamma}^{\perp}$ is the right orthogonal complement of $\boldsymbol{\Gamma}$.

Proof For positive diagonal matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{\Delta}$ and positive definite matrices $\mathscr{R}$, $\mathscr{N}$, $\mathscr{G}$, $\boldsymbol{Q}_i (i = 1, \ldots, 4)$, we take the Lyapunov–Krasovskii functional candidate

$$\boldsymbol{V} = \sum_{i=1}^{6} \boldsymbol{V}_i, \tag{16}$$

where

$$\begin{aligned}
\boldsymbol{V}_1 &= \boldsymbol{\alpha}^{\mathrm{T}}(t)\mathscr{R}\boldsymbol{\alpha}(t),\\
\boldsymbol{V}_2 &= \int_{t-h_{\mathrm{U}}}^{t} \boldsymbol{\beta}^{\mathrm{T}}(s)\mathscr{N}\boldsymbol{\beta}(s)\mathrm{d}s\\
&\quad + 2\sum_{i=1}^{n}\left(\lambda_i \int_{0}^{x_i(t)} (f_i(s) - k_i^{-}s)\mathrm{d}s\right.\\
&\quad \left. + \delta_i \int_{0}^{x_i(t)} (k_i^{+}s - f_i(s))\mathrm{d}s\right),\\
\boldsymbol{V}_3 &= \int_{t-h(t)}^{t} \boldsymbol{\beta}^{\mathrm{T}}(s)\mathscr{G}\boldsymbol{\beta}(s)\mathrm{d}s,\\
\boldsymbol{V}_4 &= \int_{t-h_{\mathrm{U}}}^{t}\int_{s}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(u)\boldsymbol{Q}_1\dot{\boldsymbol{x}}(u)\mathrm{d}u\mathrm{d}s,\\
\boldsymbol{V}_5 &= \int_{t-h_{\mathrm{U}}}^{t}\int_{s}^{t} \boldsymbol{x}^{\mathrm{T}}(u)\boldsymbol{Q}_2\boldsymbol{x}(u)\mathrm{d}u\mathrm{d}s,\\
\boldsymbol{V}_6 &= \int_{t-h_{\mathrm{U}}}^{t}\int_{s}^{t}\int_{u}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(v)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(v)\mathrm{d}v\mathrm{d}u\mathrm{d}s\\
&\quad + \int_{t-h_{\mathrm{U}}}^{t}\int_{s}^{t}\int_{u}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(v)\boldsymbol{Q}_4\dot{\boldsymbol{x}}(v)\mathrm{d}v\mathrm{d}u\mathrm{d}s.
\end{aligned} \tag{17}$$

By calculating $\dot{\boldsymbol{V}}_1$, we have

$$\begin{aligned}
\dot{\boldsymbol{V}}_1 &= 2\boldsymbol{\alpha}^{\mathrm{T}}(t)\mathscr{R}\dot{\boldsymbol{\alpha}}(t) = 2\begin{bmatrix} \boldsymbol{x}(t)\\ \boldsymbol{x}(t-h_{\mathrm{U}})\\ \boldsymbol{\eta}_1(t)\\ \boldsymbol{x}(t-h(t)) \end{bmatrix}^{\mathrm{T}} \mathscr{R}\dot{\boldsymbol{\alpha}}(t)\\
&= 2\boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Pi}_1\boldsymbol{\Psi}_{[h(t)]}\mathscr{R}\boldsymbol{\Pi}_2^{\mathrm{T}}\boldsymbol{\zeta}(t),
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\boldsymbol{\eta}_1(t) &= \left(\frac{1}{h(t)}\int_{t-h(t)}^{t} \boldsymbol{x}(s)\mathrm{d}s\right) h(t)\\
&\quad + \left(\frac{1}{h_{\mathrm{U}} - h(t)}\int_{t-h_{\mathrm{U}}}^{t-h(t)} \boldsymbol{x}(s)\mathrm{d}s\right)(h_{\mathrm{U}} - h(t)).
\end{aligned}$$

The time-derivative of $\boldsymbol{V}_2$ can be calculated as

$$\begin{aligned}
\dot{\boldsymbol{V}}_2 &= \boldsymbol{\beta}^{\mathrm{T}}(t)\mathscr{N}\boldsymbol{\beta}(t) - \boldsymbol{\beta}^{\mathrm{T}}(t-h(t))\mathscr{N}\boldsymbol{\beta}(t-h(t))\\
&\quad + 2\left[\boldsymbol{f}(\boldsymbol{x}(t)) - \boldsymbol{K}_{\mathrm{m}}\boldsymbol{x}(t)\right]^{\mathrm{T}}\boldsymbol{\Lambda}\dot{\boldsymbol{x}}(t)\\
&\quad + 2\left[\boldsymbol{K}_{\mathrm{p}}\boldsymbol{x}(t) - \boldsymbol{f}(\boldsymbol{x}(t))\right]^{\mathrm{T}}\boldsymbol{\Delta}\dot{\boldsymbol{x}}(t)\\
&= \boldsymbol{\zeta}^{\mathrm{T}}(t)[\boldsymbol{\Pi}_3\mathscr{N}\boldsymbol{\Pi}_3^{\mathrm{T}} - \boldsymbol{\Pi}_4\mathscr{N}\boldsymbol{\Pi}_4^{\mathrm{T}} + \boldsymbol{e}_9\boldsymbol{\Lambda}\boldsymbol{e}_4^{\mathrm{T}}\\
&\quad + \boldsymbol{e}_4\boldsymbol{\Lambda}\boldsymbol{e}_9^{\mathrm{T}} - \boldsymbol{e}_1\boldsymbol{K}_{\mathrm{m}}\boldsymbol{\Lambda}\boldsymbol{e}_4^{\mathrm{T}} - \boldsymbol{e}_4\boldsymbol{\Lambda}\boldsymbol{K}_{\mathrm{m}}\boldsymbol{e}_1^{\mathrm{T}}\\
&\quad + \boldsymbol{e}_1\boldsymbol{K}_{\mathrm{p}}\boldsymbol{\Delta}\boldsymbol{e}_4^{\mathrm{T}} + \boldsymbol{e}_4\boldsymbol{\Delta}\boldsymbol{K}_{\mathrm{p}}\boldsymbol{e}_1^{\mathrm{T}}\\
&\quad - \boldsymbol{e}_9\boldsymbol{\Delta}\boldsymbol{e}_4^{\mathrm{T}} - \boldsymbol{e}_4\boldsymbol{\Delta}\boldsymbol{e}_9^{\mathrm{T}}]\boldsymbol{\zeta}(t).
\end{aligned} \tag{19}$$

An upper bound of $\boldsymbol{V}_3$ can be obtained as

$$\begin{aligned}
\dot{\boldsymbol{V}}_3 &= \boldsymbol{\beta}^{\mathrm{T}}(t)\mathscr{G}\boldsymbol{\beta}(t) - (1 - \dot{h}(t))\boldsymbol{\beta}^{\mathrm{T}}\\
&\quad \times (t - h(t))\mathscr{G}\boldsymbol{\beta}(t - h(t))\\
&\leq \boldsymbol{\zeta}^{\mathrm{T}}(t)\left[\boldsymbol{\Pi}_5\mathscr{G}\boldsymbol{\Pi}_5^{\mathrm{T}} - \boldsymbol{\Pi}_6\widetilde{\mathscr{G}}\boldsymbol{\Pi}_6^{\mathrm{T}}\right]\boldsymbol{\zeta}(t).
\end{aligned} \tag{20}$$

Calculating $\dot{\boldsymbol{V}}_4$ leads to

$$\dot{\boldsymbol{V}}_4 = h_{\mathrm{U}}\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{Q}_1\dot{\boldsymbol{x}}(t) - \int_{t-h_{\mathrm{U}}}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{Q}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s. \tag{21}$$

Here, with free-weighting matrices $\mathscr{X}$ and $\mathscr{Y}$ defined in (13), by utilizing Lemma 1,[20] the integral term $-\int_{t-h_{\mathrm{U}}}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{Q}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s$ in Eq. (21) can be estimated as

$$\begin{aligned}
&-\int_{t-h_{\mathrm{U}}}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{Q}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&= -\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{Q}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&\quad -\int_{t-h_{\mathrm{U}}}^{t-h(t)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{Q}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&\leq \boldsymbol{\zeta}^{\mathrm{T}}(t)(h(t)\mathscr{X}\boldsymbol{Q}_1^{-1}\mathscr{X}^{\mathrm{T}})\boldsymbol{\zeta}^{\mathrm{T}}(t)\\
&\quad + 2\boldsymbol{\zeta}^{\mathrm{T}}(t)\mathscr{X}(\boldsymbol{e}_1^{\mathrm{T}} - \boldsymbol{e}_2^{\mathrm{T}})\boldsymbol{\zeta}(t)\\
&\quad + \boldsymbol{\zeta}^{\mathrm{T}}(t)((h_{\mathrm{U}} - h(t))\mathscr{Y}\boldsymbol{Q}_1^{-1}\mathscr{Y}^{\mathrm{T}})\boldsymbol{\zeta}^{\mathrm{T}}(t)\\
&\quad + 2\boldsymbol{\zeta}^{\mathrm{T}}(t)\mathscr{Y}(\boldsymbol{e}_2^{\mathrm{T}} - \boldsymbol{e}_3^{\mathrm{T}})\boldsymbol{\zeta}(t).
\end{aligned} \tag{22}$$

From inequality (22), an upper bound of $\dot{\boldsymbol{V}}_4$ can be

$$\begin{aligned}
\dot{\boldsymbol{V}}_4 &\leq \boldsymbol{\zeta}^{\mathrm{T}}(t)\{h_{\mathrm{U}}\boldsymbol{e}_4\boldsymbol{Q}_1\boldsymbol{e}_4^{\mathrm{T}} + (h(t)\mathscr{X}\boldsymbol{Q}_1^{-1}\mathscr{X}^{\mathrm{T}})\\
&\quad + \mathscr{X}(\boldsymbol{e}_1^{\mathrm{T}} - \boldsymbol{e}_2^{\mathrm{T}}) + (\boldsymbol{e}_1 - \boldsymbol{e}_2)\mathscr{X}^{\mathrm{T}}\\
&\quad + (h_{\mathrm{U}} - h(t))\mathscr{Y}\boldsymbol{Q}_1^{-1}\mathscr{Y}^{\mathrm{T}}\\
&\quad + \mathscr{Y}(\boldsymbol{e}_2^{\mathrm{T}} - \boldsymbol{e}_3^{\mathrm{T}}) + (\boldsymbol{e}_2 - \boldsymbol{e}_3)\mathscr{Y}^{\mathrm{T}}\}\boldsymbol{\zeta}(t).
\end{aligned} \tag{23}$$

Calculating $\dot{\boldsymbol{V}}_5$ leads to

$$\begin{aligned}
\dot{\boldsymbol{V}}_5(t) &= h_{\mathrm{U}}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}_2\boldsymbol{x}(t) - \int_{t-h(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{Q}_2\boldsymbol{x}(s)\mathrm{d}s\\
&\quad - \int_{t-h_{\mathrm{U}}}^{t-h(t)} \boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{Q}_2\boldsymbol{x}(s)\mathrm{d}s\\
&\leq \boldsymbol{\zeta}^{\mathrm{T}}(t)[h_{\mathrm{U}}\boldsymbol{e}_1\boldsymbol{Q}_2\boldsymbol{e}_1^{\mathrm{T}} - \boldsymbol{e}_6(h(t)\boldsymbol{Q}_2)\boldsymbol{e}_6^{\mathrm{T}}\\
&\quad - \boldsymbol{e}_7((h_{\mathrm{U}} - h(t))\boldsymbol{Q}_2)\boldsymbol{e}_7^{\mathrm{T}}]\boldsymbol{\zeta}(t).
\end{aligned} \tag{24}$$

The time-derivative of $\boldsymbol{V}_6$ can be obtained as

$$\dot{\boldsymbol{V}}_6(t) = (h_{\mathrm{U}}^2/2)\dot{\boldsymbol{x}}^{\mathrm{T}}(t)(\boldsymbol{Q}_3 + \boldsymbol{Q}_4)\dot{\boldsymbol{x}}(t)$$

$$
\begin{aligned}
&-\int_{t-h_{\rm U}}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&-\int_{t-h_{\rm U}}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_4\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s.
\end{aligned}\tag{25}
$$

Here, by utilizing Lemma 2, we have

$$
\begin{aligned}
&-\int_{t-h_{\rm U}}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&=-\int_{t-h(t)}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&\quad-\int_{t-h_{\rm U}}^{t-h(t)}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&\le-\int_{t-h(t)}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&\quad-\int_{t-h_{\rm U}}^{t-h(t)}\int_{s}^{t-h(t)}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&\le\boldsymbol{\zeta}^{\rm T}(t)\{-2(\boldsymbol{e}_1-\boldsymbol{e}_6)^{\rm T}\boldsymbol{Q}_3(\boldsymbol{e}_1-\boldsymbol{e}_6)\\
&\quad-2(\boldsymbol{e}_2-\boldsymbol{e}_7)^{\rm T}\boldsymbol{Q}_3(\boldsymbol{e}_2-\boldsymbol{e}_7)\}\boldsymbol{\zeta}(t).
\end{aligned}\tag{26}
$$

Using a similar method to that of inequality (22), an upper bound of the term

$$
-\int_{t-h_{\rm U}}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_4\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s
$$

can be obtained as

$$
\begin{aligned}
&-\int_{t-h_{\rm U}}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_4\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&=-\int_{t-h(t)}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_4\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&\quad-\int_{t-h_{\rm U}}^{t-h(t)}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&\le-\int_{t-h(t)}^{t}\int_{s}^{t}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_4\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&\quad-\int_{t-h_{\rm U}}^{t-h(t)}\int_{s}^{t-h(t)}\dot{\boldsymbol{x}}^{\rm T}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u){\rm d}u{\rm d}s\\
&\le(h^2(t)/2)\boldsymbol{\zeta}^{\rm T}(t)\mathscr{Z}\boldsymbol{Q}_4^{-1}\mathscr{Z}^{\rm T}\boldsymbol{\zeta}(t)\\
&\quad+2\boldsymbol{\zeta}^{\rm T}(t)(h(t)\mathscr{Z})(\boldsymbol{e}_1^{\rm T}-\boldsymbol{e}_6^{\rm T})\boldsymbol{\zeta}(t)\\
&\quad+((h_{\rm U}-h(t))^2/2)\boldsymbol{\zeta}^{\rm T}(t)\mathscr{W}\boldsymbol{Q}_4^{-1}\mathscr{W}^{\rm T}\boldsymbol{\zeta}(t)\\
&\quad+2\boldsymbol{\zeta}^{\rm T}(t)((h_{\rm U}-h(t))\mathscr{W})(\boldsymbol{e}_2^{\rm T}-\boldsymbol{e}_7^{\rm T})\boldsymbol{\zeta}(t),
\end{aligned}\tag{27}
$$

where $\mathscr{Z}$ and $\mathscr{W}$ are defined in Eq. (13). Therefore, $\dot{\boldsymbol{V}}_6$ can be estimated as

$$
\begin{aligned}
\dot{\boldsymbol{V}}_6\le\;&\boldsymbol{\zeta}^{\rm T}(t)[(h_{\rm U}^2/2)\boldsymbol{e}_4(\boldsymbol{Q}_3+\boldsymbol{Q}_4)\boldsymbol{e}_4^{\rm T}\\
&-2(\boldsymbol{e}_1-\boldsymbol{e}_6)\boldsymbol{Q}_4(\boldsymbol{e}_1^{\rm T}-\boldsymbol{e}_6^{\rm T})\\
&-2(\boldsymbol{e}_2-\boldsymbol{e}_7)\boldsymbol{Q}_4(\boldsymbol{e}_2^{\rm T}-\boldsymbol{e}_7^{\rm T})\\
&+(h^2(t)/2)\mathscr{Z}\boldsymbol{Q}_4^{-1}\mathscr{Z}^{\rm T}\\
&+(h(t)\mathscr{Z})(\boldsymbol{e}_1^{\rm T}-\boldsymbol{e}_6^{\rm T})+(\boldsymbol{e}_1-\boldsymbol{e}_6)(h(t)\mathscr{Z}^{\rm T})\\
&+((h_{\rm U}-h(t))^2/2)\mathscr{W}\boldsymbol{Q}_4^{-1}\mathscr{W}^{\rm T}\\
&+((h_{\rm U}-h(t))\mathscr{W})(\boldsymbol{e}_2^{\rm T}-\boldsymbol{e}_7^{\rm T})\\
&+(\boldsymbol{e}_2-\boldsymbol{e}_7)((h_{\rm U}-h(t))\mathscr{W}^{\rm T})]\boldsymbol{\zeta}(t).
\end{aligned}\tag{28}
$$

Note that Eq. (5) means

$$
\begin{aligned}
&\left[f_j(x_j(t))-k_j^-x_j(t)\right]\left[f_j(x_j(t))-k_j^+x_j(t)\right]\le0,\\
&(j=1,\ldots,n),
\end{aligned}\tag{29}
$$

$$
\begin{aligned}
&\left[f_j(x_j(t-h(t)))-k_j^-x_j(t-h(t))\right]\\
&\left[f_j(x_j(t-h(t)))-k_j^+x_j(t-h(t))\right]\le0,\\
&(j=1,\ldots,n),
\end{aligned}\tag{30}
$$

$$
\begin{aligned}
&\left[f_j(x_j(t-h_{\rm U}))-k_j^-x_j(t-h_{\rm U})\right]\\
&\left[f_j(x_j(t-h_{\rm U}))-k_j^+x_j(t-h_{\rm U})\right]\le0,\\
&(j=1,\ldots,n).
\end{aligned}\tag{31}
$$

From three inequalities (29)–(31), for any positive diagonal matrices $\boldsymbol{H}_1=\mathrm{diag}\{h_{1i},\ldots,h_{1n}\}$, $\boldsymbol{H}_2=\mathrm{diag}\{h_{2i},\ldots,h_{2n}\}$ and $\boldsymbol{H}_3=\mathrm{diag}\{h_{3i},\ldots,h_{3n}\}$, the following inequality holds:

$$
\begin{aligned}
0\le\;&-2\sum_{j=1}^{n}h_{1j}\left[f_j(x_j(t))-k_j^-x_j(t)\right]\\
&\times\left[f_j(x_j(t))-k_j^+x_j(t)\right]\\
&-2\sum_{j=1}^{n}h_{2j}\left[f_j(x_j(t-h(t)))-k_j^-x_j(t-h(t))\right]\\
&\times\left[f_j(x_j(t-h(t)))-k_j^+x_j(t-h(t))\right]\\
&-2\sum_{j=1}^{n}h_{3j}\left[f_j(x_j(t-h_{\rm U}))-k_j^-x_j(t-h_{\rm U})\right]\\
&\times\left[f_j(x_j(t-h_{\rm U}))-k_j^+x_j(t-h_{\rm U})\right]\\
=\;&\boldsymbol{\zeta}^{\rm T}(t)[-2\boldsymbol{e}_1\boldsymbol{K}_{\rm m}\boldsymbol{H}_1\boldsymbol{K}_{\rm p}\boldsymbol{e}_1^{\rm T}+\boldsymbol{e}_1(\boldsymbol{K}_{\rm m}+\boldsymbol{K}_{\rm p})\boldsymbol{H}_1\boldsymbol{e}_9^{\rm T}\\
&+\boldsymbol{e}_9\boldsymbol{H}_1(\boldsymbol{K}_{\rm m}+\boldsymbol{K}_{\rm p})\boldsymbol{e}_1^{\rm T}-2\boldsymbol{e}_9\boldsymbol{H}_1\boldsymbol{e}_9^{\rm T}\\
&-2\boldsymbol{e}_2\boldsymbol{K}_{\rm m}\boldsymbol{H}_2\boldsymbol{K}_{\rm p}\boldsymbol{e}_2^{\rm T}+\boldsymbol{e}_2(\boldsymbol{K}_{\rm m}+\boldsymbol{K}_{\rm p})\boldsymbol{H}_2\boldsymbol{e}_{10}^{\rm T}\\
&+\boldsymbol{e}_{10}\boldsymbol{H}_2(\boldsymbol{K}_{\rm m}+\boldsymbol{K}_{\rm p})\boldsymbol{e}_2^{\rm T}-2\boldsymbol{e}_{10}\boldsymbol{H}_2\boldsymbol{e}_{10}^{\rm T}\\
&-2\boldsymbol{e}_3\boldsymbol{K}_{\rm m}\boldsymbol{H}_3\boldsymbol{K}_{\rm p}\boldsymbol{e}_3^{\rm T}+\boldsymbol{e}_3(\boldsymbol{K}_{\rm m}+\boldsymbol{K}_{\rm p})\boldsymbol{H}_3\boldsymbol{e}_{11}^{\rm T}\\
&+\boldsymbol{e}_{11}\boldsymbol{H}_3(\boldsymbol{K}_{\rm m}+\boldsymbol{K}_{\rm p})\boldsymbol{e}_3^{\rm T}-2\boldsymbol{e}_{11}\boldsymbol{H}_3\boldsymbol{e}_{11}^{\rm T}]\boldsymbol{\zeta}(t).
\end{aligned}\tag{32}
$$

From expressions (17)–(32) and by applying S-procedure,[30] an upper bound of $\dot{\boldsymbol{V}}$ can be

$$
\dot{\boldsymbol{V}}\le\boldsymbol{\zeta}^{\rm T}(t)\left\{\boldsymbol{\Sigma}_{[h(t)]}+\boldsymbol{\Omega}_{[h(t)]}\right\}\boldsymbol{\zeta}(t),\tag{33}
$$

where $\boldsymbol{\Sigma}_{[h(t)]}$ is defined in Eq. (13) and $\boldsymbol{\Omega}_{[h(t)]}$ is defined as

$$
\begin{aligned}
\boldsymbol{\Omega}_{[h(t)]}=\;&h(t)\mathscr{X}\boldsymbol{Q}_1^{-1}\mathscr{X}^{\rm T}+(h_{\rm U}-h(t))\mathscr{Y}\boldsymbol{Q}_1^{-1}\mathscr{Y}^{\rm T}\\
&+(h^2(t)/2)\mathscr{Z}\boldsymbol{Q}_4^{-1}\mathscr{Z}^{\rm T}\\
&+((h_{\rm U}-h(t))^2/2)\mathscr{W}\boldsymbol{Q}_4^{-1}\mathscr{W}^{\rm T}.
\end{aligned}\tag{34}
$$

By Lemma 3, $\boldsymbol{\zeta}^{\rm T}(t)\left\{\boldsymbol{\Sigma}_{[h(t)]}+\boldsymbol{\Omega}_{[h(t)]}\right\}\boldsymbol{\zeta}(t)<0$ with $\boldsymbol{0}=\boldsymbol{\Gamma}\boldsymbol{\zeta}(t)$ is equivalent to

$$
(\boldsymbol{\Gamma}^{\perp})^{\rm T}\left\{\boldsymbol{\Sigma}_{[h(t)]}+\boldsymbol{\Omega}_{[h(t)]}\right\}\boldsymbol{\Gamma}^{\perp}<0.\tag{35}
$$

Then, by the use of Fact 1, inequality (35) is equivalent to

$$\begin{bmatrix} (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}} \begin{pmatrix} \boldsymbol{\Sigma}_{[h(t)]} + h(t)\mathscr{X}\boldsymbol{Q}_1^{-1}\mathscr{X}^{\mathrm{T}} \\ +(h_{\mathrm{U}} - h(t))\mathscr{Y}\boldsymbol{Q}_1^{-1}\mathscr{Y}^{\mathrm{T}} \end{pmatrix} \boldsymbol{\Gamma}^{\perp} & (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(h(t)\mathscr{Z}) & (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(h_{\mathrm{U}} - h(t))\mathscr{W} \\ \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} \\ \star & \star & -2\boldsymbol{Q}_4 \end{bmatrix} < \boldsymbol{0}. \tag{36}$$

Since the elements of inequality (36) are affinely dependent on $h(t)$, inequality (36) for $0 \leq h(t) \leq h_{\mathrm{U}}$ holds if the following two inequalities are satisfied:

$$\begin{bmatrix} (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}} \left(\boldsymbol{\Sigma}_{[h(t)=0]} + h_{\mathrm{U}}\mathscr{Y}\boldsymbol{Q}_1^{-1}\mathscr{Y}^{\mathrm{T}} \right) \boldsymbol{\Gamma}^{\perp} & 0 & (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{W}) \\ \star & -2\boldsymbol{Q}_4 & 0 \\ \star & \star & -2\boldsymbol{Q}_4 \end{bmatrix} < 0, \tag{37}$$

and

$$\begin{bmatrix} (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}} \left(\boldsymbol{\Sigma}_{[h(t)=h_{\mathrm{U}}]} + h_{\mathrm{U}}\mathscr{X}\boldsymbol{Q}_1^{-1}\mathscr{X}^{\mathrm{T}} \right) \boldsymbol{\Gamma}^{\perp} & (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{Z}) & 0 \\ \star & -2\boldsymbol{Q}_4 & 0 \\ \star & \star & -2\boldsymbol{Q}_4 \end{bmatrix} < 0. \tag{38}$$

From Fact 1, inequalities (37) and (38) are equivalent to LMIs (14) and (15), respectively. Therefore, if LMIs (14) and (15) hold, then inequality (36) holds for $0 \leq h(t) \leq h_{\mathrm{U}}$, which means that the neural networks (1) is asymptotically stable. This completes our proof.

Remark 1 In Theorem 1, $(1 - \dot{h}(t))\dot{\boldsymbol{x}}(t - h(t))$ is considered as an augmented state, which was proposed in Ref. [25]. However, unlike the method of Ref. [25], in obtaining an upper bound of

$$-\int_{t-h_{\mathrm{U}}}^{t}\int_{s}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(u)\boldsymbol{Q}_3\dot{\boldsymbol{x}}(u)\mathrm{d}u\mathrm{d}s$$

shown in (26), this double integral term is first separated by two terms and then Lemma 2 is applied to each ones, respectively. With newly introduced augmented vectors $\frac{1}{h(t)}\int_{t-h(t)}^{t}\boldsymbol{x}(s)\mathrm{d}s$ and $\frac{1}{h_{\mathrm{U}}-h(t)}\int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{x}(s)\mathrm{d}s$, more information about time-varying delays $h(t)$ may be utilized in stability condition of Eq. (4).

Remark 2 As mentioned in Section 1, an important index for checking the conservatism of stability criterion is to increase the feasible region of stability condition or obtain maximum delay bounds for guaranteeing the system stability. Therefore, how to choose Lyapunov–Krasovskii's functional and derive its time derivative upper bound are important to reduce the conservatism of stability region. In this regard, the utilization of information about time-varying delay plays a key role in increasing the stability region of time-delay systems. Also, the proposed method utilizes the convex-hull properties of time-varying delays $h(t)$. Can a Lyapunov–Krasovskii's functional which is dependent on $h(t)$ enhance the feasible region of stability criterion for neural networks with time-varying delays? With this motivation, the authors propose the following Lyapunov–Krasovskii's functional which is new and has not been considered in other literature:

$$\boldsymbol{V}_7 = \boldsymbol{\gamma}^{\mathrm{T}}(t)\left(h(t)\mathscr{P}_1 + (h_{\mathrm{U}} - h(t))\mathscr{P}_2\right)\boldsymbol{\gamma}(t), \tag{39}$$

where $\boldsymbol{\gamma}^{\mathrm{T}}(t) = \left[\boldsymbol{x}^{\mathrm{T}}(t) \;\; \boldsymbol{x}^{\mathrm{T}}(t-h(t)) \;\; \boldsymbol{x}^{\mathrm{T}}(t-h_{\mathrm{U}})\right]$, $\mathscr{P}_1$ and $\mathscr{P}_2$ are positive matrices. With the new Lyapunov–Krasovskii's functional in expression (39), a new delay-dependent stability criterion will be introduced as Theorem 2.

Theorem 2 For given positive scalars h_{U}, any ones h_{Dl} and h_{Du} which satisfy $h_{\mathrm{Dl}} < h_{\mathrm{Du}} < 1$, diagonal matrices $\boldsymbol{K}_{\mathrm{p}} = \mathrm{diag}\{k_1^+, \ldots, k_n^+\}$ and $\boldsymbol{K}_{\mathrm{m}} = \mathrm{diag}\{k_1^-, \ldots, k_n^-\}$, system (1) is asymptotically stable for $0 \leq h(t) \leq h_{\mathrm{U}}$ and $h_{\mathrm{Dl}} \leq \dot{h}(t) \leq h_{\mathrm{Du}} < 1$ if there exist positive diagonal matrices $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$, $\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1, \ldots, \delta_n\}$, $\boldsymbol{H}_j = \mathrm{diag}\{h_{1i}, \ldots, h_{1n}\}$ $(j = 1, 2, 3)$, positive definite matrices $\mathscr{P}_1 = [P_{1,ij}]_{3\times3}$, $\mathscr{P}_2 = [P_{2,ij}]_{3\times3}$, $\mathscr{R} = [R_{ij}]_{4\times4}$, $\mathscr{N} = [N_{ij}]_{3\times3}$, $\mathscr{G} = [G_{ij}]_{3\times3}$, $\boldsymbol{Q}_i$ $(i = 1, \ldots, 4)$ and any matrices $\boldsymbol{X}_i$, $\boldsymbol{Y}_i$ $(i = 1, \ldots, 5)$, $\boldsymbol{Z}_j$, $\boldsymbol{W}_j$ $(j = 1, 2)$ satisfying the following LMIs:

$$\begin{bmatrix} (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(\boldsymbol{\Omega}_1)\boldsymbol{\Gamma}^{\perp} & \boldsymbol{0} & (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{W}) & (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{Y}) \\ \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} \\ \star & \star & \star & -\boldsymbol{Q}_1 \end{bmatrix} < \boldsymbol{0}, \tag{40}$$

$$\begin{bmatrix} (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(\boldsymbol{\Omega}_2)\boldsymbol{\Gamma}^{\perp} & (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{Z}) & \boldsymbol{0} & (\boldsymbol{\Gamma}^{\perp})^{\mathrm{T}}(h_{\mathrm{U}}\mathscr{X}) \\ \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} \\ \star & \star & \star & -\boldsymbol{Q}_1 \end{bmatrix} < \boldsymbol{0}, \tag{41}$$

where

$$\begin{aligned} \boldsymbol{\Pi}_7 &= \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix}, \quad \boldsymbol{\Pi}_8 = \begin{bmatrix} \boldsymbol{e}_4 & \boldsymbol{e}_8 & \boldsymbol{e}_5 \end{bmatrix}, \\ \boldsymbol{\Omega}_1 &= \boldsymbol{\Sigma}_{[h(t)=0]} + \boldsymbol{\Pi}_7(h_{\mathrm{U}}\mathscr{P}_2)\boldsymbol{\Pi}_8^{\mathrm{T}} + \boldsymbol{\Pi}_8(h_{\mathrm{U}}\mathscr{P}_2)\boldsymbol{\Pi}_7^{\mathrm{T}} \\ &\quad + \boldsymbol{\Pi}_7\left(h_{\mathrm{Du}}\mathscr{P}_1 - h_{\mathrm{Dl}}\mathscr{P}_2\right)\boldsymbol{\Pi}_7^{\mathrm{T}}, \\ \boldsymbol{\Omega}_2 &= \boldsymbol{\Sigma}_{[h(t)=h_{\mathrm{U}}]} + \boldsymbol{\Pi}_7(h_{\mathrm{U}}\mathscr{P}_1)\boldsymbol{\Pi}_8^{\mathrm{T}} + \boldsymbol{\Pi}_8(h_{\mathrm{U}}\mathscr{P}_1)\boldsymbol{\Pi}_7^{\mathrm{T}} \\ &\quad + \boldsymbol{\Pi}_7\left(h_{\mathrm{Du}}\mathscr{P}_1 - h_{\mathrm{Dl}}\mathscr{P}_2\right)\boldsymbol{\Pi}_7^{\mathrm{T}}, \end{aligned} \tag{42}$$

and other notations are the same as ones in Eq. (13).

Proof We consider the following Lyapunv-Krasvskii's functional:

$$\boldsymbol{V} = \sum_{i=1}^{7}\boldsymbol{V}_i, \tag{43}$$

where $\boldsymbol{V}_i(i=1,\ldots,6)$ are the same as ones in Eq. (6) and

$$\boldsymbol{V}_7 = \gamma^{\mathrm{T}}(t)\left(h(t)\mathscr{P}_1 + (h_{\mathrm{U}} - h(t))\mathscr{P}_2\right)\gamma(t). \tag{44}$$

An upper bound of $\dot{\boldsymbol{V}}_7$ can be obtained as

$$\begin{aligned} \dot{\boldsymbol{V}}_7 &= \gamma^{\mathrm{T}}(t)\left(\dot{h}(t)\mathscr{P}_1 - \dot{h}(t)\mathscr{P}_2\right)\boldsymbol{\gamma}(t) \\ &\quad + 2\boldsymbol{\gamma}^{\mathrm{T}}(t)\left(h(t)\mathscr{P}_1 + (h_{\mathrm{U}} - h(t))\mathscr{P}_2\right)\dot{\gamma}(t) \\ &\leq \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Pi}_7\left(h_{\mathrm{Du}}\mathscr{P}_1 - h_{\mathrm{Dl}}\mathscr{P}_2\right)\boldsymbol{\Pi}_7^{\mathrm{T}}\boldsymbol{\zeta}(t) \\ &\quad + 2\boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Pi}_7(h(t)\mathscr{P}_1 \\ &\quad + (h_{\mathrm{U}} - h(t))\mathscr{P}_2)\boldsymbol{\Pi}_8^{\mathrm{T}}\boldsymbol{\zeta}(t). \end{aligned} \tag{45}$$

The other procedure is straightforward from the proof of Theorem 1, so we omit it.

Remark 3 In Theorem 1 of Ref. [25], the utilized number of free variables and total decision ones were $45.5n^2 + 3.5n$ and $60.5n^2 + 12.5n$, respectively. However, the number of free variables in Theorem 1 and 2 is $14n^2$. Also, the total number of decision variables of Theorem 1 and 2 are $39n^2 + 7n$ and $48n^2 + 10n$, respectively. Through two numerical examples, it will be shown that Theorem 1 and 2 with less decision variables than Theorem 1 in Ref. [25] give improved feasible regions.

Remark 4 When the information about $\dot{h}(t)$ is unknown, $(1-\dot{h}(t))\dot{\boldsymbol{x}}(t-h(t))$ cannot be considered as augmented states. The $\boldsymbol{V}_3$ and $\boldsymbol{V}_7$ in Eq. (43) cannot be chosen as Lyapunov–Krasovskii's functional either. For this case, from Theorem 2, a stability criterion for system (4) which will be introdunced as Corollary 1 can be derived.

For a simple representation of Corollary 1, $\widetilde{\boldsymbol{e}}_i \in \mathbb{R}^{10n\times n}$ $(i=1,\ldots,10)$ are defined as block entry matrices. Some of vectors and matrices are defined as

$$\begin{aligned} \widetilde{\zeta}^{\mathrm{T}}(t) &= \Big[\boldsymbol{x}^{\mathrm{T}}(t) \quad \boldsymbol{x}^{\mathrm{T}}(t-h(t)) \quad \boldsymbol{x}^{\mathrm{T}}(t-h_{\mathrm{U}}) \\ &\qquad \dot{\boldsymbol{x}}^{\mathrm{T}}(t) \quad \dot{\boldsymbol{x}}^{\mathrm{T}}(t-h_{\mathrm{U}}) \quad \frac{1}{h(t)}\int_{t-h(t)}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\mathrm{d}s \\ &\qquad \frac{1}{h_{\mathrm{U}}-h(t)}\int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{x}^{\mathrm{T}}(s)\mathrm{d}s \quad \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t)) \\ &\qquad \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t-h(t))) \quad \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}-h_{\mathrm{U}}))\Big], \\ \widetilde{\boldsymbol{\Pi}}_1 &= \begin{bmatrix} \widetilde{\boldsymbol{e}}_1 & \widetilde{\boldsymbol{e}}_3 & \widetilde{\boldsymbol{e}}_6 & \widetilde{\boldsymbol{e}}_7 \end{bmatrix}, \\ \widetilde{\boldsymbol{\Pi}}_2 &= \begin{bmatrix} \widetilde{\boldsymbol{e}}_4 & \widetilde{\boldsymbol{e}}_5 & \widetilde{\boldsymbol{e}}_1 - \widetilde{\boldsymbol{e}}_3 \end{bmatrix}, \\ \widetilde{\boldsymbol{\Pi}}_3 &= \begin{bmatrix} \widetilde{\boldsymbol{e}}_1 & \widetilde{\boldsymbol{e}}_4 & \widetilde{\boldsymbol{e}}_8 \end{bmatrix}, \\ \widetilde{\boldsymbol{\Pi}}_4 &= \begin{bmatrix} \widetilde{\boldsymbol{e}}_3 & \widetilde{\boldsymbol{e}}_5 & \widetilde{\boldsymbol{e}}_{10} \end{bmatrix}, \\ \widetilde{\boldsymbol{\Psi}}_{[h(t)]} &= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & h(t)\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & (h_{\mathrm{U}}-h(t))\boldsymbol{I} \end{bmatrix}, \\ \widetilde{\mathscr{X}}^{\mathrm{T}} &= \begin{bmatrix} \boldsymbol{X}_1^{\mathrm{T}} & \boldsymbol{X}_2^{\mathrm{T}} & \boldsymbol{X}_3^{\mathrm{T}} & \boldsymbol{X}_4^{\mathrm{T}} & \boldsymbol{X}_5^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \\ \widetilde{\mathscr{Y}}^{\mathrm{T}} &= \begin{bmatrix} \boldsymbol{Y}_1^{\mathrm{T}} & \boldsymbol{Y}_2^{\mathrm{T}} & \boldsymbol{Y}_3^{\mathrm{T}} & \boldsymbol{Y}_4^{\mathrm{T}} & \boldsymbol{Y}_5^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \\ \widetilde{\mathscr{Z}}^{\mathrm{T}} &= \begin{bmatrix} \boldsymbol{Z}_1^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Z}_2^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \\ \widetilde{\mathscr{W}}^{\mathrm{T}} &= \begin{bmatrix} \boldsymbol{0} & \boldsymbol{W}_1^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W}_2^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \\ \widetilde{\boldsymbol{\Gamma}} &= \begin{bmatrix} -\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W}_0 & \boldsymbol{W}_1 & \boldsymbol{0} \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
\widetilde{\boldsymbol{\Sigma}}_{[h(t)]} = {} & \widetilde{\boldsymbol{\Pi}}_1\widetilde{\Psi}_{[h(t)]}\mathscr{R}\widetilde{\boldsymbol{\Pi}}_2^{\mathrm{T}} + \widetilde{\boldsymbol{\Pi}}_2\mathscr{R}\widetilde{\Psi}_{[h(t)]}^{\mathrm{T}}\widetilde{\boldsymbol{\Pi}}_1^{\mathrm{T}} + \widetilde{\boldsymbol{\Pi}}_3\mathscr{N}\widetilde{\boldsymbol{\Pi}}_3^{\mathrm{T}} - \widetilde{\boldsymbol{\Pi}}_4\mathscr{N}\widetilde{\boldsymbol{\Pi}}_4^{\mathrm{T}} \\
& + \widetilde{\boldsymbol{e}}_8\boldsymbol{\Lambda}\widetilde{\boldsymbol{e}}_4^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_4\boldsymbol{\Lambda}\widetilde{\boldsymbol{e}}_8^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_1\boldsymbol{K}_{\mathrm{m}}\boldsymbol{\Lambda}\widetilde{\boldsymbol{e}}_4^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_4\boldsymbol{\Lambda}\boldsymbol{K}_{\mathrm{m}}\widetilde{\boldsymbol{e}}_1^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_1\boldsymbol{K}_{\mathrm{p}}\boldsymbol{\Delta}\widetilde{\boldsymbol{e}}_4^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_4\boldsymbol{\Delta}\boldsymbol{K}_{\mathrm{p}}\widetilde{\boldsymbol{e}}_1^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_8\boldsymbol{\Delta}\widetilde{\boldsymbol{e}}_4^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_4\boldsymbol{\Delta}\widetilde{\boldsymbol{e}}_8^{\mathrm{T}} \\
& + h_{\mathrm{U}}\widetilde{\boldsymbol{e}}_4\boldsymbol{Q}_1\widetilde{\boldsymbol{e}}_4^{\mathrm{T}} + \widetilde{\mathscr{X}}(\widetilde{\boldsymbol{e}}_1^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_2^{\mathrm{T}}) + (\widetilde{\boldsymbol{e}}_1 - \widetilde{\boldsymbol{e}}_2)\widetilde{\mathscr{X}}^{\mathrm{T}} + \widetilde{\mathscr{Y}}(\widetilde{\boldsymbol{e}}_2^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_3^{\mathrm{T}}) + (\widetilde{\boldsymbol{e}}_2 - \widetilde{\boldsymbol{e}}_3)\widetilde{\mathscr{Y}}^{\mathrm{T}} + h_{\mathrm{U}}\widetilde{\boldsymbol{e}}_1\boldsymbol{Q}_2\widetilde{\boldsymbol{e}}_1^{\mathrm{T}} \\
& - \widetilde{\boldsymbol{e}}_6(h(t)\boldsymbol{Q}_2)\widetilde{\boldsymbol{e}}_6^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_7(h_{\mathrm{U}} - h(t))\boldsymbol{Q}_2)\widetilde{\boldsymbol{e}}_7^{\mathrm{T}} + (h_{\mathrm{U}}^2/2)\widetilde{\boldsymbol{e}}_4(\boldsymbol{Q}_3 + \boldsymbol{Q}_4)\widetilde{\boldsymbol{e}}_4^{\mathrm{T}} \\
& - 2(\widetilde{\boldsymbol{e}}_1 - \widetilde{\boldsymbol{e}}_6)\boldsymbol{Q}_3(\widetilde{\boldsymbol{e}}_1^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_6^{\mathrm{T}}) - 2(\widetilde{\boldsymbol{e}}_2 - \widetilde{\boldsymbol{e}}_7)\boldsymbol{Q}_3(\widetilde{\boldsymbol{e}}_2^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_7^{\mathrm{T}}) + (h(t)\widetilde{\mathscr{Z}})(\widetilde{\boldsymbol{e}}_1^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_6^{\mathrm{T}}) \\
& + (\widetilde{\boldsymbol{e}}_1 - \widetilde{\boldsymbol{e}}_6)(h(t)\widetilde{\mathscr{Z}}^{\mathrm{T}}) + ((h_{\mathrm{U}} - h(t))\widetilde{\mathscr{W}})(\widetilde{\boldsymbol{e}}_2^{\mathrm{T}} - \widetilde{\boldsymbol{e}}_7^{\mathrm{T}}) + (\widetilde{\boldsymbol{e}}_2 - \widetilde{\boldsymbol{e}}_7)((h_{\mathrm{U}} - h(t))\widetilde{\mathscr{W}}^{\mathrm{T}}) \\
& - 2\widetilde{\boldsymbol{e}}_1\boldsymbol{K}_{\mathrm{m}}\boldsymbol{H}_1\boldsymbol{K}_{\mathrm{p}}\widetilde{\boldsymbol{e}}_1^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_1(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{H}_1\widetilde{\boldsymbol{e}}_8^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_8\boldsymbol{H}_1(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\widetilde{\boldsymbol{e}}_1^{\mathrm{T}} - 2\widetilde{\boldsymbol{e}}_8\boldsymbol{H}_1\widetilde{\boldsymbol{e}}_8^{\mathrm{T}} \\
& - 2\widetilde{\boldsymbol{e}}_2\boldsymbol{K}_{\mathrm{m}}\boldsymbol{H}_2\boldsymbol{K}_{\mathrm{p}}\widetilde{\boldsymbol{e}}_2^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_2(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{H}_2\widetilde{\boldsymbol{e}}_9^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_9\boldsymbol{H}_2(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\widetilde{\boldsymbol{e}}_2^{\mathrm{T}} - 2\widetilde{\boldsymbol{e}}_9\boldsymbol{H}_2\widetilde{\boldsymbol{e}}_9^{\mathrm{T}} \\
& - 2\widetilde{\boldsymbol{e}}_3\boldsymbol{K}_{\mathrm{m}}\boldsymbol{H}_3\boldsymbol{K}_{\mathrm{p}}\widetilde{\boldsymbol{e}}_3^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_3(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\boldsymbol{H}_3\widetilde{\boldsymbol{e}}_{10}^{\mathrm{T}} + \widetilde{\boldsymbol{e}}_{10}\boldsymbol{H}_3(\boldsymbol{K}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{p}})\widetilde{\boldsymbol{e}}_3^{\mathrm{T}} - 2\widetilde{\boldsymbol{e}}_{10}\boldsymbol{H}_3\widetilde{\boldsymbol{e}}_{10}^{\mathrm{T}}. \qquad (46)
\end{aligned}
$$

Corollary 1 For a given positive scalar h_{U}, diagonal matrices $\boldsymbol{K}_{\mathrm{p}} = \mathrm{diag}\{k_1^+, \ldots, k_n^+\}$ and $\boldsymbol{K}_{\mathrm{m}} = \mathrm{diag}\{k_1^-, \ldots, k_n^-\}$, system (1) is asymptotically stable for $0 \leq h(t) \leq h_{\mathrm{U}}$ if there exist positive diagonal matrices $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$, $\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1, \ldots, \delta_n\}$, $\boldsymbol{H}_j = \mathrm{diag}\{h_{1i}, \ldots, h_{1n}\}$ $(j = 1, 2, 3)$, positive definite matrices $\mathscr{R} = [R_{ij}]_{4\times4}$, $\mathscr{N} = [N_{ij}]_{3\times3}$, $\boldsymbol{Q}_i (i = 1, \ldots, 4)$ and any matrices $\boldsymbol{X}_i$, $\boldsymbol{Y}_i$ $(i = 1, \ldots, 5)$, $\boldsymbol{Z}_j$, $\boldsymbol{W}_j$ $(j = 1, 2)$ satisfying the following LMIs:

$$
\begin{bmatrix}
\left(\widetilde{\boldsymbol{\Gamma}}^{\perp}\right)^{\mathrm{T}}\left(\begin{array}{c}\widetilde{\boldsymbol{\Sigma}}_{[h(t)=0]}\end{array}\right)\widetilde{\boldsymbol{\Gamma}}^{\perp} & \boldsymbol{0} & \left(\widetilde{\boldsymbol{\Gamma}}^{\perp}\right)^{\mathrm{T}}(h_{\mathrm{U}}\widetilde{\mathscr{W}}) & \left(\widetilde{\boldsymbol{\Gamma}}^{\perp}\right)^{\mathrm{T}}(h_{\mathrm{U}}\widetilde{\mathscr{Y}}) \\
\star & -2\boldsymbol{Q}_4 & \boldsymbol{0} & \boldsymbol{0} \\
\star & \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} \\
\star & \star & \star & -\boldsymbol{Q}_1
\end{bmatrix} < \boldsymbol{0}, \qquad (47)
$$

$$
\begin{bmatrix}
\left(\widetilde{\boldsymbol{\Gamma}}^{\perp}\right)^{\mathrm{T}}\left(\begin{array}{c}\widetilde{\boldsymbol{\Sigma}}_{[h(t)=h_{\mathrm{U}}]}\end{array}\right)\widetilde{\Gamma}^{\perp} & \left(\widetilde{\boldsymbol{\Gamma}}^{\perp}\right)^{\mathrm{T}}(h_{\mathrm{U}}\widetilde{\mathscr{Z}}) & \boldsymbol{0} & \left(\widetilde{\boldsymbol{\Gamma}}^{\perp}\right)^{\mathrm{T}}(h_{\mathrm{U}}\widetilde{\mathscr{X}}) \\
\star & -2\boldsymbol{Q}_4 & \boldsymbol{0} & \boldsymbol{0} \\
\star & \star & -2\boldsymbol{Q}_4 & \boldsymbol{0} \\
\star & \star & \star & -\boldsymbol{Q}_1
\end{bmatrix} < \boldsymbol{0}, \qquad (48)
$$

where $\widetilde{\boldsymbol{\Sigma}}_{1[h(t)]}$, $\widetilde{\boldsymbol{\Gamma}}$, $\widetilde{\mathscr{X}}$, $\widetilde{\mathscr{Y}}$, $\widetilde{\mathscr{Z}}$, and $\mathscr{W}$ are defined in Eq. (46) and $\widetilde{\boldsymbol{\Gamma}}^{\perp}$ is the right orthogonal complement of $\widetilde{\boldsymbol{\Gamma}}$.

Proof We consider the Lyaunov–Krasovskii's functional $\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{V}_2 + \boldsymbol{V}_4 + \boldsymbol{V}_5 + \boldsymbol{V}_6$ defined in Eq. (46). From the augmented vector $\widetilde{\zeta}(t)$ defined in (:symb3) and the proof of Theorem 1, the procedure of obtaining delay-dependent stability condition presented in inequalities (47) and (48) is straightforward from the proof of Theorem 1. Therefore, we omit it.

Remark 5 In Eq. (16), if $h(t) = 0$, then h_{U} is also zero. Therefore, all integral terms including delay information will be zero. Also, the terms $1/h(t)$ and $1/(h_{\mathrm{U}} - h(t))$ can be found in the augmented vector $\boldsymbol{\zeta}(t)$. Note that

$$
\lim_{h(t)\to0}\frac{1}{h(t)}\int_{t-h(t)}^{t}\boldsymbol{x}(s)\mathrm{d}s = \dot{\boldsymbol{x}}(t),
$$

$$
\lim_{h(t)\to0}\frac{1}{h(t)}\int_{t-h(t)}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s = \dot{\boldsymbol{x}}(t),
$$

$$
\lim_{h(t)\to h_{\mathrm{U}}}\frac{1}{h_{\mathrm{U}} - h(t)}\int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{x}(s)\mathrm{d}s = \dot{\boldsymbol{x}}(t - h_{\mathrm{U}}),
$$

and

$$
\lim_{h(t)\to h_{\mathrm{U}}}\frac{1}{h_{\mathrm{U}} - h(t)}\int_{t-h_{\mathrm{U}}}^{t-h(t)}\dot{\boldsymbol{x}}(s)\mathrm{d}s = \dot{\boldsymbol{x}}(t - h_{\mathrm{U}}).
$$

As mentioned in Section 1, the proposed method is inspired by the work of Ref. [27]. However, the basic approach in this paper is different from the method in Ref. [27] in which the term $1/h(t)$ was also inserted into the augmented vector.

4. Numerical examples

Example 1 Consider neural networks (4) with the parameters

$$
\boldsymbol{A} = \mathrm{diag}\{1.5, 0.7\}, \quad \boldsymbol{W}_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix},
$$

$$\boldsymbol{W}_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix},$$
$$\boldsymbol{K}_{\rm p} = {\rm diag}\{0.3,\ 0.8\}, \qquad \boldsymbol{K}_{\rm m} = 0. \tag{49}$$

In Ref. [25], when

$$|\dot{h}(t)| \leq h_D,$$

the maximum delay bounds for various conditions of h_D were investigated such that system (4) with the parameters (49) was asymptotically stable. By using Theorem 1 and 2, one can obtain maximum delay bounds as listed in Table 1. From Table 1, one can see Theorem 1 improves the feasible region of stability criteria compared with the methods of Ref. [25]. Also, with the consideration of $\boldsymbol{V}_7$ mentioned in Remark 2, Theorem 2 provides larger delay bounds than those of Theorem 1. Therefore, the proposed $\boldsymbol{V}_7$ in Eq. (39) plays a role in enhancing the feasible region of stability criterion in Theorem 1, which supports that how to choose Lyapunov–Krasovskii's functional is very important to reduce the conservatism of stability criterion. Note that the number of decision variables in Ref. [25] were 267. However, the number of decision variables of Theorem 1 and 2 are, respectively, 170 and 212, which are much less than those in Ref. [25]. This means that our proposed methods give larger delay bounds with less computational burden and time cost than those in Ref. [25].

Table 1. Delay bounds $h_{\rm U}$ with different $h_{\rm D}$ values (Example 1).

$h_{\rm D}$	0.4	0.45	0.5	0.55
Sun *et al.*[25]	4.3814	3.6008	3.3377	3.2350
Theorem 1	4.7765	4.0144	3.7702	3.6790
Theorem 2	4.8401	4.0626	3.8083	3.7064

Example 2 Consider the neural networks (4) with the parameters

$$\boldsymbol{A} = {\rm diag}\{1.2769, 0.6231, 0.9230, 0.4480\},$$
$$\boldsymbol{W}_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$
$$\boldsymbol{W}_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$
$$\boldsymbol{K}_{\rm p} = {\rm diag}\{0.1137,\ 0.1279,\ 0.7994,\ 0.2368\},$$

and

$$\boldsymbol{K}_{\rm m} = 0. \tag{50}$$

In Ref. [20] when

$$-\infty < \dot{h}(t) \leq h_D,$$

the maximum delay bounds for guaranteeing asymptotic stability of system (1) were investigated. With more practical condition of $|\dot{h}(t)| \leq h_D$, the maximum delay bound for guaranteeing the asymptotic stability of Eq. (50) were reported in Ref. [25]. As listed in Table 2, the proposed theorems and corollary provide larger delay bounds than those of Refs. [20] and [25].

Table 2. Delay bounds $h_{\rm U}$ with different h_D values (Example 2).

$h_{\rm D}$	0.1	0.5	0.9	Unknown
Kwon *et al.*[20]	3.7525	2.7353	2.2760	2.1326
Sun *et al.*[25]	3.7008	3.1245	2.5979	–
Theorem 1	3.7816	3.2218	2.6422	–
Theorem 2	3.7854	3.2229	2.6422	–
Corollary 1	–	–	–	2.1950

5. Conclusion

In this paper, three delay-dependent stability criteria for neural networks with time-varying delays have been proposed by the use of Lyapunov method and LMI framework. In Theorem 1, by utilizing Lemma 3, the improved stability criterion has been proposed by taking different upper bounds of double integral terms as mentioned in Remark 1. With the new Lyapunov–Krasovskii's functional (39), the further improved stability criterion is proposed in Theorem 2. Finally, when information about $\dot{h}(t)$ is unknown, new delay-dependent stability criterion is proposed in Corollary 1. Through two numerical examples, the improvement of the proposed stability criteria is successfully verified. The proposed Theorems and Corollary support the fact choosing Lyapunov–Krasovskii's functionals and estimating upper bounds of time-derivative of Lyapunov–Krasovskii's functionals play an important role in reducing the conservatism of passivity criteria as mentioned in Section 1.

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