Steady-state probability density function in wave turbulence under large volume limit*

Yeontaek Choi[a)] and Sang Gyu Jo[b)†]

[a)] *Division of Fusion and Convergence of Mathematical Sciences, National Institute for Mathematical Sciences, Daejeon* 305-340, *Republic of Korea*

[b)] *Department of Physics, Kyungpook National University, Daegu* 702-701, *Republic of Korea*

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We investigate the possibility for two-mode probability density function (PDF) to have a non-zero flux steady state solution. We take the large volume limit so that the space of modes becomes continuous. It is shown that in this limit all the steady-state two- or higher-mode PDFs are the product of one-mode PDFs. The flux of this steady-state solution turns out to be zero for any finite mode PDF.

Keywords: wave turbulence, probability flux, multi-mode probability density function, Larguerre equation

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1. Introduction

The theory of wave turbulence (WT)[1] deals with an ensemble of weakly and nonlinearly interacting dispersive waves. The ensemble is described by a probability density function(PDF) $\mathscr{P}$ in the space of states of a system under investigation.[2−5] The space of states is usually parametrized by the amplitude squares $\{s_i\}$ and phases $\{\xi_i\}$ of all the possible modes which diagonalize the free Hamiltonian. In order to incorporate the weakness of nonlinear interaction, we use a smallness parameter ϵ, which the interaction hamiltonian is taken to be proportional to.

The main essence of WT is the behaviour of the PDF during time evolution. The time evolution of $\mathscr{P}$ has been derived using a perturbative method on the assumption that the time is in a range given by $2\pi/\omega \ll T \ll 2\pi/\,\mathrm{e}^2\omega$,[2−6] where ω is a typical linear frequency. It is given as a continuity equation with flux $\mathscr{F}$ determined by the interaction Hamiltonian as follows:

$$\frac{\partial \mathscr{P}}{\partial t} + \boldsymbol{\nabla} \cdot \mathscr{F} = 0. \tag{1}$$

Here, the symbol $\boldsymbol{\nabla}$ is the divergence in the parameter space. It turns out that the leading term of the flux is of the order ϵ^2 and is independent of the phase. This rationalizes the random phase assumption up to ϵ^2 order. In other words, if $\mathscr{P}(T=0)$ is independent of the phase, then $\mathscr{P}(T>0)$ is also phase-independent for all T's.

The flux $\mathscr{F}$, interpreted as the probability current density, plays an important role for the time evolution of $\mathscr{P}$. There is no doubt that it will play a central role in studying the non-equilibrium phenomena. However, from the viewpoint of the importance of its role, nothing much has been discussed on the flux.

A steady-state PDF can be obtained by solving the divergence free condition

$$\boldsymbol{\nabla} \cdot \mathscr{F} = 0. \tag{2}$$

Two types of solutions occur for this equation. One is a zero flux solution where $\mathscr{F}$ itself vanishes and the other is a nonzero flux solution where $\mathscr{F}$ does not vanish. In the latter case, non-vanishing flow in the parameter space describes the pattern of the dynamical change of parameters.

In this paper, we consider two-mode PDF $P^{(2)}(s_\mathrm{a}, s_\mathrm{b})$ obtained from $\mathscr{P}$ by integrating all modes other than a and b. An M-mode PDF $P^{(M)}(s_1, s_2, \ldots, s_M)$ is defined as

$$P^{(M)}(s_1, s_2, \ldots, s_M)$$

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†Corresponding author. E-mail: sgjo@knu.ac.kr

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$$= \left(\prod_{s_l \neq s_1, s_2, \ldots, s_M} \int_{R^+} \mathrm{d}s_l \right) \mathscr{P}\{s\}.$$

In our recent work,[7] it has been shown that a two-mode steady-state solution may have a non-vanishing flux if some conditions for the coefficients in the nonlinear Hamiltonian are satisfied. The shape of the flux is circular in the parameter space as suggested in Ref. [5]. However, if we take the infinite volume limit of the real space, some coefficients may be neglected and the conditions can never be achieved. We give a complete argument why, in this limit, nonzero flux steady solution PDF does not exist. Furthermore, an M-mode steady-state PDF is shown to be the product of one-mode steady-state PDFs.

In the following two sections, we review the derivation of the time evolution equation of the full-mode PDF and some properties of one- and two-mode PDFs. In Section 4, we search for a non-zero flux steady-state PDF in the case that the volume becomes infinite. Here, we see that a steady-state solution is just the product of two one-mode steady-state PDFs and the corresponding flux vanishes. In Section 5, we extend the number of modes and show that the same result occurs. Finally, in Section 6, we give some concluding remarks.

2. Master equation

The physical system we are going to deal with in this paper is a stochastic ensemble of dispersive waves interacting with each other nonlinearly and weakly. The dynamics of a dispersive wave are described by a complex field, $c(\boldsymbol{x}, t)$ which represents wave actions in a d-dimensional periodic box with sides L. The time evolution of the wave is given by

$$\mathrm{i}\dot{c}_l = \frac{\partial \mathscr{H}}{\partial \bar{c}_l}, \tag{3}$$

where c_l represents the l-th Fourier mode of $c(\boldsymbol{x}, t)$ as

$$c_l(t) = \frac{1}{V} \int c(\boldsymbol{x}, t)\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_l \cdot \boldsymbol{x}} \mathrm{d}^d x. \tag{4}$$

Here $V = L^d$ is the volume of the system and Hamiltonian $\mathscr{H}$ carries the unit of energy density.[1] The Hamiltonian is the sum of the free term ($\mathscr{H}_2$) and perturbative terms ($\mathscr{H}_3, \mathscr{H}_4, \ldots$), i.e.

$$\mathscr{H} = \mathscr{H}_2 + \mathscr{H}_3 + \mathscr{H}_4 + \cdots. \tag{5}$$

Each term $\mathscr{H}_j$ can be represented by the interaction coefficients T and j amplitudes c_l as

$$\mathscr{H}_j = \sum_{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \ldots, \boldsymbol{q}_n, \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_m} \left(T^{\boldsymbol{q}_1 \boldsymbol{q}_2 \ldots \boldsymbol{q}_n}_{\boldsymbol{p}_1, \boldsymbol{p}_2 \ldots \boldsymbol{p}_m} \bar{c}_{\boldsymbol{q}_1} \bar{c}_{\boldsymbol{q}_2} \ldots \bar{c}_{\boldsymbol{q}_n} c_{\boldsymbol{p}_1} c_{\boldsymbol{p}_2} \ldots c_{\boldsymbol{p}_m} + \text{c.c.} \right), \quad n + m = j,$$

where $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \ldots \boldsymbol{q}_n$ and $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots \boldsymbol{p}_m$ are wave vectors in a d-dimensional Fourier space lattice. The coefficients $T^{\boldsymbol{q}_1 \boldsymbol{q}_2 \ldots \boldsymbol{q}_n}_{\boldsymbol{p}1, \boldsymbol{p}2 \ldots \boldsymbol{p}_m}$ represent the wave–wave interactions where n waves collide to create m waves or $m \to n$. They carry delta functions due to the momentum conservation.

Here, we consider terms up to $\mathscr{H}_3$ and neglect $\mathscr{H}_4$ and the higher terms. The free Hamiltonian $\mathscr{H}_2$ is diagonalized as follows:

$$\mathscr{H}_2 = \sum_n \omega_n |c_n|^2. \tag{6}$$

The triple coupling Hamiltonian $\mathscr{H}_3$ can be written as

$$\mathscr{H}_3 = \epsilon \sum_{l,m,n} V^l_{mn} \bar{c}_l c_m c_n \delta^l_{m+n} + \text{c.c.}, \tag{7}$$

where $\epsilon \ll 1$ is a small parameter of nonlinearity. Then Eq. (3) becomes

$$\mathrm{i}\dot{a}_l = \epsilon \sum_{m,n} \Big(V^l_{mn} a_m a_n\, \mathrm{e}^{\mathrm{i}\omega^l_{mn} t} \delta^l_{m+n} + 2\bar{V}^m_{ln} \bar{a}_n a_m\, \mathrm{e}^{-\mathrm{i}\omega^m_{ln} t} \delta^m_{l+n} \Big), \tag{8}$$

where $a_j = c_j\, \mathrm{e}^{\mathrm{i}\omega_j t}$, $\omega^l_{mn} \equiv \omega_{k_l} - \omega_{k_m} - \omega_{k_m}$ and $\omega_l = \omega_{k_l}$ (dispersion relation).

Note here that under the limit $V \to \infty$, which we will take later on, summation over the modes becomes integral over the momentum space by[1,8]

$$\sum_n \to \frac{V}{(2\pi)^d} \int \mathrm{d}^d k.$$

Therefore, both $\mathscr{H}_2$ and $\mathscr{H}_3$ behave as V^{-1}. In $\mathscr{H}_3$, the triple-summation is reduced to double-summation owing to the delta function and V^2 from this double-

summation and V^{-3} from triple cs combine to give V^{-1} dependence.

Now, we write the complex mode a_l as $a_l = A_l\psi_l$ where A_l is a real positive amplitude and ψ_l is a phase factor which takes values on S^1, a unit circle centred at zero in the complex plane. We define the PDF $\mathscr{P}\{s,\xi\}$ as the probability density function for the wave intensities A_l^2 to be s_l and for the phase factors ψ_l to be ξ_l for all ls. With this PDF, the expectation of f becomes

$$\begin{aligned}&\langle f\{A^2,\psi\}\rangle\\ &= \left(\prod_l \int_0^\infty \mathrm{d}s_l \oint_{\mathcal{S}^1} \frac{|\mathrm{d}\xi_l|}{2\pi}\right) \mathscr{P}\{s,\xi\} f\{s,\xi\} \quad (9)\end{aligned}$$

for any function f depending on a_ls.

This PDF $\mathscr{P}\{s,\xi\}$ has been shown in Ref. [2] to satisfy

$$\dot{\mathscr{P}} + \sum_j \frac{\partial \mathscr{F}_j}{\partial s_j} = 0\,, \quad (10)$$

where $\mathscr{F}_j$ is the s_j-th component of the flux of probability,

$$\begin{aligned}&-\frac{\mathscr{F}_j}{4\pi\epsilon^2 s_j}\\ &= \sum_{m,n}\Big\{\big[|V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}\\ &\quad + 2|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m}\big]s_n s_m\frac{\partial\mathscr{P}}{\partial s_j}\\ &\quad + 2\mathscr{P}[|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m}\\ &\quad - |V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}]s_m\\ &\quad + 2\big[|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m}\\ &\quad - 2|V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}\big]s_n s_m\frac{\partial\mathscr{P}}{\partial s_m}\Big\}. \quad (11)\end{aligned}$$

Our $\dot{\mathcal{P}}$ turns out to be independent of ξ in the ϵ^2 order approximation. This indicates that if $\mathcal{P}(T=0)$ is independent of ξ, then $\mathcal{P}(T)$ is independent of ξ for all Ts. Note here that $\mathscr{F}_j(s_j=0)=0$ as it should be.

Before finishing this section, we define M-mode amplitude PDF as

$$P^{(M)}_{j_1,j_2,\ldots,j_M} = \left(\prod_{l\neq j_1,j_2,\ldots,j_M}\int_{\mathscr{R}^+}\mathrm{d}s_l\right)\mathscr{P}\{s\}. \quad (12)$$

In the following sections, we consider the cases where $M=1$ and $M=2$.

3. One-mode and two-mode amplitude PDF

When $M=2$, the corresponding PDF is the two-mode amplitude PDF $P^{(2)}(s_\mathrm{a},s_\mathrm{b})$ which is obtained from $\mathscr{P}\{s\}$ by integrating over all the modes except the two modes k_a and k_b as

$$P^{(2)}(s_\mathrm{a},s_\mathrm{b}) = \left(\prod_{l\neq\mathrm{a,b}}\int\mathrm{d}s_l\right)\mathscr{P}\{s\}. \quad (13)$$

The time evolution of $P^{(2)}(s_\mathrm{a},s_\mathrm{b})$ can be obtained by integrating Eq. (10) and we have

$$\dot{P}^{(2)}(s_\mathrm{a},s_\mathrm{b}) + \frac{\partial F_\mathrm{a}}{\partial s_\mathrm{a}} + \frac{\partial F_\mathrm{b}}{\partial s_\mathrm{b}} = 0, \quad (14)$$

where

$$\begin{aligned}-\frac{F_\mathrm{a}}{4\pi\epsilon^2 s_\mathrm{a}} &= (A_\mathrm{a}+B_\mathrm{a}s_\mathrm{b})P^{(2)} + (C_\mathrm{a}+D_\mathrm{a}s_\mathrm{b})\frac{\partial P^{(2)}}{\partial s_\mathrm{a}}\\ &\quad + E_\mathrm{a}s_\mathrm{b}\frac{\partial P^{(2)}}{\partial s_\mathrm{b}}, \quad (15)\\ -\frac{F_\mathrm{b}}{4\pi\epsilon^2 s_\mathrm{b}} &= (A_\mathrm{b}+B_\mathrm{b}s_\mathrm{a})P^{(2)} + (C_\mathrm{b}+D_\mathrm{b}s_\mathrm{a})\frac{\partial P^{(2)}}{\partial s_\mathrm{b}}\\ &\quad + E_\mathrm{b}s_\mathrm{a}\frac{\partial P^{(2)}}{\partial s_\mathrm{a}}, \quad (16)\end{aligned}$$

with

$$\begin{aligned}A_\mathrm{a} &= 2{\sum_{m,n}}'' v^n_{\mathrm{a},m}(n_m-n_n) + 2{\sum_m}'' v^\mathrm{b}_{\mathrm{a},m}n_m\\ &\quad + 2{\sum_{m,n}}'' v^\mathrm{a}_{m,n}n_n - 2{\sum_{m,n}}'' v^\mathrm{a}_{m,\mathrm{b}}n_m,\\ B_\mathrm{a} &= 2\Big({\sum_m}'' v^m_{\mathrm{a,b}} + {\sum_m}'' v^\mathrm{a}_{m,\mathrm{b}} - {\sum_m}'' v^\mathrm{b}_{\mathrm{a},m}\Big),\\ C_\mathrm{a} &= {\sum_{m,n}}'' v^\mathrm{a}_{m,n}n_m n_n + 2{\sum_{m,n}}'' v^n_{\mathrm{a},m}n_m n_n,\\ D_\mathrm{a} &= 2\Big({\sum_m}'' v^\mathrm{a}_{\mathrm{b},m}n_m + {\sum_m}'' v^\mathrm{b}_{\mathrm{a},m}n_m + {\sum_m}'' v^m_{\mathrm{a,b}}n_m\Big),\\ E_\mathrm{a} &= 2\Big({\sum_m}'' v^m_{\mathrm{a,b}}n_m - 2{\sum_m}'' v^\mathrm{a}_{\mathrm{b},m}n_m\Big), \quad (17)\end{aligned}$$

and $A_\mathrm{b}, B_\mathrm{b}, C_\mathrm{b}, D_\mathrm{b}$ and E_b can be obtained by exchanging a and b. In this expression two modes a and b are excluded in the summation $\sum''$ and we use the abbreviation $v^j_{m,n} = |V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}$ for simplicity.

If we further integrate $\dot{P}^{(2)}(s_\mathrm{a},s_\mathrm{b})$ with respect to s_b, we obtain the one-mode PDF $P^{(1)}(s_\mathrm{a})$ as

$$P^{(1)}(s_\mathrm{a}) = \int \mathrm{d}s_\mathrm{b}P^{(2)}(s_\mathrm{a},s_\mathrm{b}). \quad (18)$$

The time evolution of $P^{(1)}(s_{\rm a})$ can be obtained by integrating Eq. (14) with respect to $s_{\rm b}$ as follows:

$$\dot{P}^{(1)}(s_{\rm a}) + \frac{\partial F_{\rm a}}{\partial s_{\rm a}} = 0, \tag{19}$$

where

$$-\frac{F_{\rm a}}{4\pi\epsilon^2 s_{\rm a}} = \gamma_{\rm a} P^{(1)} + \eta_{\rm a}\frac{\partial P^{(1)}}{\partial s_{\rm a}}, \tag{20}$$

with

$$\gamma_{\rm a} = A_{\rm a} + B_{\rm a} n_{\rm b} - E_{\rm a}, \tag{21}$$

and

$$\eta_{\rm a} = C_{\rm a} + D_{\rm a} n_{\rm b}. \tag{22}$$

Using Eqs. (17), (21) and (22), we can express $\gamma_{\rm a}$ and $\eta_{\rm a}$ as

$$\begin{aligned}\gamma_{\rm a} &= 2{\sum_{m,n}}' v^n_{{\rm a},m}(n_m - n_n) + 2{\sum_{m,n}}' v^{\rm a}_{m,n} n_n,\\ \eta_{\rm a} &= {\sum_{m,n}}' v^{\rm a}_{m,n} n_m n_n + 2{\sum_{m,n}}' v^n_{{\rm a},m} n_m n_n.\end{aligned} \tag{23}$$

In the summation $\sum'$, mode a is excluded. It is easy to show that

$$P^{(1)}(s_{\rm a}) = \frac{1}{n_{\rm a}}\,{\rm e}^{-s_{\rm a}/n_{\rm a}} \tag{24}$$

is a steady-state solution. Here $n_{\rm a} = \eta_{\rm a}/\gamma_{\rm a}$ and this solution defines zero flux, $F_{\rm a} = 0$. If $\gamma_{\rm a}$ is negative, then $n_{\rm a}$ becomes negative and the corresponding steady-state solution cannot be normalized. Note that η is always positive. This comes from Eq. (23). In fact, $n_{\rm a}$ is the expectation value of $s_{\rm a}$ and therefore it should be positive. This indicates that if a theory predicts any negative $\gamma_{\rm a}$ mode, there may not be any steady-state solution. Therefore, we consider a system which does not allow any negative $\gamma_{\rm a}$.

We now return to the case where $M = 2$. In the large volume limit, the summation $\sum''$ becomes integral over momentum space. In this limit, double-summation terms dominate over single-summation terms and we neglect single-summation terms. This can be understood from the usual procedure

$$\sum_{k_{\rm a}} \to \frac{V}{(2\pi)^d}\int {\rm d}^d k$$

in the limit $V \to \infty$. The contribution from double-summation terms is roughly V times bigger than that from single-summation terms. The latter comes from measure-zero sets and these can be neglected. In this approximation, we have $A_j = \gamma_j$, $B_j = D_j = E_j = 0$ and $C_j = \eta_j$, $j =$ a, b. Each component of the flux of probability becomes

$$-\frac{F_j}{4\pi\epsilon^2 s_j} = \gamma_j P^{(2)} + \eta_j \frac{\partial P^{(2)}}{\partial s_j}, \quad j = {\rm a}, {\rm b}. \tag{25}$$

For a steady-state PDF, the flux should be divergence free, i.e.,

$$\frac{\partial F_j}{\partial s_j} = 0. \tag{26}$$

A special case is when the flux is zero. In this case Eq. (25) becomes

$$\gamma_j P^{(2)} + \eta_j \frac{\partial P^{(2)}}{\partial s_j} = 0. \tag{27}$$

The unique solution for this equation is

$$P^{(2)} = \frac{1}{n_{\rm a} n_{\rm b}}\,{\rm e}^{-(s_{\rm a}/n_{\rm a})}\,{\rm e}^{-(s_{\rm b}/n_{\rm b})}. \tag{28}$$

This solution is the product of two one-mode steady-state solutions, $P^{(1)}(s_{\rm a})$ and $P^{(1)}(s_{\rm b})$.

In the next section, we search for steady-state solutions of more interesting cases where the flux is non-zero.

4. Search for non-zero flux steady-state solutions

A steady-state solution for PDF can be obtained by solving

$$\boldsymbol{\nabla}\cdot F = \frac{\partial F_{\rm a}}{\partial s_{\rm a}} + \frac{\partial F_{\rm b}}{\partial s_{\rm b}} = 0. \tag{29}$$

Substituting Eq. (25) into this equation, we have

$$\begin{aligned}&\frac{\partial}{\partial s_{\rm a}}\left[s_{\rm a}\left(\gamma_{\rm a}P^{(2)} + \eta_{\rm a}\frac{\partial P^{(2)}}{\partial s_{\rm a}}\right)\right]\\ &+ \frac{\partial}{\partial s_{\rm b}}\left[s_{\rm b}\left(\gamma_{\rm b}P^{(2)} + \eta_{\rm b}\frac{\partial P^{(2)}}{\partial s_{\rm b}}\right)\right] = 0,\end{aligned} \tag{30}$$

which becomes

$$\begin{aligned}&\gamma_{\rm a}\left(P^{(2)} + n_{\rm a}\frac{\partial P^{(2)}}{\partial s_{\rm a}} + s_{\rm a}\frac{\partial P^{(2)}}{\partial s_{\rm a}} + n_{\rm a}s_{\rm a}\frac{\partial^2 P^{(2)}}{\partial s_{\rm a}^2}\right)\\ &+ \gamma_{\rm b}\left(P^{(2)} + n_{\rm b}\frac{\partial P^{(2)}}{\partial s_{\rm b}} + s_{\rm b}\frac{\partial P^{(2)}}{\partial s_{\rm b}} + n_{\rm b}s_{\rm b}\frac{\partial^2 P^{(2)}}{\partial s_{\rm b}^2}\right)\\ &= 0.\end{aligned} \tag{31}$$

To use the separation of variables method, we put $P^{(2)}(s_{\rm a}, s_{\rm b}) = q(s_{\rm a})r(s_{\rm b})$ and obtain

$$\gamma_{\rm a}[qr + n_{\rm a}\dot{q}r + s_{\rm a}\dot{q}r + n_{\rm a}s_{\rm a}\ddot{q}r]$$

$$+\gamma_{\rm b}[qr+n_{\rm b}qr'+s_{\rm b}qr'+n_{\rm b}s_{\rm b}qr'']=0, \quad (32)$$

where $\dot{} = \partial/\partial s_{\rm a}$ and $' = \partial/\partial s_{\rm b}$. Dividing this equation by qr, we have

$$\gamma_{\rm a}\left(1+\frac{n_{\rm a}\dot{q}}{q}+\frac{s_{\rm a}\dot{q}}{q}+\frac{n_{\rm a}s_{\rm a}\ddot{q}}{q}\right) + \gamma_{\rm b}\left(1+\frac{n_{\rm b}r'}{r}+\frac{s_{\rm b}r'}{r}+\frac{n_{\rm b}s_{\rm b}r''}{r}\right)=0. \quad (33)$$

Since the first and the second parentheses are functions of $s_{\rm a}$ and $s_{\rm b}$ respectively, they should be constants

$$1+\frac{n_{\rm a}\dot{q}}{q}+\frac{s_{\rm a}\dot{q}}{q}+\frac{n_{\rm a}s_{\rm a}\ddot{q}}{q}+\lambda_{\rm a}=0,$$
$$1+\frac{n_{\rm b}r'}{r}+\frac{s_{\rm b}r'}{r}+\frac{n_{\rm b}s_{\rm b}r''}{r}+\lambda_{\rm b}=0, \quad (34)$$

where $\lambda_{\rm a}$ and $\lambda_{\rm b}$ are constants satisfying

$$\lambda_{\rm a}\gamma_{\rm a}+\lambda_{\rm b}\gamma_{\rm b}=0. \quad (35)$$

Introducing new variables $x = s_{\rm a}/n_{\rm a}$ and $y = s_{\rm b}/n_{\rm b}$ and substituting $q = {\rm e}^{-x}Q(x)$ and $r = {\rm e}^{-y}R(y)$, we obtain

$$x\frac{{\rm d}^2Q}{{\rm d}x^2}+(1-x)\frac{{\rm d}Q}{{\rm d}x}+\lambda_{\rm a}Q=0,$$
$$y\frac{{\rm d}^2R}{{\rm d}y^2}+(1-y)\frac{{\rm d}R}{{\rm d}y}+\lambda_{\rm b}R=0. \quad (36)$$

These are Laguerre's differential equations. We now analyse the Laguerre equation,

$$x\frac{{\rm d}^2L_\lambda(x)}{{\rm d}x^2}+(1-x)\frac{{\rm d}L_\lambda(x)}{{\rm d}x}+\lambda L_\lambda(x)=0. \quad (37)$$

The series solution of this equation is given by

$$L_\lambda(x)=\sum_{m=0}^{\infty}C_m x^m, \quad (38)$$

where

$$C_m=(-1)^m\frac{\lambda!}{(\lambda-m)!m!m!}. \quad (39)$$

This series terminates when λ is a non-negative integer n. It is the Laguerre polynomial $L_n(x)$ which can be expressed as

$$L_n(x)=\frac{{\rm e}^x}{n!}\frac{{\rm d}^n}{{\rm d}x^n}\left(x^n\,{\rm e}^{-x}\right). \quad (40)$$

The beginning Laguerre polynomials are

$$L_0(x)=1,$$
$$L_1(x)=-x+1,$$
$$L_2(x)=\frac{1}{2}x^2-2x+1. \quad (41)$$

If the constants $\lambda_{\rm a}$ and $\lambda_{\rm b}$ are non-integral and positive, the solutions for Q and R behave for large x and y as $Q\sim x^{-\lambda_{\rm a}-1}\,{\rm e}^x$ and $R\sim y^{-\lambda_{\rm b}-1}\,{\rm e}^y$. If the constants $\lambda_{\rm a}$ and $\lambda_{\rm b}$ are negative, the solutions for Q and R behave for large x and y as $Q\sim x^{\lambda_{\rm a}}\,{\rm e}^x$ and $R\sim y^{\lambda_{\rm b}}\,{\rm e}^y$. For both cases, $P^{(2)}$ cannot be properly normalized. In order to have non-zero flux normalizable solutions, we need to require $\lambda_{\rm a}$ and $\lambda_{\rm b}$ to be non-negative integers. We assume $\gamma_{\rm a}$ and $\gamma_{\rm b}$ to be to be positive and therefore, the only possibility for $\lambda_{\rm a}$ and $\lambda_{\rm b}$ is $\lambda_{\rm a}=\lambda_{\rm b}=0$.

With these vanishing λs, Q and R become

$$Q=L_0(x),$$
$$R=L_0(y).$$

The corresponding PDF is

$$P^{(2)}=\frac{1}{n_{\rm a}n_{\rm b}}\,{\rm e}^{-(s_{\rm a}/n_{\rm a})}\,{\rm e}^{-(s_{\rm b}/n_{\rm b})}.$$

This is just the multiplication of two one-mode PDF's $P^{(1)}_{\rm a}(s_{\rm a})$ and $P^{(1)}_{\rm b}(s_{\rm b})$. Using Eq. (25), we have $F_{\rm a}=F_{\rm b}=0$, and therefore, this solution corresponds to the zero flux solution. A non-zero flux steady-state solution does not exist.

5. Multi-mode case

We now consider multi-mode amplitude PDF

$$P^{(M)}(s_{{\rm a}_1},s_{{\rm a}_2},\ldots,s_{{\rm a}_M}) = \left(\prod_{l\neq{\rm a}_1,{\rm a}_2,\ldots,{\rm a}_M}\int {\rm d}s_l\right)\mathscr{P}\{s\}. \quad (42)$$

Once M-modes $\{{\rm a}_1,{\rm a}_2,\ldots,{\rm a}_M\}$ are selected, we denote modes ${\rm a}_i$ by i for simplicity.

The time evolution of $P^{(M)}(s_1,s_2,\ldots,s_M)$ is governed by

$$\dot{P}^{(M)}(s_1,s_2,\ldots,s_M)+\sum_{i=1}^{M}\frac{\partial F_i}{\partial s_i}=0. \quad (43)$$

Here,

$$-\frac{F_i^{(M)}}{4\pi\epsilon^2 s_i}=\gamma_i P^{(M)}+\eta_i\frac{\partial P^{(M)}}{\partial s_i}, \quad (44)$$

where

$$\gamma_i=2\sum_{m,n}v^n_{i,m}(n_m-n_n)+2\sum_{m,n}v^i_{m,n}n_n,$$

$$\eta_i = \sum_{m,n} v^i_{m,n} n_m n_n + 2\sum_{m,n} v^n_{i,m} n_m n_n. \tag{45}$$

Note that we take the large volume limit and the summation $\sum$ should be understood as integral. Obviously, the contribution of single summation terms is negligible and those terms are excluded.

To find a non-zero steady-state solution for M-mode PDF, we need to solve

$$\sum_{i=1}^{M} \frac{\partial F_i}{\partial s_i} = 0. \tag{46}$$

Substituting Eq. (44) to this equation, we have

$$\sum_{i=1}^{M} \frac{\partial}{\partial s_i}\left[s_i\left(\gamma_i P^{(M)} + \eta_i \frac{\partial P^{(M)}}{\partial s_i}\right)\right] = 0. \tag{47}$$

After a little algebra we obtain

$$\sum_{i=1}^{M} \gamma_i \Bigg(P^{(M)} + \eta_i \frac{\partial P^{(M)}}{\partial s_i} + s_i \frac{\partial P^{(M)}}{\partial s_i} + \eta_i s_i \frac{\partial^2 P^{(M)}}{\partial s_i^2}\Bigg) = 0. \tag{48}$$

Here we have used the fact $\eta_i = n_i\gamma_i$.

As we did in the two-mode case, we put

$$P^{(M)}(s_1, s_2, \ldots, s_M) = \prod_{i=1}^{M} q_i(s_i), \tag{49}$$

and we obtain

$$\sum_{i=1}^{M} q_1 q_2 \ldots q_{i-1}\gamma_i \times \left(q_i + n_i \frac{\partial q_i}{\partial s_i} + s_i \frac{\partial q_i}{\partial s_i} + n_i s_i \frac{\partial^2 q_i}{\partial s_i^2}\right) q_{i+1} \ldots q_M = 0. \tag{50}$$

Dividing this equation by $q_1 q_2 \cdots q_M$, we obtain

$$\sum_{i=1}^{M} \gamma_i \left(1 + \frac{n_i}{q_i}\frac{\partial q_i}{\partial s_i} + \frac{s_i}{q_i}\frac{\partial q_i}{\partial s_i} + \frac{n_i s_i}{q_i}\frac{\partial^2 q_i}{\partial s_i^2}\right) = 0. \tag{51}$$

We notice that the i-th summand is a function of s_i only and therefore it should be a constant. We let

$$1 + \frac{n_i}{q_i}\frac{\partial q_i}{\partial s_i} + \frac{s_i}{q_i}\frac{\partial q_i}{\partial s_i} + \frac{n_i s_i}{q_i}\frac{\partial^2 q_i}{\partial s_i^2} = -\lambda_i \tag{52}$$

with

$$\sum_{i=1}^{M} \gamma_i \lambda_i = 0. \tag{53}$$

Introducing new variables $x_i = s_i/n_i$ and substituting $q_i = \mathrm{e}^{-x_i} Q_i(x_i)$, we have

$$x_i \frac{\mathrm{d}^2 Q_i}{\mathrm{d}x_i^2} + (1 - x_i)\frac{\mathrm{d}Q_i}{\mathrm{d}x_i} + \lambda_i Q_i = 0 \tag{54}$$

for all is. Again we obtain Laguerre's equations. In order to have a normalizable PDF $P^{(M)}$, we need to require λ_i to be a non-negative integer for each i.

On the assumption that γ_i are positive for all is, we therefore conclude that λ_i are zero for all is. This means that

$$Q_i = L_0(x_i) = 1 \tag{55}$$

and

$$P^{(M)}(s_1, s_2, \ldots, s_M) = \frac{1}{n_\mathrm{a} n_\mathrm{b} \cdots n_M} \mathrm{e}^{-(s_\mathrm{a}/n_\mathrm{a})} \mathrm{e}^{-(s_\mathrm{b}/n_\mathrm{b})} \ldots \mathrm{e}^{-(s_M/n_M)}. \tag{56}$$

Therefore, we can conclude that non-zero flux steady-state solution $P^{(M)}$ does not exist for any finite number M.

6. Conclusions and discussion

We have investigated the possibility of a non-zero flux steady-state PDF. If we take the large volume limit, the space of modes become continuous. In this limit, among those coefficients in Eqs. (15) and (16) the coefficients B, D and E can be neglected. We further assume that the coefficients $A_i = \gamma_i$ are all positive in order to have a normalizable PDF.

Then, all the multi-mode steady-state PDF $P^{(M)}$ turns out to be a product of one-mode zero flux PDF $P^{(1)}$'s:

$$P^{(M)}(s_1, s_2, \ldots, s_M) = P^{(1)}(s_1)P^{(1)}(s_2)\ldots P^{(1)}(s_M).$$

This also means that the corresponding fluxes F_i vanish for all is. Therefore, there cannot be any non-zero flux steady-state PDF $P^{(M)}$ for any finite integer M.

In our previous work,[7] we showed a possible non-zero flux solution $\tilde{P}^{(2)}(s_\mathrm{a}, s_\mathrm{b}) = P^{(1)}(s_\mathrm{a})P^{(1)}(s_\mathrm{b})$, which becomes a steady-state solution only if

$$\left(A_\mathrm{a} + \frac{C_\mathrm{a}}{n_\mathrm{a}}\right) + \left(A_\mathrm{b} + \frac{C_\mathrm{b}}{n_\mathrm{b}}\right) = 0. \tag{57}$$

Each component of the flux F_i is shown to be proportional to $\alpha(= A_\mathrm{a} - C_\mathrm{a}/n_\mathrm{a})$.

If we now take the large volume limit, we have

$$\lim_{V\to\infty} A_\mathrm{a} = \gamma_\mathrm{a},$$

$$\lim_{V\to\infty} C_{\rm a} = \eta_{\rm a}. \tag{58}$$

Therefore, in this limit, Eq. (58) holds and our $\tilde{P}^{(2)}(s_{\rm a}, s_{\rm b})$ describes a steady-state solution. However, the flux, proportional to α which converges to zero under the large volume limit, vanishes. The solution becomes a zero-flux steady-state solution.

One final comment we want to add here is about the sign of γ. In this paper, we assumed it to be positive. However, in principle it would be either negative or zero. In either case, the corresponding one-mode PDF becomes non-normalizable and should be abandoned. Therefore, the analysis of a system which allows negative or zero γs should be done with caution.

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