Exact analytical solutions of three-dimensional Gross–Pitaevskii equation with time–space modulation*

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(Received 23 November 2010; revised manuscript received 23 December 2010)

By the generalized sub-equation expansion method and symbolic computation, this paper investigates the $(3+1)$-dimensional Gross–Pitaevskii equation with time- and space-dependent potential, time-dependent nonlinearity, and gain or loss. As a result, rich exact analytical solutions are obtained, which include bright and dark solitons, Jacobi elliptic function solutions and Weierstrass elliptic function solutions. With computer simulation, the main evolution features of some of these solutions are shown by some figures. Nonlinear dynamics of a soliton pulse is also investigated under the different regimes of soliton management.

Keywords: Gross–Pitaevskii equation, soliton solutions, Bose–Einstein condensate, symbolic computation

PACS: 03.75.–b, 05.45.Yv, 31.15.–p **DOI:** 10.1088/1674-1056/20/5/050315

1. Introduction

The celebrated nonlinear Schrödinger (NLS) equation known for its numerous important physical applications stimulates a great deal of attention. In particular, the potential of optical solitons applied successfully in long distance telecommunication, more and more attentions have been abstracted at different versions of NLS equation describing management of the propagation of picosecond optical solitons in mono-mode fibres. Further studies of the integrable inhomogeneous models lead to the concepts of self-similar solitons and nonautonomous solitons.[1−3] Another kind of NLS equation called the Gross–Pitaevskii (GP) equation is used to describe a dilute gas of weakly interacting atomic particles. In this case the NLS equation is supplemented by an external trap potential, which confines the gas particles. The GP equation has become an important theoretical tool in recent Bose–Einstein condensate (BEC) experiments.[4−10] For example, it enables prediction of the outcome of experiments and helps with designing and calibrating the apparatus. Also, several soliton like phenomena such as vortices, dark solitons, or their decay into vortex rings have recently been observed.[11] Moreover, periodic wave solutions in spinor BEC governed by a system of three coupled mean-field equations, different shapes of bright solitons and fascinating interactions between two solitons in BEC are achieved and the bright possibilities for soliton experimental observability are discussed.[12−14] Very recently, great interest has been focused on the three-dimensional (3D) GP describing the dynamics of a weakly interacting Bose gas at zero temperature. The 3D GP model with time–space modulation[15−18]

$$\mathrm{i}\,h\frac{\partial \Psi}{\partial t} = \left[-\frac{h^2}{2m}\boldsymbol{\nabla}^2 + V_{\mathrm{ext}}(t,\boldsymbol{r}) + G(t)|\,\Psi|^2\right]\Psi + \mathrm{i}\,\Gamma(t)\,\Psi, \tag{1}$$

where $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z)$, $\boldsymbol{r} = (x, y, z)$, $\Psi \equiv \Psi(t, \boldsymbol{r})$ denotes the order parameter with $N = \int |\Psi|^2 \mathrm{d}\boldsymbol{r}$ being the number of atoms in the condensate, $G(t) = 4\pi h^2 a_{\mathrm{s}}(t)/m$ is the interaction function with $a_{\mathrm{s}}(t)$ being the s-wave scattering length modulated by a Feshbach resonance and $\Gamma(t)$ is the gain or loss term, which is phenomenologically incorporated to account for the interaction of atomic or thermal clouds. We note that the dissipative dynamics originating from the interaction between the radial and axial degrees of freedom also has been studied recently.[19] Here the potential is chosen as a harmonic trap $V_{\mathrm{ext}}(t, \boldsymbol{r}) = (m/2)[\boldsymbol{r} - \boldsymbol{e}(t)]\boldsymbol{\omega}^2(t)[\boldsymbol{r} - \boldsymbol{e}(t)]$ with $\boldsymbol{\omega}(t) =$

*Project supported by Zhejiang Provincial Natural Science Foundations of China (Grant No. Y6090592), National Natural Science Foundation of China (Grant Nos. 11041003 and 10735030), Ningbo Natural Science Foundation (Grant Nos. 2010A610095, 2010A610103, and 2009B21003), and K.C. Wong Magna Fund in Ningbo University of China.

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diag$[\omega_x(t), \omega_y(t), \omega_z(t)]$ being a diagonal matrix of the trap frequencies in three directions and $\boldsymbol{e}(t) = [e_1(t), e_2(t), e_3(t)]$ corresponding to its centre.

Using the suitable scales and variables, $\boldsymbol{r} = a_z\boldsymbol{r}', t = \tilde{\omega}_z^{-1}t', \boldsymbol{e}(t) = a_z\boldsymbol{e}'(t), \Psi = \sqrt{N/a_z^3}\psi$, $a_z = [\hbar/m\tilde{\omega}_z]^{1/2}$ and $\tilde{\omega}_z = \int \omega_z(t)\mathrm{d}t$, we arrive at the dimensionless 3D GP equation after dropping the primes

$$\mathrm{i}\frac{\partial \Psi}{\partial t} = \left[-\frac{1}{2}\boldsymbol{\nabla}^2 + \nu(t,\boldsymbol{r}) + g(t)|\Psi|^2 \right]\Psi + \mathrm{i}\,\gamma(t)\,\Psi, \quad (2)$$

where $g(t) = 4\pi N^2 a_\mathrm{s}(t)/a_z^4$, $\gamma(t) = \Gamma(t)/(\hbar\tilde{\omega}_z)$ and

$$\nu(t,\boldsymbol{r}) = \frac{1}{2}[\boldsymbol{r} - \boldsymbol{e}(t)]\boldsymbol{\alpha}^2(t)[\boldsymbol{r} - \boldsymbol{e}(t)], \quad (3)$$

with $\boldsymbol{\alpha}(t) = [\alpha_1(t), \alpha_2(t), \alpha_3(t)] = \tilde{\omega}_z^{-1}\boldsymbol{\omega}(t)$. Equation (2) is associated with $\delta\mathcal{L}/\delta\psi^*$ in which the Lagrangian density can be written as

$$\begin{aligned}\mathcal{L} =\ & \mathrm{i}(\psi\psi_t^* - \psi^*\psi_t) + |\boldsymbol{\nabla}\psi|^2 + g(t)|\psi|^4 \\ & + 2[\nu(t,\boldsymbol{r}) + \mathrm{i}\,\gamma(t)]|\psi|^2. \end{aligned} \quad (4)$$

The analytical expressions for the evolution of soliton can help physicists and engineers to discuss and examine the sensitivity of the model and different interactions of several physical parameters. In the course of obtaining exact soliton solutions to nonlinear partial differential equations (NLPDEs), with the availability of symbolic computation systems like *Mathematica* or *Maple*, many sophisticated effective direct methods have been proposed, which include the tanh-function method,[20] the extended tanh-function method,[21] Jacobi elliptic function method,[22] extended Jacobi elliptic function method,[23,24] the project Ricatti equation method,[25−29] the sub-equation method[30−32] and so on. Among them, the sub-equation method is an effective and powerful method, whose remarkable characteristic is that we can have many different ansatzs and thus end up with abundant exact solutions for many NLPDEs.

The paper is organized as follows. In Section 2, we will extend the generalized sub-equation method to investigate the 3D GP equation (2). After thorough analysis and some tedious calculation by symbolic computation, we successfully present a series of exact analytical solutions for Eq. (2), which include solitary wave solutions, rational solutions, triangular periodic solutions, Jacobi elliptic function solutions and Weierstrass elliptic function solutions and so on. Furthermore, some figures are given to show the main properties of bright solitons, Jacobi elliptic functions solutions. In Section 3, nonlinear dynamics of the soliton pulses is investigated from the different regimes of soliton management. Finally, some conclusions are given briefly.

2. Exact analytical solutions of 3D GP equation with time–space modulation

We now extend the generalized sub-equation method[30] to investigate 3D GP equation (2). Considering a vortex phase $(x^2 + y^2 + z^2)$ set to reflect a rotation-invariable feature of the 3D generalized NLS equation, the solutions of Eq. (2) can be assumed as follows:

$$\begin{aligned}\Psi(\boldsymbol{r},t) =\ & \left[\frac{A_0(t) + A_1(t)\phi(\xi) + B_1(t)\phi'(\xi)}{1 + a_1(t)\phi(\xi) + b_1(t)\phi'(\xi)} \right. \\ & \left. + M(t)\phi^{-1}(\xi)\right]\exp[\mathrm{i}\Theta(\boldsymbol{r},t)], \\ \Theta(\boldsymbol{r},t) =\ & \Theta_0(\boldsymbol{r},t) + k(t), \\ \Theta_0(\boldsymbol{r},t) =\ & \boldsymbol{\lambda}\cdot\boldsymbol{r}^2 + \boldsymbol{\mu}\cdot\boldsymbol{r},\ \xi = \boldsymbol{p}\cdot\boldsymbol{r} + q(t), \end{aligned} \quad (5)$$

where $\boldsymbol{\lambda} = (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ denotes the vector of the curvature of the wave-front, the spatial frequency shift vector $\boldsymbol{\mu} = (\mu_1(t), \mu_2(t), \mu_3(t))$ is relative to the parallel motion of the wave-front, $k(t)$ is the homogeneous phase shift and $\boldsymbol{p} = (p_1(t), p_2(t), p_3(t))$ denotes the vector of the inverse spatial widths of the localized solutions along x, y, z directions. The $A_0(t)$, $A_1(t)$, $B_1(t)$, $a_1(t)$, $b_1(t)$, $M(t)$, $k(t)$, $q(t)$ and $\lambda_i(t)$, $\mu_i(t)$, $p_i(t)$, $(i = 1, 2, 3)$ are functions of t to be determined, the variable $\phi = \phi(\xi)$ satisfies

$$\begin{aligned}\phi'^2(\xi) =\ & \left(\frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi}\right)^2 = h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) \\ & + h_3\phi^3(\xi) + h_4\phi^4(\xi), \end{aligned} \quad (6)$$

where h_i $(i = 0, 1, 2, 3, 4)$ are constants.

Substituting Eqs. (5) and (6) into Eq. (2), removing the exponential term, collecting coefficients of monomials of $\phi(\xi)$, $\phi'(\xi)$ and x, y, z of the resulting system, then separating each coefficients to the real part and imaginary part, we obtain an ordinary differential equation (ODE) system with respect to differentiable functions $g(t)$, $M(t)$, $A_0(t)$, $A_1(t)$, $B_1(t)$, $a_0(t)$, $a_1(t)$, $b_1(t)$, $\lambda_i(t)$, $\mu_i(t)$, $p_i(t)$, $(i = 1, 2, 3)$, $k(t)$ and $q(t)$, for simplification we omit the ODE system.

Solving the ODE system with the symbolic computation system—*Maple*, we find that only the nonlinearity and the vector of the inverse spatial widths

of the localized solutions along x, y, z directions fulfill a certain constraint, GP equation (2) possesses rich exact analytical solution as follows.

Family 1 When $h_1 = h_3 = 0$, i.e., $\phi'^2(\xi) = h_0 + h_2\phi^2(\xi) + h_4\phi^4(\xi)$, a series of exact analytical solutions are obtained as follows:

$$\begin{aligned}
\Psi_{11} &= F\phi(\xi)\exp[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k_1(t))],\\
g(t) &= \frac{h_4}{F^2}\sum_{i=1}^{3}p_i^2(t),\\
k_1'(t) &= \frac{1}{2}\sum_{i=1}^{3}[h_2p_i^2(t)-\mu_i^2(t)-\alpha_i^2(t)e_i^2(t)], \qquad (7)\\
\Psi_{12} &= F\phi^{-1}(\xi)\exp[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k_1(t))],\\
g(t) &= \frac{h_0}{F^2}\sum_{i=1}^{3}p_i^2(t), \qquad (8)\\
\Psi_{13} &= F\left[\phi(\xi)\pm\sqrt{\frac{h_0}{h_4}}\phi^{-1}(\xi)\right]\\
&\quad\times\exp[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k_3(t))],\\
g(t) &= \frac{h_4}{F^2}\sum_{i=1}^{3}p_i^2(t),\\
k_3'(t) &= \frac{1}{2}\sum_{i=1}^{3}\left[\left(h_2\pm 3h_4\sqrt{\frac{h_0}{h_4}}\right)p_i^2(t)\right.\\
&\quad\left.-\mu_i^2(t)-\alpha_i^2(t)e_i^2(t)\right], \qquad (9)\\
\Psi_{14} &= F\cdot\frac{1+\epsilon\sqrt[4]{\frac{h_4}{h_0}}\phi(\xi)}{1-\epsilon\sqrt[4]{\frac{h_4}{h_0}}\phi(\xi)}\exp[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k_4(t))],\\
\epsilon &= \pm 1,\\
g(t) &= \frac{1}{F^2}\left(\frac{h_0}{2}\sqrt{\frac{h_4}{h_0}}+\frac{h_2}{4}\right)\sum_{i=1}^{3}p_i^2(t),\\
k_4'(t) &= \frac{1}{2}\sum_{i=1}^{3}\left[\left(3h_0\sqrt{\frac{h_4}{h_0}}-\frac{h_2}{2}\right)p_i^2(t)\right.\\
&\quad\left.-\mu_i^2(t)-\alpha_i^2(t)e_i^2(t)\right], \qquad (10)
\end{aligned}$$

where

$$\begin{aligned}
\Theta_0(\boldsymbol{r},t) &= x^2\lambda_1(t)+y^2\lambda_2(t)+z^2\lambda_3(t)\\
&\quad+x\mu_1(t)+y\mu_2(t)+z\mu_3(t),\\
\xi(\boldsymbol{r},t) &= p_1(t)x+p_2(t)y+p_3(t)z\\
&\quad-\int_0^t\sum_{i=1}^{3}p_i(t)\,\mu_i(t)\,\mathrm{d}t+q_0,\\
\lambda_i(t) &= -\frac{p_i'(t)}{2p_i(t)},\\
\mu_i(t) &= \left(\int_0^t\frac{e_i(t)\,(\alpha_i(t))^2}{p_i(t)}\mathrm{d}t+c_i\right)p_i(t),\\
F &= f_0\exp\left[\int_0^t\sum_{i=1}^{3}\frac{p_i'(t)}{2p_i(t)}+\gamma(t)\,\mathrm{d}t\right], \qquad (11)
\end{aligned}$$

and f_0, q_0, h_0, h_2, h_4, c_i $(i = 1, 2, 3)$ are arbitrary constants, $e_i(t)$ $(i = 1, 2, 3)$ are arbitrary functions of t and $\{p_i(t), \alpha_i(t)\ (i = 1, 2, 3)\}$ satisfy the following conditions:

$$\begin{aligned}
&\alpha_i^2(t)\,p_i^2(t)-p_i(t)\,p_i''(t)+2\,(p_i'(t))^2=0,\\
&(i=1,2,3). \qquad (12)
\end{aligned}$$

Notice In the following paper, $\{F, \boldsymbol{\lambda}, \boldsymbol{\mu}, \xi, \Theta_0(\boldsymbol{r}, t)\}$ are all determined by Eq. (11) and $\{p_i(t), \alpha_i(t)\,(i = 1, 2, 3)\}$ satisfy the conditions (12).

The corresponding $\phi(\xi)$ may be read from the Jacobi elliptic function solution in Table 1.

Table 1. Jacobi elliptic functions.

Solution	h_0	h_2	h_4	$\phi(\xi)$	$m=0$	$m=1$
1	1	$-(1+m^2)$	m^2	$\mathrm{sn}(\xi,m)$	sin	tanh
2	$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{cn}(\xi,m)$	cos	sech
3	m^2-1	$2-m^2$	-1	$\mathrm{dn}(\xi,m)$	1	sech
4	m^2	$-(1+m^2)$	1	$\mathrm{ns}(\xi,m)$	csc	coth
5	$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}(\xi,m)$	sec	cosh
6	-1	$2-m^2$	m^2-1	$\mathrm{nd}(\xi,m)$	1	cosh
7	1	$2-m^2$	$1-m^2$	$\mathrm{sc}(\xi,m)$	tan	sinh
8	$1-m^2$	$2-m^2$	1	$\mathrm{cs}(\xi,m)$	cot	csch
9	1	$-(1+m^2)$	m^2	$\mathrm{cd}(\xi,m)$	cos	1
10	m^2	$-(1+m^2)$	1	$\mathrm{dc}(\xi,m)$	sec	1

Family 2 When $h_0 = h_1 = 0$ and the nonlinearity and the vector of the inverse spatial widths of the localized solutions along x, y, z directions fulfill different constraints, we will obtain different forms of solution.

$$\begin{aligned}
\Psi_{21} &= F\left(1+\frac{4h_4}{h_3}\phi\left(\xi\right)\right)\exp[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k(t))],\\
h_3^2 &= 4h_4h_2,\\
g\left(t\right) &= h_4\frac{\sum_{i=1}^{3}p_i^2\left(t\right)}{F^2},\\
k'(t) &= -\frac{1}{2}\sum_{i=1}^{3}\left[\frac{h_2}{2}p_i^2(t)+\mu_i^2(t)+\alpha_i^2(t)e_i^2(t)\right], \quad (13)\\
\Psi_{22} &= F\frac{h_3\phi'(\xi)}{h_3+4h_4\phi(\xi)}\exp[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k(t))],\\
h_3^2 &= 4h_4h_2,\\
g\left(t\right) &= \frac{4h_4\sum_{i=1}^{3}p_i^2\left(t\right)}{h_2F^2},
\end{aligned}$$

$$\begin{aligned}
k'(t) &= \frac{1}{2}\sum_{i=1}^{3}[h_2p_i^2(t)-\mu_i^2(t)-\alpha_i^2(t)e_i^2(t)], \quad (14)\\
\Psi_{23} &= F\frac{2h_2\phi(\xi)}{2h_2+h_3\phi(\xi)}\exp[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k(t))],\\
h_3^2 &\neq 4h_4h_2,\\
g\left(t\right) &= \frac{(4h_4h_2-h_3^2)\sum_{i=1}^{3}p_i^2\left(t\right)}{4h_2F^2\left(t\right)},\\
k'(t) &= \frac{1}{2}\sum_{i=1}^{3}\left[h_2p_i^2(t)-\mu_i^2(t)-\alpha_i^2(t)e_i^2(t)\right]. \quad (15)
\end{aligned}$$

At the same time, two solutions of elliptic equation (6) with $h_0=h_1=0$ are derived as follows:

$$\phi_1(\xi)=\frac{4h_2\,C_0\left(\mathrm{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2}{2(1-\,\Delta_1)+2\,(1+\Delta_1)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)-(2\,h_3\,C_0+1-\Delta_1)\left(\mathrm{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2}, \quad (16)$$

$$\phi_2(\xi)=\frac{4h_2\,C_0\left(\mathrm{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2}{2\Delta_3-2\,(\Delta_2+{C_0}^2)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)-(2\,h_3\,C_0+\Delta_3)\left(\mathrm{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)^2}, \quad (17)$$

where $h_2>0$, $C_0=\exp(\sqrt{h_2}C_1)$ is an arbitrary constant, and $\Delta_1={C_0}^2(4\,h_2h_4-{h_3}^2)$, $\Delta_2=(4\,h_2h_4-{h_3}^2)$, $\Delta_3={C_0}^2-\Delta_2$.

Family 3 When $h_2=h_4=0$ and $\{h_0,h_1,h_3\}$ satisfy different conditions, we can obtain many exact analytical solutions for Eq.(2). For simplification, we only list two types of these solutions as follows:

$$\begin{aligned}
\Psi_{31} &= F\frac{4h_0\phi(\xi)}{4h_0+h_1\phi(\xi)}\exp\left[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k(t))\right],\\
g\left(t\right) &= \frac{5h_1^4\sum_{i=1}^{3}p_i^2\left(t\right)}{256h_0^3F^2},\\
k'(t) &= -\frac{1}{2}\sum_{i=1}^{3}\left[\frac{3h_1^2}{8h_0}{p_i}^2\left(t\right)+\mu_i^2(t)+\alpha_i^2(t)e_i^2(t)\right],\\
h_3 &= -\frac{h_1^3}{8h_0^2}>0, \quad (18)\\
\Psi_{32} &= F\frac{1+\rho_2\phi\left(\xi\right)-3\rho_1\phi'\left(\xi\right)}{1+\rho_1\phi'\left(\xi\right)}\\
&\quad\times\exp\left[\mathrm{i}(\Theta_0(\boldsymbol{r},t)+k(t))\right],\\
g\left(t\right) &= \frac{h_3\sum_{i=1}^{3}{p_i}^2\left(t\right)}{6\rho_2F^2},
\end{aligned}$$

$$\begin{aligned}
k'(t) &= -\frac{1}{2}\sum_{i=1}^{3}\left[\frac{3h_3}{\rho_2}{p_i}^2\left(t\right)+\mu_i^2(t)+{\alpha_i}^2\left(t\right){e_i}^2\left(t\right)\right],\\
h_0 &= \frac{{\rho_2}^3+48\,{\rho_1}^2h_3}{{\rho_2}^3{\rho_1}^2},\\
h_1 &= -\frac{{\rho_2}^3+72\,{\rho_1}^2h_3}{6{\rho_2}^2{\rho_1}^2}, h_3>0. \quad (19)
\end{aligned}$$

Relatively, elliptic equation (6) with $h_2=h_4=0$ admits a Weierstrass elliptic doubly periodic-type solution:

$$\phi_(\xi)=\wp(\frac{\sqrt{h_3}}{2}\xi,g_2,g_3),$$

where

$$g_2=-4\frac{h_1}{h_3},\quad g_3=-4\frac{h_0}{h_3}. \quad (20)$$

In order to understand the significance of these solutions in Families 1–3, we can investigate their features by computer simulations. For simplification, we only take Ψ_{11} and Ψ_{23} as two examples to illustrate features of solutions by computer simulations.

For the convenience of analysing different dynamical regimes described by the given model, we specify the magnitude of main physical parameters,

which are feasible in experiments. We consider a condensed sodium sample trapped in the state $|3S_{1/2}, F = 1, m_F = -1\rangle$, which has the scattering length $a_s = 2.75$ nm.[33] The other parameters can be taken as $N = 1.2 \times 10^6$ and $\widetilde{\omega}_z = (2\pi) \times 21$ Hz, which lead to $a_z = 4.55$ μm and $\sqrt{N/a_z^3} = 1.13 \times 10^2$ μm$^{-3/2}$. To make sure that the frequencies $\alpha_i(t)$ and nonlinearity $g(t)$ are bounded for realistic cases, we choose $p_i(t)$ and the gain or loss coefficient $\gamma(t)$ as the periodic functions

$$\boldsymbol{p} = \mathrm{dn}(t, n)\boldsymbol{b}, \quad \gamma(t) = \gamma_0 \mathrm{cn}(t, k), \quad \gamma_0 \in R, \quad (21)$$

where $\boldsymbol{b} = (b_1, b_2, b_3)$ is a real constant vector describing the inverse of the width of the potential and the frequency, $n \in [0, 1]$ and $k \in [0, 1]$ are the modules of Jacobi elliptic functions. Then the function F modulating the amplitude of solution $\Psi(\boldsymbol{r}, t)$ can be found by

$$F = F_1 \sqrt{p_1(t)} \sqrt{p_2(t)} \sqrt{p_3(t)} \exp\left[\int_0^t \gamma(s)\mathrm{d}s\right]. \quad (22)$$

It is easy to see that

$$\gamma(t) = \gamma_0 \mathrm{sech}(t) > 0$$

corresponds to the dissipative case when $k = 1$ and $\gamma_0 > 0$ (we will focus on this condition next). In practical systems, the modulations of $p_i(t)$, $g(t)$ and $\gamma(t)$ depend on the use of the optical lattice and Feshbach-resonance techniques, i.e., we can achieve $\Gamma(t)$ and $a_s(t)$ by exerting particular time-dependent optical field and magnetic field. For simplification, we take $b_i = 1$, i.e., $\alpha_1(t) = \alpha_2(t) = \alpha_3(t)$ corresponds to the isotropic potential and consider that the centre of the potential locates at the origin $e_i = 0$. We can also change p_i to obtain an anisotropic potential and use nonzero e_i to obtain moving bright solitons as those discussed in 1D case.

Figures 1 and 2 present the evolution of the Jacobi elliptic function solutions Ψ_{11} with different parameters, here $U = |\Psi_{11}|^2$ denotes the intensity of solution. From Figs. 1 and 2, one can verify that the Jacobi elliptic functions $\mathrm{cn}(\xi)$ and $\mathrm{sn}(\xi)$ possess the properties: 1) when $m = 1$, Ψ_{11} presents the solitary wave; 2) when $0 < m < 1$, solution Ψ_{11} presents the periodic wave. In Figs. 2(a) and 2(b), the intensities are invariable and the time shift is changing while the Jacobi elliptic periodic wave and the solitary wave keep their shapes along the fibre. The time–space evolution of dispersion Jacobi elliptic periodic wave and bright solitary waves for the solutions Ψ_{11} are shown in Figs. 1(c), 1(d), 2(c) and 2(d). Figures 3(a)–3(d) show the evolution of the Jacobi elliptic function solutions Ψ_{23} with different parameters.

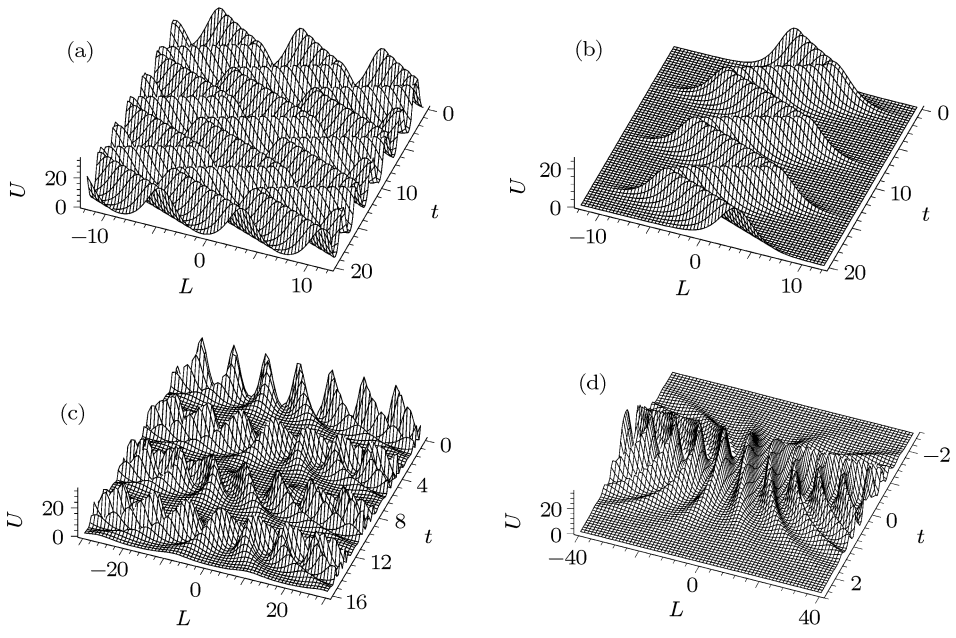


Fig. 1. Evolution plots of intensity $U = |\Psi_{11}|^2$ of solution presented as functions of $L = k_0 x + l_0 y + m_0 z$ and t in Table 1: (a) $m = 0.9$, $n = 0.1$; (b) $m = 1$, $n = 0.1$; (c) $m = 0.9$, $n = 0.9$; (d) $m = 0.9$, $n = 1$. In these cases, $k = 1$, $F_1 = 8$, $\gamma_0 = -0.1$, $h_4 = -0.5$, $h_2 = 0.2$, $q_0 = 0$, $c_i = 1$, $b_i = 1$, $\phi(\xi) = \mathrm{cn}(\xi)$.

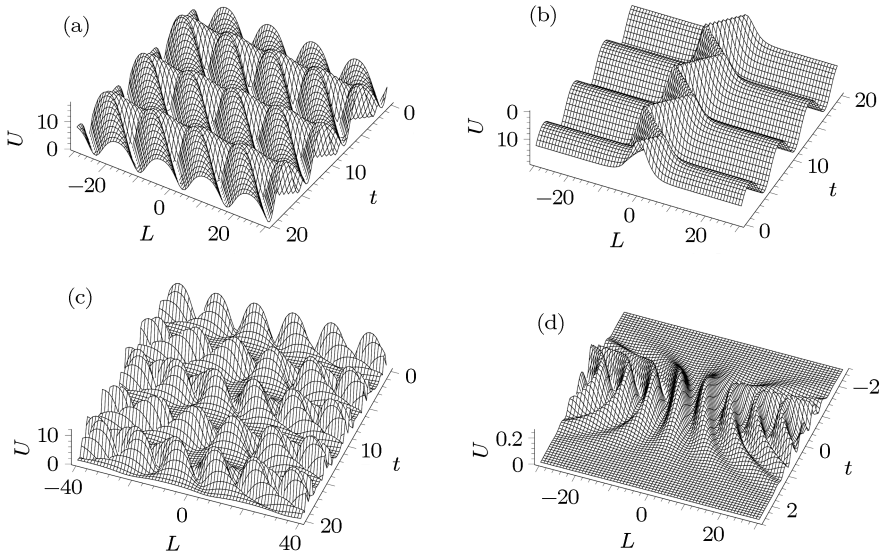


Fig. 2. Evolution plots of intensity $U = |\Psi_{11}|^2$ of solution presented as functions of $L = k_0x + l_0y + m_0z$ and t in Table 1: (a) $m = 0.9$, $n = 0.1$, $k = 0.1$; (b) $m = 1$, $n = 0.1$, $k = 0.1$; (c) $m = 0.9$, $n = 0.9$, $k = 0.9$; (d) $m = 0.1$, $n = 1$, $k = 0.1$. In these cases, $F_1 = 8$, $\gamma_0 = -0.2$, $h_4 = -0.5$, $h_2 = 0.2$, $q_0 = 0$, $c_i = 1$, $b_i = 1$, $\phi(\xi) = \mathrm{sn}(\xi)$.

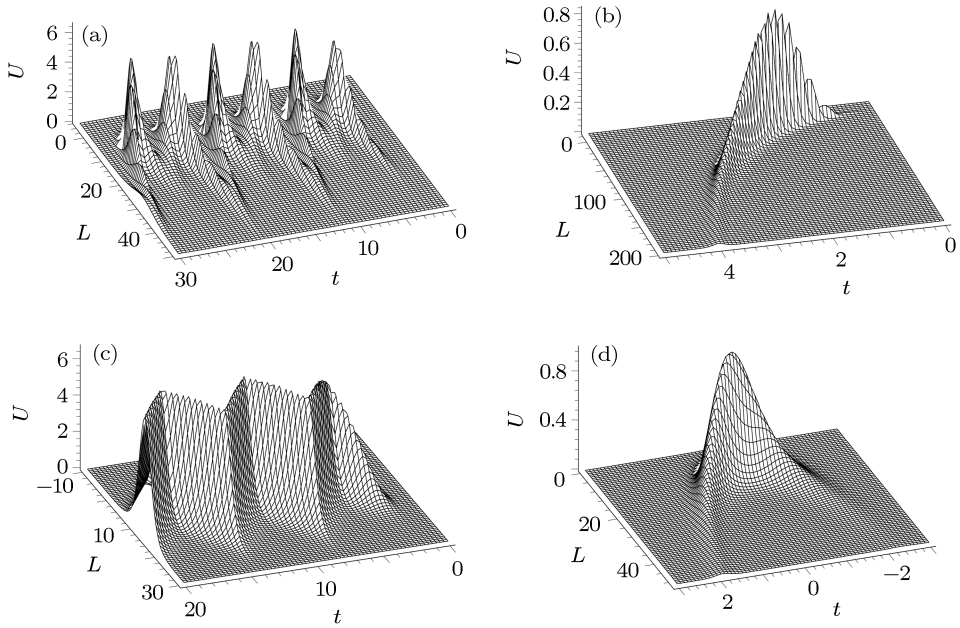


Fig. 3. Evolution plots of intensity $U = |\Psi_{23}|^2$ of solution (15) presented as functions of $L = k_0x + l_0y + m_0z$ and t. (a) $n = 0.9$, $h_2 = 0.1$, $h_4 = -1$, $\gamma_0 = 5$, $F_3 = -0.1$; (b) $n = 1$, $h_2 = 1$, $h_4 = -1$, $\gamma_0 = 5$, $F_3 = -0.1$; (c) $n = 0.1$, $k = 1$, $h_2 = 0.1$, $h_4 = -1$, $\gamma_0 = 5$, $F_3 = -0.1$, (d) $n = 1$, $h_2 = 0.1$, $h_4 = -10$, $\gamma_0 = 0.01$, $F_3 = -100$; In these cases, $k = 1$, $h_3 = 20$, $c_i = 1$, $b_i = 1$, $F_2 = 0$.

3. Different regimes of soliton management

In the following, the bright soliton $\phi(\xi) = \mathrm{sech}(\xi)$ of Eq. (7) is taken as an example to find solutions for specified soliton management conditions. In reality, one can consider more cases for each regime at the same time and derive richer results.

(I) Soliton time duration management. Consider Eq. (2) with $g(t) =$ constant and with varying phase modulation and gain/loss parameters. Suppose that soliton pulse width is determined by the known control function $p_i(t) = \Omega_i(t)$, then we can derive solutions of Eq. (2) in the form of bright solitons represented by Eq. (7), where

$$
\begin{aligned}
p_i(t) &= \Omega_i(t),\ \lambda_i(t) = -\frac{\Omega_i'(t)}{2\Omega_i(t)},\quad \alpha_i^2(t) = \frac{-2(\Omega'(t))^2 + \Omega_i(t)\Omega_i''(t)}{\Omega_i^2(t)},\quad F = C_0\sqrt{\sum_{i=1}^{3}\Omega_i^2(t)},\\
q(t) &= \int_0^t \sum_{i=1}^{3}\left(\Omega_i^2(t)\int_0^t \frac{e_i(t)\left(2\left[\Omega_i'(t)\right]^2 - \Omega_i''(t)\,\Omega_i(t)\right)}{\Omega_i^3(t)}\mathrm{d}t - \Omega_i^2(t)\,C_i\right)\mathrm{d}t' + q_0,\\
\mu_i(t) &= \left(\int_0^t \frac{\left(-2\left(\Omega_i'(t)\right)^2 + \Omega_i''(t)\,\Omega_i(t)\right)e_i(t)}{\Omega_i^3(t)}\mathrm{d}t + C_i\right)\Omega_i(t),\\
\gamma(t) &= \frac{1}{|\sum_{i=1}^{3}\Omega_i|^2}\sum_{i=1}^{3}\Omega_i(t)\Omega_i'(t) - \sum_{i=1}^{3}\frac{\Omega_i'(t)}{2\Omega_i(t)}.
\end{aligned}
\tag{23}
$$

Here integration constants C_i, q_0 are determined by initial conditions. Finally, we can easily calculate $k_1(t)$ by Eq. (7).

(II) Soliton amplification management. Consider Eq. (1) with $g(t) = g_0 =$ constant and with varying phase modulation parameters. Suppose that the gain/loss coefficient is determined by the known control function $\gamma(t) = \Lambda(t)$, where the control function $\Lambda(t)$ is required only to be once integrable. There are then an infinite number of solutions of the bright soliton represented by Eq. (7), where the main control functions $p_i(t)$ are given by

$$
\begin{aligned}
&g_0 f_0^2\left(\exp\left[\int_0^t\sum_{i=1}^{3}\frac{p_i'(t)}{2p_i(t)} + \Lambda(t)\mathrm{d}t\right]\right)^2\\
&= h_4\sum_{i=1}^{3}p_i^2(t)
\end{aligned}
\tag{24}
$$

and Eq. (12). The F is given by Eq. (11).

(III) Soliton trap frequency management. Consider Eq. (2) with $g(t) = g_0 =$ constant and with varying phase modulation and gain/loss parameters. Suppose that trap frequency management function $\alpha_i(t)$ (we call them control function here) are the given arbitrary analytical functions. There are then infinite number of solutions of Eq. (2) of the bright soliton represented by Eq. (7), where the main functions $p_i(t)$ can be expressed in functions of $\alpha_i(t)$, which are determined by Eq. (12), $F = c_0\sqrt{\sum_{i=1}^{3}p_i^2(t)}$ and c_0 is an arbitrary constant (both positive and negative).

From Eqs. (11) and (12) we can calculate $\mu_i(t)$ and $q(t)$ successively. Finally, the amplification or absorption coefficient $\gamma(t)$ are given by

$$
\gamma(t) = \frac{1}{|\boldsymbol{p}|^2}\sum_{i=1}^{3}p_i(t)p_i'(t) - \sum_{i=1}^{3}\frac{p_i'(t)}{2p_i(t)}. \tag{25}
$$

Let us consider the following example. The fundamental set of trap frequency managed solutions can be expressed in trigonometric and hyperbolic functions. Assume that the centre of the potential locates at the origin $e_i = 0$ and the trap frequency coefficient of GP model Eq. (2) is a periodically varying control function:

$$
\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \sin t. \tag{26}
$$

Then solutions represented by Eq. (7) are given by

$$
\begin{aligned}
F &= \pm\sqrt{\frac{h_4}{g_0}\sum_{i=1}^{3}p_i^2(t)},\\
q(t) &= -\int_0^t\sum_{i=1}^{3}p_i(t)\,\mu_i(t)\,\mathrm{d}t + q_0,\\
k(t) &= \int_0^t\frac{1}{2}\sum_{i=1}^{3}\left[h_2p_i^2(t) - \mu_i^2(t)\right]\mathrm{d}t + k_0,\\
\gamma(t) &= \frac{\sum_{i=1}^{3}p_i(t)p_i'(t)}{\sum_{i=1}^{3}p_i^2(t)} - \sum_{i=1}^{3}\lambda_i(t),\\
\lambda_i(t) &= \frac{1}{2}\,\frac{\mathscr{M}_{\mathrm{c}}'\left(\frac{1}{2},\frac{1}{4},t\right) + \sigma_i\,\mathscr{M}_{\mathrm{s}}'\left(\frac{1}{2},\frac{1}{4},t\right)}{\sigma_i\,\mathscr{M}_{\mathrm{s}}\left(\frac{1}{2},\frac{1}{4},t\right) + \mathscr{M}_{\mathrm{c}}\left(\frac{1}{2},\frac{1}{4},t\right)},\\
p_i(t) &= \frac{\xi_i}{\sigma_i\,\mathscr{M}_{\mathrm{s}}\left(\frac{1}{2},\frac{1}{4},t\right) + \mathscr{M}_{\mathrm{c}}\left(\frac{1}{2},\frac{1}{4},t\right)},\\
\mu_i(t) &= \frac{\tau_i}{\sigma_i\,\mathscr{M}_{\mathrm{s}}\left(\frac{1}{2},\frac{1}{4},t\right) + \mathscr{M}_{\mathrm{c}}\left(\frac{1}{2},\frac{1}{4},t\right)},
\end{aligned}
\tag{27}
$$

where integration constants $\sigma_i, \xi_i, \tau_i, q_0, k_0$ are determined by initial conditions. The even and odd general Mathieu functions $\mathscr{M}_{\mathrm{c}}$ and $\mathscr{M}_{\mathrm{s}}$ appear frequently in physical problems involving elliptical shapes or periodic potentials (notice: with regard to the definition of Mathieu functions $\mathscr{M}_{\mathrm{c}}$ and $\mathscr{M}_{\mathrm{s}}$, one can refer to the help of *Maple*). These functions were first introduced by Mathieu when he analysed the solutions to the Mathieu differential equation

$$
y'' + (a - 2q\cos(2x))y = 0, \tag{28}
$$

which arises from the separation of the 2D or 3D wave equation modeling the motion of an elliptic membrane.

4. Summary and discussion

In this paper, we investigate the (3 + 1)-dimensional GP equation with time- and space-dependent potential, time-dependent nonlinearity and gain or loss with the generalized sub-equation expansion method to obtain rich exact analytical solutions, which include bright and dark solitons, Jacobi elliptic function solutions and Weierstrass elliptic function solutions. With computer simulation, the main evolution features of the analytical solutions obtained are investigated by computer simulations. Nonlinear dynamics of a soliton pulse is also investigated from the different regimes of soliton management. Our results can be applied to many physical fields, such as Bose–Einstein condensate, nonlinear optics, plasma physics, and open up opportunities for further studies on relative experiments.

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