Dense coding with a two-qubit Heisenberg XYZ chain under the influence of phase decoherence*

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(Received 3 July 2010; revised manuscript received 17 January 2011)

We investigate the joint effects of phase decoherence, Dzyaloshinskii–Moriya (DM) interaction and inhomogeneity of the external magnetic field (b) on dense coding in a two-qubit anisotropic Heisenberg XYZ spin chain. Analytical expressions are obtained for the dense coding capacity. It is found that valid dense coding is always possible with this model when the system is initially prepared in the maximum entangled state. Moreover, optimal dense coding can be implemented for this initial state as long as the mean spin–spin coupling constant J_+ of the XY plane is larger than b and the DM interaction despite the intrinsic decoherence. Non-maximal entangled initial states are found to be undesirable for dense coding with this model.

Keywords: dense coding, intrinsic decoherence, DM interaction

PACS: 03.67.–a, 03.65.Ud, 75.10.Pq **DOI:** 10.1088/1674-1056/20/5/050305

1. Introduction

Quantum dense coding,[1,2] as a significant application of quantum information, is made possible by an entangled quantum state, which is an intriguing nonclassical effect. In the initial dense coding protocol,[1] the sender could transmit two bits of classical information to the receiver by sending a single qubit if they shared an Einstein–Podolsky–Rosen (EPR) state.[3] A great deal of work on dense coding has been done both experimentally[4] and theoretically.[5,6] Generally, in dense coding, the sender performs one of the local unitary transformations $U_i \in U(d)$ on the d-dimensional quantum system to put the initially shared entangled state ρ into state $\rho_i = (U_i \otimes I_d)\rho(U_i^\dagger \otimes I_d)$ with a prior probability p_i $(i = 0, 1, \ldots, i_{\max})$. The sender then sends his quantum system to the receiver. Upon receiving this quantum system, the receiver performs a suitable measurement on ρ_i to extract the information. The optimal amount of information that can be conveyed is known to be bounded by the Holevo quantity,[7] i.e., $\chi = S(\overline{\rho}) - S(\rho)$, where $S(\rho) = -\mathrm{Tr}(\rho \,\mathrm{lb}\rho)$ denotes the von Neuman entropy and $\overline{\rho} = \sum_{i=0}^{i_{\max}} p_i\rho_i$ is the average density matrix of the signal ensemble. Since the Holevo quantity is asymptotically achievable,[8,9] we can use $\chi = S(\overline{\rho}) - S(\rho)$ as the definition of the capacity of the dense coding. Ref. [10] considered optimal dense coding where the shared entangled state is a general mixed one. It also proved that the capacity of the dense coding is maximized when the sender prepares the signal states by mutually orthogonal unitary transformations with prior probabilities.

Entanglement in solid-state systems at finite temperature has attracted much interest in recent years[11,12] since the thermal entanglement was found,[13] which is the simplest and most interesting type of quantum entanglement. As a simple but realistic solid-state system, the Heisenberg model is an ideal candidate for the generation and manipulation of entangled states. Thermal entanglement in the various Heisenberg models has been investigated, especially for the implementation of teleportation and optimal dense coding.[14−19] Some work has also been done by introducing the decoherence effect caused by the unavoidable interaction with the surrounding environment of the system.[20,21] The entanglement and the teleportation in two and three qubit Heisenberg models with intrinsic decoherence have been studied in Refs. [22]–[25]. They found that the average fidelity of teleportation depends mainly on the type of the ini-

*Project supported by the National Natural Science Foundation of China (Grant No. 10064004), the Priority Subjects Program for Theoretical Physics of Xinjiang Normal University (XJNU), China, and the Science and Technology Innovative Foundation for Graduate Students of XJNU, China (Grant No. 20101203).

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tial states of the channel, but it is independent of the anisotropy of the model.

In this paper, we will focus our attention on the interplay between the intrinsic decoherence and system parameters for the implementation of optimal dense coding with a Heisenberg XYZ spin chain in the presence of a nonuniform magnetic field along the z-axis. Our concern on the nonuniformity of the magnetic field stems from the existence of unwanted inhomogeneous Zeeman coupling in reality[26] and also the demand for such a coupling for faster gate operations in some schemes.[27] The present study on the dense coding in the Heisenberg spin chain will shed light on the joint influence of phase decoherence and system parameters on the dense coding quality. More importantly, we will clarify the possibilities of valid dense coding and optimal dense coding separately, with and without DM interaction and magnetic field inhomogeneity.

2. Model and description of intrinsic decoherence

The Hamiltonian of a two-qubit Heisenberg XYZ chain with DM interaction in the presence of inhomogeneous external magnetic field is written as

$$\begin{aligned} H = & \frac{1}{2}(J_x\sigma_1^x\sigma_2^x + J_y\sigma_1^y\sigma_2^y + J_z\sigma_1^z\sigma_2^z \\ & + (B+b)\sigma_1^z + (B-b)\sigma_2^z \\ & + D(\sigma_1^x\sigma_2^y - \sigma_1^y\sigma_2^x)), \end{aligned} \tag{1}$$

where $\sigma_{1,2}^{\alpha}$ ($\alpha = x, y, z$) are the Pauli operators, J_α are the coupling constants for the spin–spin interaction, and D is the DM coupling. The external magnetic fields and DM interaction are assumed to be along the z-direction. The B is the homogeneous part of the magnetic field, while the b describes the inhomogeneity. Without loss of generality, we choose $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ as the standard basis, on which the eigenstates of the above Hamiltonian are

$$\begin{aligned} |\phi_1^{\pm}\rangle &= \sqrt{\frac{D^2+J_+^2}{2(\eta^2 \pm b\eta)}}\left(\frac{\mathrm{i}(b\pm\eta)}{D+\mathrm{i}\,J_+}|10\rangle + |01\rangle\right), \\ |\phi_2^{\pm}\rangle &= \sqrt{\frac{J_-^2}{2(\mu^2 \pm B\mu)}}\left(\frac{(B\pm\mu)}{\mathrm{i}\,J_-}|11\rangle + |00\rangle\right), \end{aligned} \tag{2}$$

and the corresponding eigenvalues are

$$E_1^{\mp} = \mp\eta - \frac{1}{2}J_z, \quad E_2^{\mp} = \mp\mu + \frac{1}{2}J_z, \tag{3}$$

where $\eta = \sqrt{J_+^2 + D^2 + b^2}$, $\mu = \sqrt{J_-^2 + B^2}$, and $J_+ = (J_x + J_y)/2$ is the mean coupling constant of the XY plane, $J_- = (J_x - J_y)/2$ is the anisotropy.

According to the Milburn's equation[25] followed by the assumption that a system does not evolve continuously under unitary transformation for sufficiently short time steps, the master equation for pure phase decoherence is given by

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\,\mathrm{i}[H,\rho] - \frac{1}{2\gamma}[H,[H,\rho(t)]], \tag{4}$$

where γ is the phase (intrinsic) decoherence rate. The formal solution of the above master equation can be expressed as[20−25]

$$\rho(t) = \sum_{k=0}^{\infty}\frac{l^k}{k!}M^k(t)\rho(0)M^{\dagger k}(t), \tag{5}$$

where $\rho(0)$ is the density operator of the initial system and M^k(t) is defined by

$$M^k(t) = H^k\exp(-\,\mathrm{i}\,Ht)\exp\left(-\frac{t}{2\gamma}H^2\right). \tag{6}$$

Then the time evolution of the density operator for the system mentioned above is

$$\begin{aligned} \rho(t) = & \sum_{mn}\exp\left[-\frac{\gamma t}{2}(E_m - E_n)^2 - \mathrm{i}(E_m - E_n)t\right] \\ & \times\langle\phi_m|\rho(0)|\phi_n\rangle|\phi_m\rangle\langle\phi_n|, \end{aligned} \tag{7}$$

where $E_{m,n}$ and $\phi_{m,n}$ are the eigenvalues and the corresponding eigenvectors respectively.

We assume that the system is initially prepared in the Bell state $|\psi(0)\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. From Eq. (7), the time evolution of $\rho(t)$ for this initial state can be obtained as

$$\begin{aligned} \rho(t) = & \tau_+|01\rangle\langle 01| + \tau_-|10\rangle\langle 10| \\ & + \delta_+|01\rangle\langle 10| + \delta_-|10\rangle\langle 01|, \end{aligned} \tag{8}$$

where

$$\tau_{\pm} = \frac{\pm bJ_+ + \eta^2 \mp \sqrt{b^2+D^2}\mathrm{e}^{-2t\gamma\eta^2}\sqrt{D^2+J_+^2}\cos(\varphi)}{2\eta^2},$$

$$\delta_{\pm} = \mathrm{e}^{\pm\,\mathrm{i}\,\theta}\left(\frac{b\sqrt{b^2+D^2}\mathrm{e}^{-2t\gamma\eta^2}\cos(\varphi)}{2\eta^2} + \frac{J_+\sqrt{D^2+J_+^2}}{2\eta^2} \mp \frac{\mathrm{i}\sqrt{b^2+D^2}\mathrm{e}^{-2t\gamma\eta^2}\eta\sin(\varphi)}{2\eta^2}\right),$$

where

$$\theta = \arctan\left(\frac{D}{J_+}\right), \quad \text{and} \quad \varphi = \arctan\left(\frac{D\eta}{bJ_+}\right) + 2t\eta.$$

3. Dense coding with the model

Now we carry out dense coding by using the entangled states of the above spin chain as a channel. The set of the mutually orthogonal unitary transformation[5,16] of the optimal dense coding for the two-qubit is

$$\begin{aligned} &U_{00}|j\rangle = |j\rangle, \quad U_{01}|j\rangle = |j+1(\mathrm{mod}2)\rangle, \\ &U_{10}|j\rangle = \mathrm{e}^{\sqrt{-1}\frac{2\pi}{2}j}|j\rangle, \\ &U_{11}|j\rangle = \mathrm{e}^{\sqrt{-1}\frac{2\pi}{2}j}|j+1(\mathrm{mod}2)\rangle, \end{aligned} \tag{9}$$

where $|j\rangle$ is the single qubit computational basis ($|j\rangle$ $=|0\rangle, |1\rangle$). After performing these unitary transformations, the average state of the signal ensemble becomes

$$\overline{\rho^*(t)} = \frac{1}{4}\sum_{i=0}^{3}(U_i \otimes I)\rho(t)(U_i^\dagger \otimes I), \tag{10}$$

where we have $0 \to 00, 1 \to 01, 2 \to 10, 3 \to 11$ and $\rho(t)$ is the density matrix of the initial signal state. Through straightforward algebra, we have

$$\begin{aligned} \overline{\rho^*}(t) = \frac{1}{2}[&\tau_-|00\rangle\langle 00| + \tau_+|01\rangle\langle 01| \\ &+ \tau_-|10\rangle\langle 10| + \tau_+|11\rangle\langle 11|]. \end{aligned} \tag{11}$$

If the sender performs the set of mutually orthogonal unitary transformations, the maximum dense coding capacity χ can be written as

$$\chi = S(\overline{\rho^*}) - S(\rho). \tag{12}$$

Thus the analytical expression of the maximal dense coding capacity can be obtained as

$$\begin{aligned} \chi = &-\frac{1}{\ln 2}\left(\tau_+ \ln\left[\frac{1}{2}\tau_+\right] + \tau_- \ln\left[\frac{1}{2}\tau_-\right]\right) \\ &-\frac{1}{\ln 4}\left(\Gamma_- \ln\left[-\frac{1}{2}\Gamma_-\right] - \Gamma_+ \ln\left[\frac{1}{2}\Gamma_+\right]\right), \end{aligned} \tag{13}$$

where

$$\Gamma_\pm = (\pm\tau_- \pm \tau_+ + \sqrt{4\delta_-\delta_+ + (\tau_- - \tau_+)^2}).$$

From the expression (13), one can easily find that the optimal dense coding capacity χ for the initial state is invariant for the nonuniform part of the magnetic field B, the spin–spin coupling along z-axis J_z and the anisotropy of the XY plane J_-. For valid dense coding, the parameters should be chosen to satisfy $\chi > 1$, while for optimal dense coding χ must be a maximum, i.e. χ=2, so that the sender can transmit 2 bits of classical information by sending 1 qubit.

As the first observation of our study, we plot the time evolution of the dense coding capacity χ with and without inhomogeneity b for the different DM interactions in Fig. 1. It is found that for the case of $b = 0$ and $D = 0$, with other parameters fixed, the dense coding capacity χ reaches a maximum value of 2 and keeps invariant as time elapses despite the existence of the intrinsic decoherence. The reason is that the channel takes the form of $(|01\rangle + |10\rangle)/\sqrt{2}$ in this case, which is the two-qubit maximally entangled state and independent of time t. For the case of $b = 0$ and $D \neq 0$, the dense coding capacity χ oscillates with time and finally reaches different stable values between 1 and 2 for the different DM interactions as time approaches infinity. Larger DM interactions cause bigger collapse of the dense coding capacity. As long as the inhomogeneity of the magnetic field is introduced, the overall quality of the dense coding significantly decreases.

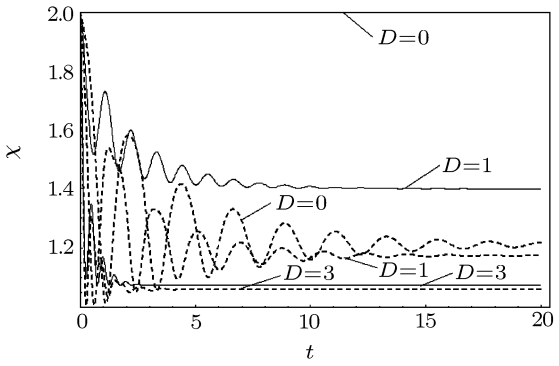


Fig. 1. Dense coding capacity χ values each as a function of time t for different DM interaction D values. The other parameters are set as $\gamma = 0.05$, $J_+ = 1$, $b = 0$ (solid line), and $b = 1$ (dashed line).

As the next step, we give the intuitional description of the influence of the intrinsic decoherence on optimal dense coding in terms of three-dimensional diagram. From Fig. 2, it is easy to find that the χ oscillates periodically with time for the small phase decoherence rates. As the decoherence effect increases, though the oscillatory behaviour of the dense coding capacity suffers a significant suppression, χ approaches a certain stable value between 1 and 2 for a long time, which still allows a valid dense coding.

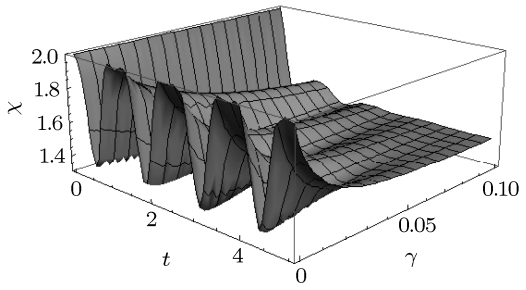


Fig. 2. Dense coding capacity χ as a function of time t and phase decoherence rate, on the assumption that $J_+ = 2$, $D = 1$ and $b = 1$.

Subsequently, we plot the time evolution of the χ with the mean coupling constant J_+ of the XY plane with and without inhomogeneity of the magnetic field in Fig. 3, which shows clearly that the optimal dense coding capacity χ collapses very quickly to a certain value at the very beginning and then oscillates with the time evolution when the J_+ is small, and reaches again different stable values for the different J_+ values for a long time. As the mean coupling constant J_+ increases, χ can be enhanced to and kept at a maximum value of 2 despite the intrinsic decoherence as long as the J_+ is large enough compared with b or D.

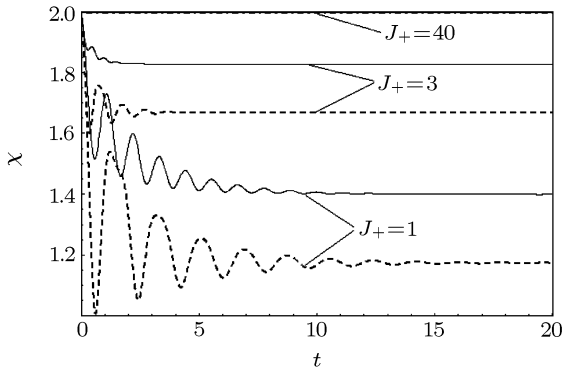


Fig. 3. Dense coding capacity χ values each as a function of time for different values of coupling parameter J_+. The other parameters are set as $\gamma = 0.05$, $D = 1$, $J_+ = 1$, $b = 0$ (solid line), and $b = 1$ (dashed line).

In all the above cases, the dense coding capacity reaches a stable value ($\chi_{\rm s}$) as time approaches infinity, on which we mainly focus our attention in the following. The algebra lies behind the long-term behaviour of the dense coding capacity can be obtained from the expression (13) as

$$\chi_{\rm s} = -\frac{1}{\ln 2}\left(\Lambda_+ \ln\left[\frac{1}{2}\Lambda_+\right] + \Lambda_- \ln\left[\frac{1}{2}\Lambda_-\right]\right) - \frac{1}{\ln 4}(\Omega_- \ln\left[-\frac{1}{2}\Omega_-\right] - \Omega_+ \ln\left[\frac{1}{2}\Omega_+\right]), \quad (14)$$

where

$$\Lambda_\pm = \frac{\pm bJ_+ + \eta^2}{2\eta^2}, \quad \text{and} \quad \Omega_\pm = \pm 1 + \frac{J_+}{\eta}.$$

The above analytical expression allows us to describe the decoherence-irrelevant behaviours of the dense coding capacity versus different parameters. In particular, the joint effects of the constructive and the destructive parameters on dense coding are depicted in Fig. 4(a) (J_+ and b) and Fig. 4(b) (J_+ and D), in which only the positive values of the parameters are considered due to the symmetries of χ for a long time. Again, both figures show that in order to implement optimal dense coding, the mean coupling constant J_+ must be large enough compared with the other parameters as long as the initial state is the Bell state. Other types of initial states, such as the Werner state and unentangled state have also been taken into consideration, and found to be useless even for valid dense coding under the intrinsic decoherence. Furthermore, two surface plots are presented in each of these figures to compare the destructive effects of b and D on dense coding, and the inhomogeneity of the magnetic field is found to be more destructive than the DM interaction.

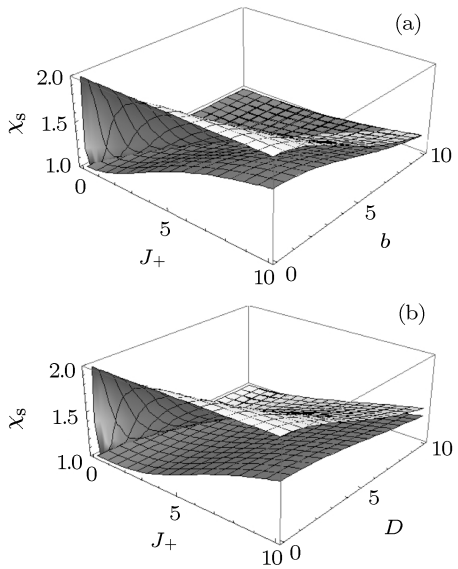


Fig. 4. (a) Stable dense coding capacity $\chi_{\rm s}$ as a function of inhomogeneity b and mean coupling constant J_+, where $D = 0$ (for the upper plot) and $D = 5$ (for the lower plot). (b) $\chi_{\rm s}$ as a function of DM interaction D and mean coupling constant J_+, where $b = 0$ (for the upper plot) and $b = 5$ (for the lower plot).

4. Conclusion

We have investigated the joint effects of intrinsic decoherence, DM interaction and magnetic field inho-

mogeneity on dense coding in a two-qubit Heisenberg XYZ model. Based on the master equation, we have obtained the analytical expression for the dense coding capacity and discussed its behaviour under various parameters for the initially maximum entangled state. A stable valid dense coding capacity, controllable by the system parameters, is always possible for this case. Except the significant influence on the short-period oscillation of the dense coding capacity, the intrinsic decoherence turned out to be irrelevant as far as the long-term behaviour is concerned, which can be described through the relevant analytic expression and plots. Optimal dense coding can always be implemented with this model as long as the DM interaction and the magnetic field inhomogeneity are not introduced. More importantly, the large mean spin–spin coupling constant J_+ compared with b and D also guarantees the optimal dense coding. Moreover, our qualitative results show that the destructive influence of b on the dense coding is greater than that of D. As a final but nontrivial result, non-maximal entangled initial states turned out to be useless for dense coding with this model under the intrinsic decoherence.

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