Conservation issue of pairwise quantum discord and entanglement of two coupled qubits in a two-mode vacuum cavity*

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The conservation issues of pairwise quantum discord and entanglement of two qubits coupled to a two-mode vacuum cavity are investigated by considering the dipole–dipole interaction between two qubits. It is found that the sum of the square of the pairwise quantum discords and the sum of the square of the pairwise concurrences are both conserved in the strong dipole–dipole interaction limit. However, in the middle dipole–dipole and weak dipole–dipole interaction limits, the sum of the square of the pairwise concurrences is still conserved while the sum of the square of the pairwise discords is not. The crucial reason for this is that the quantum discords are not equivalent if the measurements are performed on different subsystems in a general situation. So it is very important for quantum computation depending on the quantum discord to select the target performed by the measurements.

Keywords: quantum discord, quantum entanglement, dipole–dipole (DD) interaction

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1. Introduction

The phenomenon of quantum entanglement, first considered by Einstein–Podolsky–Rosen (EPR) and Schrödinger,[1,2] is a unique phenomenon and the most incredible feature of quantum physics. In the field of quantum information theory, quantum entanglement has been a significant physical resource and played an important role in quantum computation and quantum information research.[3] It can be said that if there were no entanglement, there would be no quantum information. Unfortunately, the entanglement is very frangible to the system–environment interaction.[4] Considering an initial entangled state exposed to local noisy environments, the phenomenon of entanglement sudden death (ESD) often occurs.[5,6]

According to Ref. [7], we know that the two-qubit entanglement is not truly lost, but instead redistributed among the interacting parties. For example, in the double Jaynes–Cummings model (identical JC systems), with the disappearance of the entanglement between two qubits, entanglement between two cavities occurs. This means that the entanglement is transferred from the qubits to the cavities, but the total amount of the entanglement is conserved, as demonstrated by Yönaç *et al.*[8] This conservation rule has been extended to a more general case (non-identical JC systems) by Chan *et al.*[9] They showed that the non-local pairwise entanglement still conserves the sum of the square of the non-local pairwise concurrences (SSPC) for the two-qubit initial entangled state $|\Psi\rangle = \cos\alpha|10\rangle + \sin\alpha|01\rangle$, but this conservation rule is broken for $|\Phi\rangle = \cos\alpha|11\rangle + \sin\alpha|00\rangle$ because the sudden death of the two-qubit entanglement and the sudden birth of the two-cavity entanglement are not simultaneous.

However, entanglement is not the only type of useful quantum correlation for quantum technology. Quantum discord, first introduced by Ollivier and Zurek as a measure of the 'quantumness',[10] is another type of quantum correlation. Generally speaking, quantum discord is not equivalent to entangle-

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ment, especially for mixed states. Even when entanglement is absent, quantum discord is still available. This indicates that the quantum information tasks can be processed without entanglement.[11] In the past few years, many authors[12−31] have focused their attention on the study of quantum discord and its applications in different physical models. This research shows that quantum discord also plays a significant role in quantum computation and quantum information. In contrast with quantum entanglement suffering a sudden death in some cases, quantum discord is more robust against the influence of the environment and free of sudden death.

No detailed investigation concerning the conservation of quantum discord is available at present. In this paper, we use a system consisting of two coupled qubits interacting with a two-mode vacuum cavity to study the conservation rules of entanglement and discord in the following three cases: strong dipole–dipole (DD) interaction, middle DD interaction and weak DD interaction. We find that the SSPC is always a constant and equals the square of the initial entanglement during the time evolution in the above three cases, while the sum of the square of the pairwise discords (SSPD) is not conserved in both the middle and weak DD interaction limits. This difference is due to the fact that quantum discord is a physical quantity that depends on the target performed by the measurements. Generally, the quantum discords are not equivalent if the measurements are performed on different subsystems.

The rest of the present paper is organized as follows. In Section 2, the physical model is introduced and solved. In Section 3, we review the concurrence and quantum discord of two qubits. In Section 4, we present the numerical results of the evolutions of SSPD and SSPC in three DD interaction limits. Finally, we draw our conclusions from the present study in Section 5.

2. Physical model and its exact solution

We consider two two-level atoms resonantly interacting with an ideal cavity supporting two modes with frequencies ω_1 and ω_2. Considering that the distance between the atoms is less than the wave length of the cavity, the DD interaction between the atoms can not be neglected. In the rotation wave approximation, the Hamiltonian for the total system can be written as (with $\hbar = 1$)[32]

$$\begin{aligned} H =& \omega_1(S_Z^{\rm A} + a^\dagger a) + \omega_2(S_Z^{\rm B} + b^\dagger b) \\ &+ \varepsilon(S_+^{\rm A}S_-^{\rm B} + S_-^{\rm A}S_+^{\rm B}) + g_1(aS_+^{\rm A} + a^\dagger S_-^{\rm A}) \\ &+ g_2(bS_+^{\rm B} + b^\dagger S_-^{\rm B}), \end{aligned} \tag{1}$$

where $S_Z^{\rm A}(S_Z^{\rm B})$, $S_+^{\rm A}(S_+^{\rm B})$ and $S_-^{\rm A}(S_-^{\rm B})$ denote the atomic inversion, the raising and the lowering operators, respectively, with $S_Z^{\rm A}(S_Z^{\rm B}) = (1/2)(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$, $S_+^{\rm A}(S_+^{\rm B}) = |\uparrow\rangle\langle\downarrow|$ and $S_-^{\rm A}(S_-^{\rm B}) = |\downarrow\rangle\langle\uparrow|$; $a^\dagger$ (a) and $b^\dagger(b)$ are the creation (annihilation) operators for the modes of the cavities labeled by a and b, respectively; $g_1(g_2)$ is the coupling constant between the atom A(B) and the cavity mode a(b); ε refers to the strength of DD interaction between the atoms. For simplicity, we assume $\omega_1 = \omega_2 = \omega_0$ and $g_1 = g_2$.

We restrict our analysis to the case where both cavity modes are prepared initially in the vacuum state $|0\rangle_{\rm a} \otimes |0\rangle_{\rm b}$ and the initial state of the two atoms is

$$\rho_{\rm AB}(0) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

According to Hamilton (1), the states $|\downarrow\downarrow 00\rangle$, $|\uparrow\downarrow 00\rangle$ and $|\downarrow\uparrow 00\rangle$ will evolve respectively as follows:

$$\begin{aligned} |\downarrow\downarrow 00\rangle &\rightarrow |\downarrow\downarrow 00\rangle, \\ |\uparrow\downarrow 00\rangle &\rightarrow c_1|\uparrow\downarrow 00\rangle + c_2|\downarrow\uparrow 00\rangle + c_3|\downarrow\downarrow 10\rangle \\ &\quad + c_4|\downarrow\downarrow 01\rangle, \\ |\downarrow\uparrow 00\rangle &\rightarrow b_1|\uparrow\downarrow 00\rangle + b_2|\downarrow\uparrow 00\rangle + b_3|\downarrow\downarrow 10\rangle \\ &\quad + b_4|\downarrow\downarrow 01\rangle, \end{aligned}$$

where

$$\begin{aligned} c_1 &= b_2 = \alpha(t) + g^2\gamma(t), \\ c_2 &= b_1 = -{\rm i}\varepsilon\beta(t), \\ c_3 &= b_4 = -{\rm i}g\beta(t) - {\rm i}g^3\eta(t), \\ c_4 &= b_3 = -g\varepsilon\gamma(t), \end{aligned} \tag{3}$$

with

$$\begin{aligned} \alpha(t) &= \frac{a^2\cos(at) - b^2\cos(bt)}{a^2 - b^2}, \\ \beta(t) &= \frac{a\sin(at) - b\sin(bt)}{a^2 - b^2}, \end{aligned}$$

$$\gamma(t) = \frac{\cos(at) - \cos(bt)}{-a^2 + b^2},$$
$$\eta(t) = \begin{cases} \dfrac{b\sin(at) - a\sin(bt)}{-a^3 b + ab^3}, & b \neq 0, \\ \dfrac{at - \sin(at)}{a^3}, & b = 0, \end{cases} \quad (4)$$

and

$$p = \varepsilon^2 + 2g^2,$$
$$q = \varepsilon^4 + 4g^2\varepsilon^2,$$
$$a = \sqrt{\frac{p + \sqrt{q}}{2}},$$
$$b = \sqrt{\frac{p - \sqrt{q}}{2}}. \quad (5)$$

Then, we can straightforwardly obtain the reduced density matrices $\rho_{\rm AB}(t)$, $\rho_{\rm ab}(t)$, $\rho_{\rm Ab}(t)$ and $\rho_{\rm aB}(t)$. According to these density matrixes, we can obtain corresponding discords and concurrences.

3. Quantum discord and concurrence of two qubits

The quantum discord is defined as[10]

$$D(\rho_{XY}) = I(\rho_{XY}) - C(\rho_{XY}), \quad (6)$$

where $I(\rho_{XY})$ is the quantum mutual information and $C(\rho_{XY})$ is the classical correlation and defined as

$$C(\rho_{XY}) = S(\rho_X) - \min_{\{\Pi_k\}} S(\rho_{X|Y}), \quad (7)$$

where $S(\rho) = -{\rm Tr}\rho\,{\rm lb}\,\rho$ is the von Neumann entropy. In Eq. (7), the minimum is taken over the set of projective measurements $\{\Pi_k = |B_k\rangle\langle B_k|, k = 1, 2\}$, where $|B_1\rangle = \cos\theta|0\rangle + {\rm e}^{{\rm i}\phi}\sin\theta|1\rangle$ and $|B_2\rangle = {\rm e}^{-{\rm i}\phi}\sin\theta|0\rangle - \cos\theta|1\rangle$, and $S(\rho_{X|Y}) = \sum_k p_k S(\rho_k)$ is the conditional entropy of X given the knowledge of the state of Y, with $\rho_k = {\rm Tr}_Y(I \otimes \Pi_k)\rho_{XY}(I \otimes \Pi_k)/p_k$ and $p_k = {\rm Tr}_{XY}(I \otimes \Pi_k)\rho_{XY}(I \otimes \Pi_k)$. In this paper, these pairwise discords of the partitions as $D_{\rm AB}$, $D_{\rm ab}$, $D_{\rm Ab}$ and $D_{\rm aB}$ performing on the second subsystem that is in the Y position are obtained.

It is convenient to use concurrence E to quantify the bipartite entanglement. According to Ref. [33], the concurrence is defined as

$$E = \max\left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right), \quad (8)$$

where λ_i $(i = 1, 2, 3, 4)$ are the eigenvalues of the density matrix $\boldsymbol{\rho}(\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y)\boldsymbol{\rho}^*(\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y)$ in decreasing order and $\boldsymbol{\sigma}_y$ is the Pauli matrix in the y direction.

4. Numerical results and discussion

From Fig. 1, we note that both the sum of the square of these pairwise discords of the partitions as $D_{\rm AB}$, $D_{\rm ab}$, $D_{\rm Ab}$ and $D_{\rm aB}$ (SSPD), i.e. SSPD $= D_{\rm AB}^2 + D_{\rm ab}^2 + D_{\rm Ab}^2 + D_{\rm aB}^2$, and the sum of the square of these pairwise concurrences of the partitions as $E_{\rm AB}$, $E_{\rm ab}$, $E_{\rm Ab}$ and $E_{\rm aB}$ (SSPC), i.e. SSPC $= E_{\rm AB}^2 + E_{\rm ab}^2 + E_{\rm Ab}^2 + E_{\rm aB}^2$, are conserved in the strong DD interaction limit. It is not difficult to understand the conservations of SSPD and SSPC. Since the DD interaction is much stronger than the couplings between atoms and cavity modes, it is safe to neglect the interactions between atoms and cavity. From Eqs. (3), (4) and (5), if $\varepsilon \gg g$, then $c_1 = b_2 = 1$ while $c_i = 0, (i = 2, 3, 4)$ and $b_j = 0, (j = 1, 3, 4)$, which means that the initial state just has an additional phase factor induced by the DD interaction. Consequently, SSPD and SSPC are both constant and equal to the square of the initial value, respectively.

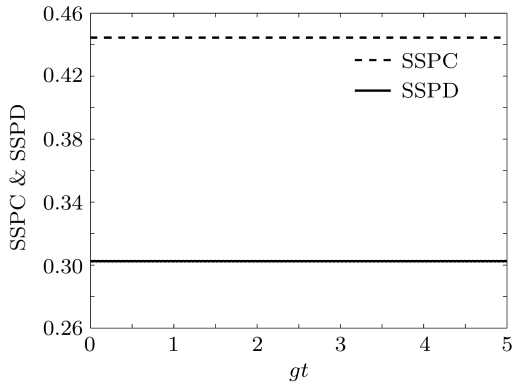


Fig. 1. Evolutions of SSPD (solid line) and SSPC (dashed line) versus dimensionless quantity gt with $\varepsilon = 120g$.

With the DD decreasing, the SSPC is still conserved, while the conservation rule of SSPD breaks down (see Figs. 2 and 3). Especially, in the weak DD interaction limit, the SSPD shows a large amplitude oscillating behaviour as displayed in Fig. 3. The difference is due to the fact that concurrence is independent of measurement, while quantum discord is highly dependent on the part being measured.

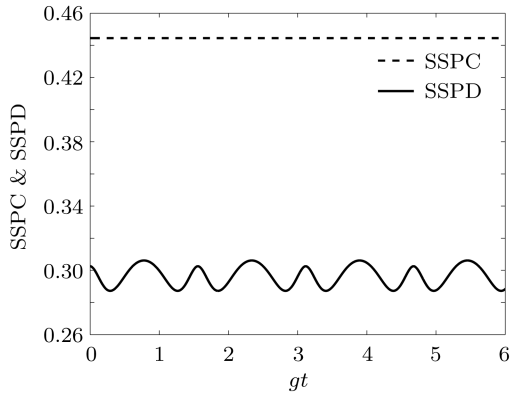


Fig. 2. Evolutions of SSPD (solid line) and SSPC (dashed line) versus dimensionless quantity gt with $\varepsilon = 3.5g$.

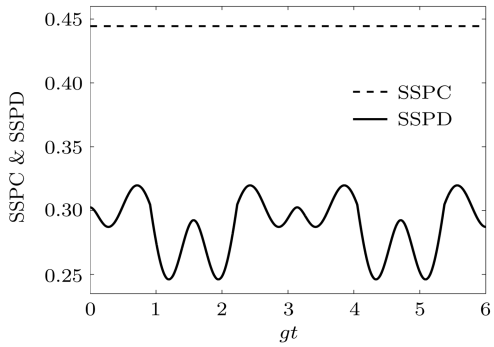


Fig. 3. Evolutions of SSPD (solid line) and SSPC (dashed line) versus dimensionless quantity gt with $\varepsilon = 0.01g$.

Due to the strong atom-mode couplings, Eq. (2) evolves inevitably into a more mixed state, because the focus of study has been on identical and resonant cavities, where $\omega_1 = \omega_2 = \omega_0$ and $g_1 = g_2$. This makes atoms A and B undistinguishable, then the expression of discord for state $\rho_{\rm AB}(t)$ is invariant under permutation of the individual labels A $\leftrightarrow$ B, i.e. $D^{\rm A}_{\rm AB} = D^{\rm B}_{\rm AB}$, where $D^{\rm A}_{\rm AB}$ and $D^{\rm B}_{\rm AB}$ represent the quantum discords measured on mode A and atom B, respectively. Similarly, $D^{\rm a}_{\rm ab} = D^{\rm b}_{\rm ab}$. However, in Fig. 4 it can be seen that the values of quantum discord ($D_{\rm Ab}$) measured on mode A and atom b are different, i.e. $D^{\rm A}_{\rm Ab} \neq D^{\rm b}_{\rm Ab}$. Similarly, $D^{\rm a}_{\rm aB} \neq D^{\rm B}_{\rm aB}$. Thus the partitions $D_{\rm Ab}$ and $D_{\rm aB}$ will have deviations as we can not perform the measurements on the same part. These deviations induce the oscillations of the solid line in Figs. 2 and 3.

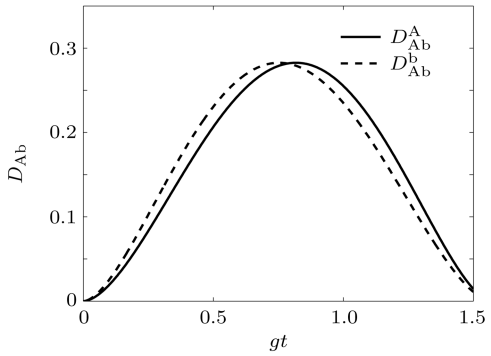


Fig. 4. Evolutions of $D^{\rm A}_{\rm Ab}$ (solid line) and $D^{\rm b}_{\rm Ab}$ (dashed line) versus dimensionless quantity gt with $\varepsilon = 0.01g$.

5. Conclusion

We have investigated the system which consists of two coupled two-level atoms resonantly interacting with an ideal two-mode cavity and studied the evolutions of pairwise entanglement and quantum discord in the following three cases: strong dipole–dipole (DD) interaction, middle DD interaction and weak DD interaction. The result shows that the entanglement is always conserved, while the quantum discord breaks the conservation rule in both middle and weak DD interaction regimes. This difference is due to the fact that quantum discord is a physical quantity depending on the target performed by the measurements. Therefore the quantum discords performing on different subsystems are different. However, Refs. [26], [34] and [35] have involved that the measurements of classical correlations are equal for some cases, i.e. $D^X = D^Y$, irrespective of whether the measurement is performed on subsystem X or Y. In fact, the quantum discord measured on part X (D^X) is not equal to that measured on part Y (D^Y) for a general case. So far, the conditions under which the quantum discords are not equivalent if the measurements are performed on different subsystems are not clear. Because of discord depending on the measuring objects, if we choose the appropriate measurement of the object, we will obtain the best quantum discord.

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