Ground eigenvalue and eigenfunction of a spin-weighted spheroidal wave equation in low frequencies*

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Spin-weighted spheroidal wave functions play an important role in the study of the linear stability of rotating Kerr black holes and are studied by the perturbation method in supersymmetric quantum mechanics. Their analytic ground eigenvalues and eigenfunctions are obtained by means of a series in low frequency. The ground eigenvalue and eigenfunction for small complex frequencies are numerically determined.

Keywords: spin-weighted spheroidal wave equation, perturbation method in supersymmetric quantum mechanics, super-potential, eigenvalue and eigenfunction

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1. Introduction

Angular spin-weighted spheroidal wave functions were introduced by Teukolsky[1] in a paper dealing with the linear perturbation of a rotating Kerr black hole. They determine the linear stability of a Kerr black hole. These functions, which are denoted by $S_n(m,s,c;\theta), n=0,1,2,\ldots$ in this paper, satisfy the angular spin-weighted spheroidal wave equations

$$\begin{aligned}&\left(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}+s-2cs\cos\theta+c^2\cos^2\theta\right.\\&\left.-\frac{(m+s\cos\theta)^2}{\sin^2\theta}\right)S_n(m,s,c;\theta)\\&=-E_n(m,s,c)S_n(m,s,c;\theta),\\&n=0,1,2,\ldots,\end{aligned}\tag{1}$$

where the independent variable θ lies in the interval $(0,\pi)$, m, s and c are constants. In the famous paper,[1] these three constants have their own physical meaning, m is the magnetic normal mode, s the spin of the perturbation field and c the rotating frequency of the Kerr black hole. The boundary conditions for these eigenfunctions $S_n(m,s,c;\theta)$, $n=0,1,2,\ldots$, with eigenvalues $E_n(m,s,c)$ are finite at $\theta=0,\pi$. For simplicity, in the following we assume that $2s$ is either zero or a positive integer. Although the spins could be $0,\pm1/2,\pm1,\pm3/2$ or ±2, it is easy to obtain a relation between the eigenvalues and eigenfunctions for $-s$ and s from the viewpoint of an integral equation.[2,3] The parameter c is a complex number. We can refer to Refs. [4]–[6] for more detailed information about angular spin-weighted spheroidal wave equations.

In this paper, we will give the ground eigenvalue and eigenfunction of Eq. (1) for small absolute values of c. This work is an extension of our preceding work,[7] in which we study the ground eigenvalue and eigenfunction of spheroidal wave equations where the spin $s=0$ in Eq. (1). As in Ref. [7], we use the perturbation method in supersymmetric quantum mechanics (SUSYQM) to present the ground eigenfunction by a product and the ground eigenvalue by a series, which will be useful for their application.

The ground eigenvalue and eigenfunction of Eq. (1) are as follows:

$$\begin{aligned}E_0(m,s,c)=&(m-s)(m+s+1)-\frac{2s^2}{m+1}\\&-\sum_{n=1}^{\infty}c^{n+1}\times\sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)}a_{k,i}(m,s)a_{n+1-k,\ell-i}(m,s)(D_\ell-D_{\ell+2}),\end{aligned}\tag{2}$$

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and

$$S_0(m,s,c;\theta)=\exp\left[-\int^{\theta}W(m,s,c;\phi)\,\mathrm{d}\phi\right]=\sin^m\theta\tan^s\frac{\theta}{2}\prod_{n=1}^{\infty}\exp\left[\sum_{\ell=0}^{n-1}a_{n,\ell}(m,s)\frac{\cos^{\ell+1}\theta}{\ell+1}c^n\right], \quad (3)$$

where the coefficients $a_{n,\ell}(m,s)$ and D_ℓ satisfy simple recurrence relations. However, we should emphasize that the finiteness of the ground eigenfunction at $\theta=0,\pi$ induces a condition on the parameters m and s, which is $m\geq s$ for non-negative integers m and $2s$. To present our main results clearly, we give the ground state eigenvalue and eigenfunction without detailed proof in Section 2. The detailed proofs are shown in the Appendix.

2. Ground state eigenvalue and eigenfunction

In order to use the SUSYQM approach, we should transform Eq. (1) into a standard Schrödinger equation.[8−10] Applying the transformation[10]

$$S_n(m,s,c;\theta)=\frac{\psi_n(m,s,c;\theta)}{\sin^{1/2}\theta},\quad 0<\theta<\pi,\ n=0,1,\ldots \quad (4)$$

to Eq. (1) yields

$$-\frac{\mathrm{d}^2\psi_n(m,s,c;\theta)}{\mathrm{d}\theta^2}+V(m,s,c;\theta)\psi_n(m,s,c;\theta)=E_n(m,s,c)\psi_n(m,s,c;\theta),\ n=0,1,2,\ldots, \quad (5)$$

where $V(m,s,c;\theta)$ is

$$V(m,s,c;\theta)=-\frac{1}{4}\left(1+\frac{1}{\sin^2\theta}\right)-s+\frac{(m+s\cos\theta)^2}{\sin^2\theta}+2sc\cos\theta-c^2\cos^2\theta. \quad (6)$$

The boundary condition of $S_n(m,s,c;\theta)$ transforms into the term

$$\lim_{\theta\to0,\pi}\frac{\psi_n(m,s,c;\theta)}{\sin^{1/2}\theta},\ n=0,1,\ldots \quad (7)$$

exists and is finite. When c is small, the ground state eigenvalue $E_0(m,s,c)$ can be expanded into

$$E_0(m,s,c)=\sum_{n=0}^{\infty}E_{0n}(m,s)c^n, \quad (8)$$

and the superpotential $W(m,s,c;\theta)$ into

$$W(m,s,c;\theta)=\sum_{n=0}^{\infty}W_{0n}(m,s;\theta)c^n, \quad (9)$$

which satisfies

$$W^2(m,s,c;\theta)-W'(m,s,c;\theta)=V(m,s,c;\theta)-E_0(m,s,c). \quad (10)$$

In Eq. (10) and in what follows, we use the prime symbol to a function to represent its derivative with respect to θ. Hence Eq. (5) can be rewritten as follows:

$$\begin{aligned}&-\mathrm{d}^2\psi_n(m,s,c;\theta)/\mathrm{d}\theta^2\\&+[V(m,s,c;\theta)-E_0(m,s,c)]\psi_n(m,s,c;\theta)\\&=[E_n(m,s,c)-E_0(m,s,c)]\psi_n(m,s,c;\theta),\\&n=0,1,\ldots.\end{aligned} \quad (11)$$

According to the theory of supersymmetric quantum mechanics, the ground state eigenfunction is[8]

$$\psi_0(m,s,c;\theta)=\exp\left[-\int^{\theta}W(m,s,c;\phi)\,\mathrm{d}\phi\right], \quad (12)$$

correspondingly, boundary condition (7) now turns into the term

$$\lim_{\theta\to0,\pi}\frac{\exp\left[-\int^{\theta}W_{00}(m,s;\phi)\,\mathrm{d}\phi\right]}{\sin^{1/2}\theta} \quad (13)$$

that exists and is finite, and for any $n\geq1$,

$$\lim_{\theta\to0,\pi}\exp\left[-c^n\int^{\theta}W_{0n}(m,s;\phi)\,\mathrm{d}\phi\right] \quad (14)$$

also exists and is finite.

Substituting Eqs. (6), (8), and (9) into Eq. (10) and equating the coefficients of c^n on both sides of Eq. (10), it is not difficult to obtain

$$\begin{aligned}&W'_{00}(m,s;\theta)-W^2_{00}(m,s;\theta)\\&=E_{00}(m,s)+\frac{1}{4}\left(1+\frac{1}{\sin^2\theta}\right)\\&\quad+s-\frac{(m+s\cos\theta)^2}{\sin^2\theta},\end{aligned} \quad (15)$$

$$\begin{aligned}&W'_{01}(m,s;\theta)-2W_{00}(m,s;\theta)W_{01}(m,s;\theta)\\&=E_{01}(m,s)-2s\cos\theta,\end{aligned} \quad (16)$$

$$\begin{aligned}&W'_{02}(m,s;\theta)-2W_{00}(m,s;\theta)W_{02}(m,s;\theta)\\&=E_{02}(m,s)+\cos^2\theta+W^2_{01}(m,s;\theta),\end{aligned} \quad (17)$$

for any $n\geq3$,

$$\begin{aligned}&W'_{0n}(m,s;\theta)-2W_{00}(m,s;\theta)W_{0n}(m,s;\theta)\\&=E_{0n}(m,s)+\sum_{k=1}^{n-1}W_{0k}(m,s;\theta)W_{0,n-k}(m,s;\theta).\end{aligned} \quad (18)$$

Obviously, the values of $E_{0n}(m,s)$ and $W_{0n}(m,s;\theta)$, $n=0,1,\ldots$, can be determined by the above equations (15)–(18) together with the boundary conditions (13) and (14). The evaluations of these $E_{0n}(m,s)$ and $W_{0n}(m,s;\theta)$ are presented, but their detailed proofs are shown in the Appendix. To begin with, the values of $E_{00}(m,s)$ and $W_{00}(m,s;\theta)$ are

$$E_{00}(m,s)=(m-s)(m+s+1) \tag{19}$$

and

$$W_{00}(m,s;\theta)=-\frac{s}{\sin\theta}-(m+1/2)\cot\theta, \tag{20}$$

respectively. Substituting the last equation into Eq. (13), one obtains

$$\exp\left[-\int^{\theta}W_{00}(m,s;\phi)\,\mathrm{d}\phi\right] =\tan^{s}\frac{\theta}{2}\sin^{m+1/2}\theta, \tag{21}$$

with

$$\exp\left[-\int^{\theta}W_{00}(m,s;\phi)\,\mathrm{d}\phi\right] \to\begin{cases}\theta^{m+s+1/2}, & \text{if } \theta\to 0;\\ (\pi-\theta)^{m-s+1/2}, & \text{if } \theta\to\pi\ .\end{cases} \tag{22}$$

Therefore, parameters m and s are needed by boundary condition (13) to satisfy the following condition:

$$m\geq s, \tag{23}$$

since we have assumed that m and $2s$ are zero or positive integers. Now substituting Eq. (20) into Eqs. (16), (17) and (18), then it follows from the boundary condition (14) that

$$E_{01}(m,s)=-\frac{2s^2}{m+1}, \tag{24}$$

$$W_{01}(m,s;\theta)=-\frac{s}{m+1}\sin\theta, \tag{25}$$

$$E_{0,n+1}(m,s)=-\sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)}a_{k,i}(m,s)a_{n+1-k,\ell-i}(m,s)(D_{\ell}-D_{\ell+2}), \tag{26}$$

and

$$W_{0,n+1}(m,s;\theta)=\sin\theta\sum_{\ell=0}^{n}a_{n+1,\ell}(m,s)\cos^{\ell}\theta \tag{27}$$

for any $n\geq 1$, where the coefficients $a_{n,\ell}(m,s)$ in the above equations satisfy the following simple recurrence relations for $n\geq 1$:

$$\begin{aligned}a_{n+1,0}(m,s)=&\sum_{k=1}^{n}\sum_{\ell=1}^{n-1}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)}a_{k,i}(m,s)a_{n+1-k,\ell-i}(m,s)d_{\ell,0}\\ &-\sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)}a_{k,i}(m,s)a_{n+1-k,\ell-i}(m,s)d_{\ell+2,0},\end{aligned} \tag{28}$$

$$\begin{aligned}a_{n+1,p}(m,s)=&\sum_{k=1}^{n}\sum_{\ell=p+1}^{n-1}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)}a_{k,i}(m,s)a_{n+1-k,\ell-i}(m,s)d_{\ell,p}\\ &-\sum_{k=1}^{n}\sum_{\ell=p-1}^{n-1}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)}a_{k,i}(m,s)a_{n+1-k,\ell-i}(m,s)d_{\ell+2,p},\end{aligned} \tag{29}$$

for $1\leq p\leq n-1$, and

$$\begin{aligned}a_{n+1,n}(m,s)=&-\sum_{k=1}^{n}a_{k,k-1}(m,s)a_{n+1-k,n-k}(m,s)d_{n+1,n}\\ =&-\frac{1}{2m+n+2}\sum_{k=1}^{n}a_{k,k-1}(m,s)a_{n+1-k,n-k}(m,s)\end{aligned} \tag{30}$$

with the initial condition

$$a_{1,0}(m,s) = -\frac{s}{m+1}. \tag{31}$$

In all the above equations concerning $a_{n+1,k}, k = 0,1,\ldots,n$, quantities $d_{l,i}$ appear, which satisfy the following recurrence relations:

$$d_{\ell,k} = \frac{1}{2m+\ell+1}\Big[-2sd_{\ell-1,k} + (\ell-1)d_{\ell-2,k}\Big], \quad 0 \le k \le \ell-3, \ell = 3,4,\ldots, \tag{32}$$

$$d_{\ell,\ell-2} = -\frac{2s}{(2m+\ell)(2m+\ell+1)}, \tag{33}$$

$$d_{\ell,\ell-1} = \frac{1}{2m+\ell+1}, \tag{34}$$

with the initial condition

$$d_{1,0} = \frac{1}{2m+2},\ d_{2,0} = -\frac{s}{(m+1)(2m+3)},\quad d_{2,1} = \frac{1}{2m+3}. \tag{35}$$

There are also some coefficients D_ℓ to determine the energy $E_{0,n+1}$ in Eq. (26), they satisfy the following recurrence relations:

$$D_\ell = -\frac{2s}{2m+\ell+1}D_{\ell-1} + \frac{\ell-1}{2m+\ell+1}D_{\ell-2}, \quad \ell = 2,3,\ldots, \tag{36}$$

with the initial condition

$$D_0 = 1,\ D_1 = -\frac{s}{m+1}. \tag{37}$$

Finally, substituting all the above evaluations (19)–(27) of $E_{0n}(m,s)$ and $W_{0n}(m,s;\theta)$ into Eqs. (4), (8), (9) and (12), the ground state eigenvalue and eigenfunction are obtained directly as follows:

$$\begin{aligned} E_0(m,s,c) =& (m-s)(m+s+1) - \frac{2s^2}{m+1} \\ &- \sum_{n=1}^{\infty} c^{n+1} \sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\sum_{i=0,\ell-i\le n-k}^{\min(\ell,k-1)} a_{k,i}(m,s) \\ &\times a_{n+1-k,\ell-i}(m,s)(D_\ell - D_{\ell+2}), \end{aligned} \tag{38}$$

and

$$\begin{aligned} S_0(m,s,c;\theta) =& \exp\left[-\int^{\theta} W(m,s,c;\phi)\,\mathrm{d}\phi\right] \\ =& \sin^m\theta\tan^s\frac{\theta}{2} \\ &\times \prod_{n=1}^{\infty}\exp\left[\sum_{\ell=0}^{n-1} a_{n,\ell}(m,s)\frac{\cos^{\ell+1}\theta}{\ell+1}c^n\right], \end{aligned} \tag{39}$$

which we have presented in Section 1.

The above expression of the ground state eigenvalue and eigenfunction are in good forms. In particular, we can evaluate them up to any order of c with the help of computer since the coefficients $a_{n,\ell}(m,s)$, $d_{\ell,k}$ and D_ℓ, satisfy the simple recurrence relations from expressions (28)–(37). Therefore, we also give a numerical determination of the ground eigenvalue and eigenfunction for small complex frequencies.

As a generalization of our preceding work,[7] our present ground eigenvalue and eigenfunction of the angular spin-weighted spheroidal wave equations should be reduced into the results in Ref. [7] when the spin is zero. Indeed, we will show in the rest of this section that this is true. It is easy to see from Eqs. (19)–(25) that

$$E_{00}(m,s=0) = m(m+1), \tag{40}$$

$$W_{00}(m,s=0;\theta) = -(m+1/2)\cot\theta, \tag{41}$$

$$E_{01}(m,s=0) = 0, \tag{42}$$

and

$$W_{01}(m,s=0;\theta) = 0. \tag{43}$$

As a matter of fact, we will show in Appendix A that

$$E_{0,2n+1}(m,s=0) = 0,\ n = 0,1,\ldots, \tag{44}$$

and

$$W_{0,2n+1}(m,s=0;\theta) = 0,\ n = 0,1,\ldots, \tag{45}$$

when the spin is zero.

Furthermore, we can obtain

$$a_{2n+1,p}(m,0) = 0,\ n = 0,1,2,\ldots; 0 \le p \le 2n, \tag{46}$$

$$a_{2n,2p}(m,0) = 0,\ n = 1,2,\ldots; 0 \le p \le n-1, \tag{47}$$

and for $1 \le p \le n-1, n = 1,2,\ldots$, the following relations:

$$\begin{aligned} a_{2n+2,2p+1}(m,0) =& \sum_{k=1}^{n}\sum_{\ell=p+1}^{n}\sum_{i=0,\ell-i\le n+1-k}^{\min(\ell-1,k-1)} a_{2k,2i+1}(m,0)a_{2(n+1-k),2\ell-2i-1}(m,0)d_{2\ell,2p+1} \\ &- \sum_{k=1}^{n}\sum_{\ell=p}^{n}\sum_{i=0,\ell-i\le n+1-k}^{\min(\ell-1,k-1)} a_{2k,2i+1}(m,0)a_{2(n+1-k),2\ell-2i-1}(m,0)d_{2\ell+2,2p+1}, \end{aligned}$$

$$= \frac{\left(\frac{3}{2}+m\right)_p}{\left(\frac{3}{2}\right)_p} \sum_{k=1}^{n} \sum_{\ell=p+1}^{n} \frac{\left(\frac{1}{2}\right)_l}{\left(\frac{3}{2}+m\right)_l} \sum_{i=0,\ell-i\le n+1-k}^{\min(\ell-1,k-1)} a_{2k,2i+1}(m,0)a_{2(n+1-k),2\ell-2i-1}(m,0)$$
$$- \frac{\left(\frac{3}{2}+m\right)_p}{\left(\frac{3}{2}\right)_p} \sum_{k=1}^{n} \sum_{\ell=p}^{n} \frac{\left(\frac{1}{2}\right)_{l+1}}{\left(\frac{3}{2}+m\right)_{l+1}} \sum_{i=0,\ell-i\le n+1-k}^{\min(\ell-1,k-1)} a_{2k,2i+1}(m,0)a_{2(n+1-k),2\ell-2i-1}(m,0), \quad (48)$$

and

$$a_{2n+2,2n+1}(m,0) = -\sum_{k=1}^{n} a_{2k,2k-1}(m,0)a_{2(n+1-k),2n+1-2k}(m,0)d_{2n+2,2n+1}$$
$$= -\frac{1}{2m+2n+3} \sum_{k=1}^{n} a_{2k,2k-1}(m,0)a_{2(n+1-k),2n+1-2k}(m,0), \quad (49)$$

with initial condition

$$a_{2,1}(m,0) = \frac{1}{2m+3}. \quad (50)$$

It is clear that if we define

$$a_{n,k}(m) = (2m+3)^n a_{2n,2k+1}(m,0), \quad (51)$$

and

$$W_{0n}(m;\theta) = W_{0,2n}(m,s=0;\theta) = \sin\theta \sum_{k=0}^{n-1} a_{2n,2k+1}(m,0)\cos^{2k+1}\theta$$
$$= \frac{1}{(2m+3)^n} \sin\theta\cos\theta \sum_{k=0}^{n-1} a_{n,k}(m)\cos^{2k}\theta,\ n=1,2,\ldots, \quad (52)$$

then we obtain from the definition (9) of $W(m,s,c;\theta)$ that

$$W(m,s=0,c;\theta) = \sum_{n=0}^{\infty} W_{0n}(m,s=0;\theta)c^n = \sum_{n=0}^{\infty} W_{0,2n}(m,s=0;\theta)c^{2n}$$
$$= -\left(m+\frac{1}{2}\right)\cot\theta + \sin\theta \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} a_{2n,2k+1}(m,0)\cos^{2k+1}\theta$$
$$= -\left(m+\frac{1}{2}\right)\cot\theta + \sin\theta\cos\theta \sum_{n=1}^{\infty} \frac{1}{(2m+3)^n} \sum_{k=0}^{n-1} a_{n,k}(m)\cos^{2k}\theta, \quad (53)$$

where these $a_{n,k}(m)$ satisfy some recurrence relations from Eqs. (48)–(51) as follows:

$$a_{n+1,p}(m) = \frac{\left(\frac{3}{2}+m\right)_p}{\left(\frac{3}{2}\right)_p} \sum_{k=1}^{n} \sum_{\ell=p+1}^{n} \frac{\left(\frac{1}{2}\right)_l}{\left(\frac{3}{2}+m\right)_l} \sum_{i=0,\ell-i\le n+1-k}^{\min(\ell-1,k-1)} a_{k,i}(m)a_{n+1-k,\ell-i-1}(m)$$
$$- \frac{\left(\frac{3}{2}+m\right)_p}{\left(\frac{3}{2}\right)_p} \sum_{k=1}^{n} \sum_{\ell=p}^{n} \frac{\left(\frac{1}{2}\right)_{l+1}}{\left(\frac{3}{2}+m\right)_{l+1}} \sum_{i=0,\ell-i\le n+1-k}^{\min(\ell-1,k-1)} a_{k,i}(m)a_{n+1-k,\ell-i-1}(m), \quad (54)$$

and

$$a_{n+1,n}(m) = -\frac{1}{2m+2n+3} \sum_{k=1}^{n} a_{k,k-1}(m)a_{n+1-k,n-k}(m), \quad (55)$$

with initial condition

$$a_{1,0}(m) = \frac{1}{2m+3}. \quad (56)$$

Hence we obtain the super-potential $W(m, s=0, c; \theta)$ as expression (53) and the recurrence relations (105), (55) and (56) in the case of $s=0$, and they are the same as those in Ref. [7], as it should be. When $s=0$, it follows from Eq. (39) that the ground eigenfunction is

$$\begin{aligned} S_0(m, s=0, c; \theta) &= \exp\left[-\int^{\theta} W(m, s=0, c; \phi)\,\mathrm{d}\phi\right] = \sin^m \frac{\theta}{2} \prod_{n=1}^{\infty} \exp\left[\sum_{\ell=0}^{n-1} a_{n,\ell}(m,0) \frac{\cos^{\ell+1}\theta}{\ell+1} c^n\right] \\ &= \sin^m \frac{\theta}{2} \prod_{n=1}^{\infty} \exp\left[\sum_{\ell=0}^{n-1} a_{2n,2\ell+1}(m,0) \frac{\cos^{2\ell+2}\theta}{2\ell+2} c^{2n}\right] \\ &= \sin^m \frac{\theta}{2} \prod_{n=1}^{\infty} \exp\left[\frac{1}{(2m+3)^n} \sum_{\ell=0}^{n-1} a_{n,\ell}(m) \frac{\cos^{2\ell+2}\theta}{2\ell+2} c^{2n}\right], \end{aligned} \tag{57}$$

and from Eq. (38) that the ground eigenvalue is

$$\begin{aligned} E_0(m, s=0, c) &= m(m+1) - \sum_{n=1}^{\infty} c^{n+1} \sum_{k=1}^{n} \sum_{\ell=0}^{n-1} \sum_{i=0, \ell-i\le n-k}^{\min(\ell, k-1)} a_{k,i}(m,0) a_{n+1-k,\ell-i}(m,0)(D_\ell - D_{\ell+2}) \\ &= m(m+1) - \sum_{n=1}^{\infty} c^{2n} \sum_{k=1}^{n-1} \sum_{\ell=0}^{n-1} \sum_{i=0, \ell-i\le n-k}^{\min(\ell, k-1)} a_{2k,2i+1}(m,0) a_{2n-2k,2\ell-2i-1}(m,0)(D_{2\ell} - D_{2\ell+2}) \\ &= m(m+1) - \sum_{n=1}^{\infty} a_{2n,1}(m,0) c^{2n} = m(m+1) - \sum_{n=1}^{\infty} \frac{a_{n,0}(m)}{(2m+3)^n} c^{2n}, \end{aligned} \tag{58}$$

since

$$D_{2\ell} = \frac{(2l-1)!!(2m+1)!!}{(2m+2\ell+1)!!}, \ell = 1, 2, 3, \ldots, \tag{59}$$

with the initial condition

$$D_0 = 1, \tag{60}$$

and

$$D_{2\ell+1} = 0,\ \ell = 0, 1, 2, \ldots, \tag{61}$$

which were proved in detail in Appendix.

Therefore, we obtain the same results in case of $s=0$ as those in Ref. [7], as it should be.

Appendix A: Proof of the formulae for the eigenvalue and superpotential

In this appendix, the detailed evaluation of $E_{0n}(m, s)$ and $W_{0n}(m, s; \theta)$, $n = 0, 1, 2, \ldots$ are given. They satisfy Eqs. (15)–(18), and their determination depends also on boundary conditions (13)–(14). We have given $E_{00}(m, s)$ and $W_{00}(m, s; \theta)$ in the last section. Here we will calculate the other terms. In order to simplify the calculation, it is useful to introduce functions $f_n(m, s; \theta), n = 1, 2, \ldots$, as follows:

$$f_1(m, s; \theta) \equiv E_{01}(m, s) - 2s\cos\theta, \tag{A1}$$

$$f_2(m, s; \theta) \equiv E_{02}(m, s) + \cos^2\theta + W_{01}^2(m, s; \theta), \tag{A2}$$

and for $n \ge 3$,

$$\begin{aligned} f_n(m, s; \theta) \equiv & E_{0n}(m, s) + \sum_{k=1}^{n-1} W_{0k}(m, s; \theta) \\ & \times W_{0,n-k}(m, s; \theta). \end{aligned} \tag{A3}$$

Hence Eqs. (16)–(18) can be concisely written as

$$\begin{aligned} & W'_{0n}(m, s; \theta) + [2s\csc\theta + (2m+1)\cot\theta] W_{0n}(m, s; \theta) \\ & = f_n(m, s; \theta),\ \ n = 1, 2, \ldots. \end{aligned} \tag{A4}$$

So it is easy to give the solutions $W_{0n}(m, s; \theta)$ of the above equations as

$$\begin{aligned} W_{0n}(m, s; \theta) &= \exp\left[2\int^{\theta} W_{00}(m, s; \phi)\,\mathrm{d}\phi\right] \int^{\theta} f_n(m, s; \varphi) \exp\left[-2\int^{\varphi} W_{00}(m, s; \phi)\,\mathrm{d}\phi\right] \mathrm{d}\varphi \\ &= \frac{1}{\tan^{2s}\dfrac{\theta}{2} \sin^{2m+1}\theta} \int^{\theta} f_n(m, s; \varphi) \tan^{2s}\frac{\varphi}{2} \sin^{2m+1}\varphi\,\mathrm{d}\varphi, \quad n = 1, 2, \ldots. \end{aligned} \tag{A5}$$

Before evaluating $W_{0n}(m,s;\theta)$, some helpful formulas are given for latter use. Their derivations are easy to obtain from the theory of special function.[11,12] If we define functions

$$A_\ell(m,s;\theta) \equiv \int^\theta \cos^\ell\phi \tan^{2s}\frac{\phi}{2}\sin^{2m+1}\phi\,\mathrm{d}\phi, \quad \ell = 0,1,2,\ldots, \tag{A6}$$

then they satisfy the following recurrence relations:

$$\begin{aligned} A_\ell(m,s;\theta) =& -\frac{2s}{2m+\ell+1}A_{\ell-1}(m,s;\theta) \\ &+\frac{\ell-1}{2m+\ell+1}A_{\ell-2}(m,s;\theta) \\ &+\frac{\cos^{\ell-1}\theta}{2m+\ell+1}\tan^{2s}\frac{\theta}{2}\sin^{2m+2}\theta, \\ & \ell = 2,3,\ldots, \end{aligned} \tag{A7}$$

$$\begin{aligned} A_1(m,s;\theta) =& -\frac{2s}{2m+2}A_0(m,s;\theta) \\ &+\frac{1}{2m+2}\tan^{2s}\frac{\theta}{2}\sin^{2m+2}\theta, \end{aligned} \tag{A8}$$

and

$$\begin{aligned} A_0(m,s;\theta) &\equiv \int^\theta \tan^{2s}\frac{\phi}{2}\sin^{2m+1}\phi\,\mathrm{d}\phi \\ &= \int^\theta (1-\cos\phi)^{2s}\sin^{2(m-s)+1}\phi\,\mathrm{d}\phi \\ &= -\sum_{\ell=0}^{2s}\frac{(-2s)_\ell}{\ell!}\sum_{k=0}^{\infty}\frac{(s-m)_k\cos^{2k+\ell+1}\theta}{k!(2k+\ell+1)}, \end{aligned} \tag{A9}$$

where $(a)_n = a\times(a+1)\times\ldots\times(a+n-1)$ for any constant a. It follows from Eqs. (A1)–(A8) that all $W_{0n}(m,s;\theta)$, $n>0$ will contain the term

$$\frac{A_0(m,s;\phi)}{\tan^{2s}\frac{\phi}{2}\sin^{2m+1}\phi}.$$

However, these terms in W_{0n} might make the ground eigenfunction $S_0(m,c;\theta) = \exp\left[-\int^\theta W(m,\phi)d\phi\right]$ blow up at the boundaries $\theta = 0,\ \pi$. For ease of understanding, we now calculate the integral

$$\begin{aligned} &\int^\theta \frac{A_0(m,s;\phi)}{\tan^{2s}\frac{\phi}{2}\sin^{2m+1}\phi}\,\mathrm{d}\phi \\ &= -\sum_{\ell=0}^{2s}\sum_{k=0}^{\infty}\frac{(-2s)_\ell(s-m)_k}{\ell!k!}\frac{1}{(2k+\ell+1)}\times\frac{1}{2^{2m}} \\ &\times\left[\sum_{n=0,n\neq m+s}^{2m}\frac{1}{2(n-m-s)}\binom{2m-2k-\ell-1}{n}{}_2F_1(-n,-2k-\ell-1;2m-2k-\ell-n;-1)\tan^{2(n-m-s)}\left(\frac{\theta}{2}\right)\right. \\ &\left.+\binom{2m-2k-\ell-1}{m+s}{}_2F_1(-n,-2k-\ell-1;m-2k-\ell-s;-1)\ln\tan\left(\frac{\theta}{2}\right)\right], \end{aligned} \tag{A10}$$

where ${}_2F_1(a,b;c;x)$ is hypergeometric function.[11,12]

When $a>0$ and k is positive integer, one obtains

$$\exp\left[a\tan^k\left(\frac{\theta}{2}\right)\right] \to \begin{cases} \infty, & \text{if } \theta\to 0; \\ 1, & \text{if } \theta\to\pi\ , \end{cases} \tag{A11}$$

and

$$\exp\left[a\tan^{-k}\left(\frac{\theta}{2}\right)\right] \to \begin{cases} 1, & \text{if } \theta\to 0; \\ \infty, & \text{if } \theta\to\pi\ . \end{cases} \tag{A12}$$

Thus to let

$$\lim_{\theta\to 0,\pi}\exp\left[a\int^\theta\frac{A_0(m,s;\phi)}{\tan^{2s}\frac{\phi}{2}\sin^{2m+1}\phi}\,\mathrm{d}\phi\right] \tag{A13}$$

exist for any complex a, from Eq. (A10) there must be $a\equiv 0$. This is an important property which we will use repeatedly in the following. Now assume that

$$\begin{aligned} A_\ell(m,s;\theta) =& D_\ell A_0(m,s;\theta) \\ &+\left(\sum_{k=0}^{\ell-1}d_{\ell,k}\cos^k\theta\right)\tan^{2s}\frac{\theta}{2}\sin^{2m+1}\theta, \\ & \ell = 1,2,\ldots, \end{aligned} \tag{A14}$$

from Eqs. (A7) and (A8), one obtains

$$D_\ell = -\frac{2s}{2m+\ell+1}D_{\ell-1}+\frac{\ell-1}{2m+\ell+1}D_{\ell-2}; \quad \ell = 2,3,\ldots, \tag{A15}$$

with the initial conditions

$$D_0 = 1, \quad D_1 = -\frac{s}{m+1}; \tag{A16}$$

and

$$d_{\ell,k} = \frac{1}{2m+\ell+1}\Big[-2sd_{\ell-1,k}+(\ell-1)d_{\ell-2,k}\Big], \quad 0\le k\le \ell-3, \ell =, 4,\ldots, \tag{A17}$$

$$d_{\ell,\ell-2} = -\frac{2s}{(2m+\ell)(2m+\ell+1)}, \tag{A18}$$

$$d_{\ell,\ell-1} = \frac{1}{2m+\ell+1}, \tag{A19}$$

with the initial conditions

$$d_{1,0} = \frac{1}{2m+2}, \quad d_{2,0} = -\frac{s}{(m+1)(2m+3)}, \quad d_{2,1} = \frac{1}{2m+3}. \tag{A20}$$

Hence

$$\begin{aligned} W_{01}(m,s;\theta) &= \frac{1}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta}\int^{\theta}\Big[E_{01}(m,s)-2s\cos\varphi\Big]\tan^{2s}\frac{\varphi}{2}\sin^{2m+1}\varphi\,\mathrm{d}\varphi \\ &= \frac{1}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta}\Big[E_{01}(m,s)A_0(m,s;\theta)-2sA_1(m,s;\theta)\Big] \\ &= \frac{1}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta}\left[E_{01}(m,s)+\frac{2s^2}{m+1}\right]A_0(m,s;\theta)-\frac{s}{m+1}\sin\theta. \end{aligned} \tag{A21}$$

In order to satisfy boundary conditions (14), the coefficient of $A_0(m,s;\theta)$ in Eq. (A21) must be zero, which is

$$E_{01}(m,s) = -\frac{2s^2}{m+1}, \tag{A22}$$

so

$$W_{01}(m,s;\theta) = -\frac{s}{m+1}\sin\theta. \tag{A23}$$

Similarly, we obtain

$$\begin{aligned} W_{02}(m,s;\theta) &= \frac{1}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta}\int^{\theta}\left[E_{02}(m,s)+\cos^2\varphi+W_{01}^2(m,s;\varphi)\right]\tan^{2s}\frac{\varphi}{2}\sin^{2m+1}\varphi\,\mathrm{d}\varphi \\ &= \frac{1}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta}\int^{\theta}\left[E_{02}(m,s)+\cos^2\varphi+\frac{s^2}{(m+1)^2}\sin^2\varphi\right]\tan^{2s}\frac{\varphi}{2}\sin^{2m+1}\varphi\,\mathrm{d}\varphi \\ &= \left[E_{02}(m,s)+\frac{s^2}{(m+1)^2}\right]\frac{A_0(m,s;\theta)}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta}+\left[1-\frac{s^2}{(m+1)^2}\right]\frac{A_2(m,s;\theta)}{\tan^{2s}\frac{\theta}{2}\sin^{2m+1}\theta} \\ &= \left[E_{02}(m,s)+\frac{s^2}{(m+1)^2}+\frac{(m+s+1)(m-s+1)(m+2s^2+1)}{(2m+3)(m+1)^3}\right]\frac{A_0(m,s;\theta)}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta} \\ &\quad+\frac{(m+s+1)(m-s+1)}{(2m+3)(m+1)^2}\left(-\frac{s}{m+1}+\cos\theta\right)\sin\theta, \end{aligned} \tag{A24}$$

boundary conditions (14) require that the coefficient of $A_0(m,s;\theta)$ in Eq. (A24) must be zero, which is

$$E_{02}(m,s) = -\frac{s^2}{(m+1)^2}-\frac{(m+s+1)(m-s+1)(m+2s^2+1)}{(2m+3)(m+1)^3}, \tag{A25}$$

so

$$W_{02}(m,s;\theta) = \frac{(m+s+1)(m-s+1)}{(2m+3)(m+1)^2}\left(-\frac{s}{m+1}+\cos\theta\right)\sin\theta. \tag{A26}$$

We now surmise the following conclusion and then prove it by mathematical induction.

Conclusion: From Eq. (A4) and boundary condition (14), we obtain

(i) for any $n \geq 1$, $W_{0n}(m;\theta)$ has a from

$$W_{0n}(m,s;\theta) = \sin\theta \times g_n(m,s;\theta), \tag{A27}$$

where $g_n(m,s;\theta)$ is a polynomial of $\cos\theta$, its highest degree is $n-1$ and, for odd n, s is a factor of $g_n(m,s;\theta)$ or $g_n(m,s;\theta) \propto s$. In particular, if

$$g_n(m,s;\theta) = \sum_{\ell=0}^{n-1} a_{n,\ell}(m,s)\cos^\ell\theta, \tag{A28}$$

then $a_{n,\ell}$ satisfy the relations. There are relations (28)–(31) for $n \geq 1$, and the coefficients $d_{\ell,p}$ occurring in these relations are given in Eqs. (A17)–(A20). Hence the super-potential is

$$W(m,s,c;\theta) = \sum_{n=0}^{\infty} W_{0n}(m,s;\theta)c^n$$
$$= -\frac{s}{\sin\theta} - (m+\frac{1}{2})\cot\theta + \sin\theta \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1} a_{n,\ell}(m,s)\cos^\ell\theta c^n, \tag{A29}$$

(ii) for any $n \geq 1$,

$$E_{0,n+1}(m,s) = -\sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)} a_{k,i}(m,s) \times a_{n+1-k,\ell-i}(m,s)(D_\ell - D_{\ell+2}), \tag{A30}$$

where the coefficients D_ℓ are given by Eqs. (A15) and (A16).

We will draw the conclusion in the following. It follows from the preceding evaluation that this conclusion holds for $n = 1, 2$. Now, if we assume that it holds for $1 \leq N \leq n$ for $n > 2$, then from Eq. (A3), we have

$$\begin{aligned} f_{n+1}(m,s;\theta) &= E_{0,n+1}(m,s) + \sum_{k=1}^{n} W_{0,k}(m,s;\theta)W_{0,n+1-k}(m,s;\theta) \\ &= E_{0,n+1}(m,s) + \sin^2\theta \sum_{k=1}^{n} g_k(m,s;\theta)g_{n+1-k}(m,s;\theta) \\ &= E_{0,n+1}(m,s) + \sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k} a_{k,i}(m,s)a_{n+1-k,j}(m,s)(\cos^{i+j}\theta - \cos^{i+j+2}\theta), \end{aligned} \tag{A31}$$

which is a polynomial in $\cos\theta$ with degree $n+1$.

From the known Eq. (A23), one has $g_1(m,s,\theta) \propto s$. Obviously, when $n+1$ is odd, then for any $1 \leq k \leq n$, either k or $n+1-k$ is odd. By induction it is easy to obtain

$$\sum_{k=1}^{n} W_{0,k}(m,s;\theta)W_{0,n+1-k}(m,s;\theta) = \sin^2\theta \sum_{k=1}^{n} g_k(m,s;\theta)g_{n+1-k}(m,s;\theta), \quad \propto s. \tag{A32}$$

Substituting Eq. (A31) into Eq. (A5) yields

$$\begin{aligned} W_{0,n+1}(m,s;\theta) &= \frac{1}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta}\int^{\theta} f_{n+1}(m,s;\varphi)\tan^{2s}\frac{\varphi}{2}\sin^{2m+1}\varphi\,\mathrm{d}\varphi \\ &= \frac{A_0(m,s;\theta)}{\tan^{2s}\dfrac{\theta}{2}\sin^{2m+1}\theta}\left[E_{0,n+1}(m,s) + \sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k} a_{k,i}(m,s)a_{n+1-k,j}(m,s)(D_{i+j} - D_{i+j+2})\right] \\ &\quad + \sin\theta\sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k} a_{k,i}(m,s)a_{n+1-k,j}(m,s)\left(\sum_{p=0}^{i+j-1} d_{i+j,p} - \sum_{p=0}^{i+j+1} d_{i+j+2,p}\right)\cos^p\theta \end{aligned} \tag{A33}$$

to satisfy condition (14). From Eq. (A13) , the coefficient of $A_0(m,s;\theta)$ in Eq. (A33) must be zero, which is

$$E_{0,n+1}(m,s) = -\sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)} a_{k,i}(m,s)a_{n+1-k,\ell-i}(m,s)(D_\ell - D_{\ell+2}), \tag{A34}$$

so

$$W_{0,n+1}(m,s;\theta) = \sin\theta\sum_{\ell=0}^{n} a_{n+1,\ell}(m,s)\cos^\ell\theta \equiv \sin\theta g_{n+1}(m,s;\theta), \tag{A35}$$

where coefficients D_ℓ and $d_{\ell,p}$ are given by Eqs. (A15)–(A20). From Eqs. (A32) and (A33), $g_{n+1}(m,s;\theta) \propto s$ for odd number $n+1$. Then we establish the conclusion by mathematical induction. It follows from the fact that $g_{n+1}(m,s;\theta) \propto s$ for odd number $n+1$ that

$$a_{2n+1,p}(m,s) \propto s,\ n=0,1,2,\ldots;\ 0 \le p \le 2n, \tag{A36}$$

then

$$a_{2n+1,p}(m,0) = 0,\ n=0,1,2,\ldots;\ 0 \le p \le 2n. \tag{A37}$$

We will prove by mathematical induction that

$$a_{2n,2p}(m,0) = 0,\ n=1,2,\ldots;\ 0 \le p \le n-1. \tag{A38}$$

To begin with, it follows from (63) that

$$a_{2,0}(m,0) = 0. \tag{A39}$$

Secondly, when $s=0$, we obtain from the definitions (A15) to (A20) of D_ℓ and $d_{\ell,p}$ that

$$D_{2\ell} = \frac{(2l-1)!!(2m+1)!!}{(2m+2\ell+1)!!},\quad \ell = 1,2,3,\ldots, \tag{A40}$$

with the initial conditions

$$D_0 = 1, \tag{A41}$$

$$D_{2\ell+1} = 0,\ \ell = 0,1,2,\ldots; \tag{A42}$$

and

$$\begin{aligned}
d_{2\ell,2p+1} &= \frac{\left(\frac{1}{2}\right)_\ell}{\left(\frac{3}{2}+m\right)_\ell}\frac{\left(\frac{3}{2}+m\right)_p}{\left(\frac{3}{2}\right)_p},\ p=0,1,\ldots,\ell-1,\\
d_{2\ell,2p} &= 0,\ p=0,1,\ldots,\ell-1,\\
d_{2\ell+1,2p+1} &= 0,\ p=0,1,\ldots,\ell-1,\\
d_{2\ell+1,2p} &= \frac{\ell!(m+2)_{p-1}}{2(m+2)_\ell p!},\ p=1,2,\ldots,\ell,\\
d_{2\ell+1,0} &= \frac{\ell!}{2(m+1)(m+2)_\ell}.
\end{aligned} \tag{A43}$$

Since we have proved $a_{2,0}(m,0)=0$, we can assume that Eq. (A38) holds for all positive integers smaller than n, then

$$\begin{aligned}
a_{2n+2,2p}(m,0) =& \sum_{k=1}^{n}\sum_{\ell=p+1}^{2n}\sum_{i=0,\ell-i\le 2n-2k+1}^{\min(\ell,2k-1)} a_{2k,i}(m,0)a_{2(n+1-k),\ell-i}(m,0)d_{\ell,2p}\\
&-\sum_{k=1}^{n}\sum_{\ell=p-1}^{2n}\sum_{i=0,\ell-i\le 2n-2k+1}^{\min(\ell,2k-1)} a_{2k,i}(m,0)a_{2(n+1-k),\ell-i}(m,0)d_{\ell+2,2p}.
\end{aligned} \tag{A44}$$

It follows from Eq. (A43) that the non-zero terms in the last equality must be those that correspond to the odd numbers of ℓ, but then either $a_{2k,i}(m,0)=0$ or $a_{2(n+1-k),2\ell+1-i}(m,0)=0$ hold for any i due to our assumption, hence we establish Eq. (A38). As for $a_{2n,2p+1}(m,0)$, we can obtain the recurrence relations (48), (49) and (50). Therefore, when $s=0$, our results are reduced into those of an angular spheroidal wave equation when spin is zero, as indicated in Ref. (7).

Finally, we obtain the ground eigenvalue as

$$\begin{aligned}
E_0(m,s,c) &= \sum_{n=0}^{\infty} E_{0n}(m,s)c^n\\
&= (m-s)(m+s+1) - \sum_{n=0}^{\infty} c^{n+1}\sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\sum_{i=0,\ell-i\le n-k}^{\min(\ell,k-1)} a_{k,i}(m,s)a_{n+1-k,\ell-i}(m,s)(D_\ell - D_{\ell+2}),
\end{aligned} \tag{A45}$$

the super-potential as

$$\begin{aligned}
W(m,s,c;\theta) &= \sum_{n=0}^{\infty} W_{0n}(m,s;\theta)c^n\\
&= -\frac{s}{\sin\theta} - \left(m+\frac{1}{2}\right)\cot\theta + \sin\theta\sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1} a_{n,\ell}(m,s)\cos^\ell\theta c^n,
\end{aligned} \tag{A46}$$

and the ground eigenfunction as

$$\psi_0(m,s,c;\theta) = \exp\left[-\int^{\theta} W(m,s,c;\phi)\,\mathrm{d}\phi\right]$$
$$= \tan^s\frac{\theta}{2}\sin^{m+\frac{1}{2}}\theta\prod_{n=1}^{\infty}\exp\left[\sum_{\ell=0}^{n-1}a_{n,\ell}(m,s)\frac{\cos^{\ell+1}\theta}{\ell+1}c^n\right]. \quad \text{(A47)}$$

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