Hawking effect and quantum nonthermal radiation of an arbitrarily accelerating charged black hole using a new tortoise coordinate transformation*

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(Received 19 October 2010; revised manuscript received 26 November 2010)

Using a new tortoise coordinate transformation, this paper investigates the Hawking effect from an arbitrarily accelerating charged black hole by the improved Damour–Ruffini method. After the tortoise coordinate transformation, the Klein–Gordon equation can be written as the standard form at the event horizon. Then extending the outgoing wave from outside to inside of the horizon analytically, the surface gravity and Hawking temperature can be obtained automatically. It is found that the Hawking temperatures of different points on the surface are different. The quantum nonthermal radiation characteristics of a black hole near the event horizon is also discussed by studying the Hamilton–Jacobi equation in curved spacetime and the maximum overlap of the positive and negative energy levels near the event horizon is given. There is a dimensional problem in the standard tortoise coordinate and the present results may be more reasonable.

Keywords: tortoise coordinate transformation, black hole, Hawking effect, quantum nonthermal radiation

PACS: 97.60.Lf, 04.70.–s, 04.70.Dy **DOI:** 10.1088/1674-1056/20/4/049701

1. Introduction

In the 1970s, Hawking discovered that a black hole has thermal properties and can radiate a thermal spectrum.[1,2] Then, the four laws of black hole thermodynamics were successfully built. To reveal the thermal properties of various black holes is an important subject in black hole physics. In the past few decades great progress has been made investigating the thermal properties of scalar fields or Dirac particles in some static or stationary black holes.[3−8] Because a realistic black hole in astrophysics can radiate or absorb matter surrounding it, it is dynamical and evolves in time. Thus the thermal properties of non-stationary spacetimes are more meaningful than that of static or stationary black holes, and it is worth while studying more about them. A well-known method to determine the location and the temperature of the event horizon of a dynamical black hole is to calculate vacuum expectation value of the renormalized energy momentum tensor.[9,10] However, this method is so complicated that it is of limited use and meets great difficulties in many cases. In recent years, the study of the Hawking radiation of some non-stationary black holes using the generalized tortoise coordinate transformation has attracted more attention and made great progress.[11−16] The tortoise coordinate transformation method was initially proposed by Damour and Ruffini[3] to deal with the Hawking effect of scalar fields or Dirac particles in some static spherically symmetric black holes and stationary axisymmetric spacetimes, which has been generalized by Zhao *et al.*[17,18] to the non-static and non-stationary cases and successfully applied to investigate the quantum thermal effect of scalar particles in various dynamical black holes.[11] For a charged black hole or a rotating black hole, Starobinsky and Unruh have proved that there is not only thermal radiation, but also nonthermal radiation.[19,20] There has also been much work done on the nonthermal radiation of black holes.[21−24]

In this paper, the Hawking effect and quantum nonthermal radiation of an arbitrarily accelerating charged black hole are discussed according to the Klein–Gordon equation and the Hamilton–Jacobi

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10873003 and 11045005) and the Natural Science Foundation of Zhejiang Province, China (Grant No. Y6090739).

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equation using an improved tortoise coordinate transformation.

The organization of this paper is as follows. In Section 2, we will give the line element of the arbitrarily accelerating charged spacetime and the surface equation of horizon. In Section 3, the Hawking effect and the surface gravity in the black hole are calculated using the improved tortoise coordinate transformation. In Section 4, the quantum nonthermal radiation of the black hole is calculated. Some conclusions and discussions are given in the last section.

2. The arbitrarily accelerating charged black hole

The line element of an arbitrarily accelerating charged black hole can be expressed in the advanced Eddington–Finkelstein time coordinate as

$$\begin{aligned} \mathrm{d}s^2 &= g_{00}\mathrm{d}\upsilon^2 + 2g_{01}\mathrm{d}\upsilon\mathrm{d}r + 2g_{02}\mathrm{d}\upsilon\mathrm{d}\theta \\ &\quad + 2g_{03}\mathrm{d}\upsilon\mathrm{d}\varphi + g_{22}\mathrm{d}\theta^2 + g_{33}\mathrm{d}\varphi^2, \end{aligned} \tag{1}$$

where

$$\begin{aligned} g_{00} &= -\Big[1 - \frac{2m}{r} - 2ar\cos\theta + \frac{Q^2}{r^2} - \frac{4aQ^2\cos\theta}{r} \\ &\quad - r^2\left(f^2 + h^2\sin^2\theta\right)\Big], \\ g_{01} &= g_{10} = 1,\ g_{02} = g_{20} = r^2 f, \\ g_{03} &= g_{30} = r^2 h\sin^2\theta,\ g_{22} = r^2, \\ g_{33} &= r^2\sin^2\theta, \\ f &= -a\sin\theta + b\sin\varphi + c\cos\varphi, \\ h &= \cot\theta\left(b\cos\varphi - c\sin\varphi\right). \end{aligned}$$

The parameters $m = m(\upsilon)$, $Q = Q(\upsilon)$, $a = a(\upsilon)$, $b = b(\upsilon)$ and $c = c(\upsilon)$ are the functions of the advanced Eddington–Finkelstein time coordinate υ, m is the mass of black hole, Q is the electric charge of black hole, a is the magnitude of acceleration, b and c refer to the change rate of the direction of acceleration.

The null hypersurface condition

$$g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0 \tag{2}$$

can be rewritten as

$$\begin{aligned} &1 - \frac{2m}{r_{\mathrm{H}}} - 2ar_{\mathrm{H}}\cos\theta_0 + \frac{Q^2}{r_{\mathrm{H}}^2} - \frac{4aQ^2\cos\theta_0}{r_{\mathrm{H}}} - 2\dot{r}_{\mathrm{H}} \\ &+ 2f_{\mathrm{H}}r'_{\mathrm{H}} + 2h_{\mathrm{H}}r'_{\mathrm{H}\varphi} + \frac{r'^2_{\mathrm{H}}}{r_{\mathrm{H}}^2} + \frac{r'^2_{\mathrm{H}\varphi}}{r_{\mathrm{H}}^2\sin^2\theta_0} = 0, \end{aligned} \tag{3}$$

where $\dot{r}_{\mathrm{H}} = \partial r_{\mathrm{H}}/\partial\upsilon$, $r'_{\mathrm{H}} = \partial r_{\mathrm{H}}/\partial\theta$, $r'_{\mathrm{H}\varphi} = \partial r_{\mathrm{H}}/\partial\varphi$.

Equation (3) determines the local event horizon of an arbitrarily accelerating charged black hole.

3. Hawking effect from the black hole horizon

The Klein–Gordon equation in arbitrarily accelerating charged spacetime is

$$\begin{aligned} &g^{11}\frac{\partial^2\Phi}{\partial r^2} + 2g^{10}\frac{\partial^2\Phi}{\partial r\partial\upsilon} + 2g^{12}\frac{\partial^2\Phi}{\partial r\partial\theta} + 2g^{13}\frac{\partial^2\Phi}{\partial r\partial\varphi} \\ &+ g^{22}\frac{\partial^2\Phi}{\partial\theta^2} + g^{33}\frac{\partial^2\Phi}{\partial\varphi^2} + f_\upsilon\frac{\partial\Phi}{\partial\upsilon} \\ &+ f_r\frac{\partial\Phi}{\partial r} + f_\theta\frac{\partial\Phi}{\partial\theta} + f_\varphi\frac{\partial\Phi}{\partial\varphi} + f_0\Phi = 0, \end{aligned} \tag{4}$$

where

$$\begin{aligned} g^{10} &= 1, \\ g^{11} &= 1 - \frac{2m}{r} - 2ar\cos\theta + \frac{Q^2}{r^2} - \frac{4aQ^2\cos\theta}{r}, \\ g^{12} &= -f,\ \ g^{13} = -h,\ \ g^{22} = \frac{1}{r^2},\ \ g^{33} = \frac{1}{r^2\sin^2\theta}, \\ f_\upsilon &= \frac{2}{r}g^{10} - 2\mathrm{i}eA_1g^{10}, \\ f_r &= \frac{2}{r}g^{11} + \frac{\partial g^{11}}{\partial r} + \frac{\partial g^{12}}{\partial\theta} + \frac{\partial g^{13}}{\partial\varphi} + g^{12}\cot\theta \\ &\quad - 2\mathrm{i}e\left(A_0g^{10} + A_1g^{11} + A_2g^{12} + A_3g^{13}\right), \\ f_\theta &= \frac{2}{r}g^{12} + g^{22}\cot\theta - 2\mathrm{i}e\left(A_1g^{12} + A_2g^{22}\right), \\ f_\varphi &= \frac{2}{r}g^{13} - 2\mathrm{i}e\left(A_1g^{13} + A_3g^{33}\right), \\ f_0 &= -e^2g^{\mu\nu}A_\mu A_\nu - \mu^2 \end{aligned} \tag{5}$$

with parameters e, A_μ and μ. The parameter e is the electric charge of Klein–Gordon particle, A_μ is its electromagnetic four-vector, and μ is the mass of Klein–Gordon particle.

For an arbitrarily accelerating charged black hole, a new tortoise coordinate can be written as[25]

$$\begin{aligned} r_* &= r + \frac{1}{2\kappa\left(\upsilon_0,\theta_0,\varphi_0\right)}\ln\left[\frac{r - r_{\mathrm{H}}\left(\upsilon,\theta,\varphi\right)}{r_{\mathrm{H}}\left(\upsilon,\theta,\varphi\right)}\right], \\ \upsilon_* &= \upsilon - \upsilon_0,\ \theta_* = \theta - \theta_0,\ \varphi_* = \varphi - \varphi_0, \end{aligned} \tag{6}$$

where κ is a constant under the tortoise coordinate transformation, υ_0 is the moment when the particle escapes from the event horizon of the black hole and depicts the evolution of the black hole, both θ_0 and φ_0 are the locations where the particle escapes from the event horizon of the black hole and depict the shape of the black hole.

According to the tortoise coordinate transformation, the Klein–Gordon equation(4) can be rewritten as

$$\frac{g^{22}r^2r_{\rm H}'^2+g^{33}r^2r_{{\rm H}\varphi}'^2+[1+2\kappa(r-r_{\rm H})]\left\{-2rr_{\rm H}\left(\dot r_{\rm H}+g^{12}r_{\rm H}'+g^{13}r_{{\rm H}\varphi}'\right)+g^{11}r_{\rm H}^2[1+2\kappa(r-r_{\rm H})]\right\}}{2\kappa r_{\rm H}^2(r-r_{\rm H})[1+2\kappa(r-r_{\rm H})]}\frac{\partial^2\Phi}{\partial r_*^2}$$
$$+2\frac{\partial^2\Phi}{\partial r_*\partial\upsilon_*}+\left\{2g^{12}-\frac{2g^{22}rr_{\rm H}'}{r_{\rm H}[1+2\kappa(r-r_{\rm H})]}\right\}\frac{\partial^2\Phi}{\partial r_*\partial\theta_*}+\left\{2g^{13}-\frac{2g^{33}rr_{{\rm H}\varphi}'}{r_{\rm H}[1+2\kappa(r-r_{\rm H})]}\right\}\frac{\partial^2\Phi}{\partial r_*\partial\varphi_*}$$
$$+\left\{\frac{1}{r_{\rm H}^2(r-r_{\rm H})[1+2\kappa(r-r_{\rm H})]}[-g^{11}r_{\rm H}^2+2\dot r_{\rm H}r_{\rm H}^2+2g^{12}r_{\rm H}'r_{\rm H}^2+2g^{13}r_{{\rm H}\varphi}'r_{\rm H}^2-g^{22}rr_{\rm H}''r_{\rm H}(r-r_{\rm H})\right.$$
$$+g^{22}rr_{\rm H}'^2(r-r_{\rm H})-g^{22}rr_{\rm H}'^2r_{\rm H}-g^{33}rr_{{\rm H}\varphi}''r_{\rm H}(r-r_{\rm H})+g^{33}rr_{{\rm H}\varphi}'^2(r-r_{\rm H})-g^{33}rr_{{\rm H}\varphi}'^2r_{\rm H}-r_{\rm H}(r-r_{\rm H})r$$
$$\left.\times\left(f_\upsilon\dot r_{\rm H}+f_\theta r_{\rm H}'+f_\varphi r_{{\rm H}\varphi}'\right)]+f_r\right\}\frac{\partial\Phi}{\partial r_*}+\frac{2\kappa(r-r_{\rm H})}{1+2\kappa(r-r_{\rm H})}$$
$$\times\left(g^{22}\frac{\partial^2\Phi}{\partial\theta_*^2}+g^{33}\frac{\partial^2\Phi}{\partial\varphi_*^2}+f_\upsilon\frac{\partial\Phi}{\partial\upsilon_*}+f_\theta\frac{\partial\Phi}{\partial\theta_*}+f_\varphi\frac{\partial\Phi}{\partial\varphi_*}+f_0\Phi\right)=0. \tag{7}$$

From the null hypersurface condition Eq. (3) of an arbitrarily accelerating charged spacetime, the numerator of the coefficient on the term $\partial^2\Phi/\partial r_*^2$ approaches zero at the horizon $r_{\rm H}$. Therefore, we can calculate the limit of the coefficient by using the L'Hospital law. Let the limit be equal to an undetermined constant A

$$\lim_{r\to r_{\rm H},\upsilon\to\upsilon_0,\theta\to\theta_0,\varphi\to\varphi_0}\left\{\frac{g^{22}r^2r_{\rm H}'^2+g^{33}r^2r_{{\rm H}\varphi}'^2}{2\kappa r_{\rm H}^2(r-r_{\rm H})[1+2\kappa(r-r_{\rm H})]}\right.$$
$$\left.+\frac{-2rr_{\rm H}\left(\dot r_{\rm H}+g^{12}r_{\rm H}'+g^{13}r_{{\rm H}\varphi}'\right)+g^{11}r_{\rm H}^2[1+2\kappa(r-r_{\rm H})]}{2\kappa r_{\rm H}^2(r-r_{\rm H})}\right\}=A.$$

While κ is selected as

$$\kappa=\frac{\dfrac{2m}{r_{\rm H}^2}-\dfrac{r_{\rm H}'^2}{2r_{\rm H}^3}-\dfrac{r_{{\rm H}\varphi}'^2}{2r_{\rm H}^3\sin^2\theta_0}-\dfrac{1}{2r_{\rm H}}-\dfrac{3Q^2}{2r_{\rm H}^3}+\dfrac{8aQ^2\cos\theta_0}{2r_{\rm H}^2}}{2r_{\rm H}\left(a\cos\theta_0+\dfrac{m}{r_{\rm H}^2}+\dfrac{r_{\rm H}'^2}{2r_{\rm H}^3}+\dfrac{r_{{\rm H}\varphi}'^2}{2r_{\rm H}^3\sin^2\theta_0}-\dfrac{Q^2}{2r_{\rm H}^3}+\dfrac{2aQ^2\cos\theta_0}{r_{\rm H}^2}\right)}, \tag{8}$$

we have $A=1$.

When r approaches to $r_{\rm H}$, the Klein–Gordon equation (7) can be transformed into

$$\frac{\partial^2\Phi}{\partial r_*^2}+2\frac{\partial^2\Phi}{\partial r_*\partial\upsilon_*}+B\frac{\partial^2\Phi}{\partial r_*\partial\theta_*}+C\frac{\partial^2\Phi}{\partial r_*\partial\varphi_*}+(D+{\rm i}2\omega_0)\frac{\partial\Phi}{\partial r_*}=0, \tag{9}$$

where

$$B=\lim_{r\to r_{\rm H},\upsilon\to\upsilon_0,\theta\to\theta_0,\varphi\to\varphi_0}\left(2g^{12}-2g^{22}r_{\rm H}'\right)=-2f_{\rm H}-\frac{2r_{\rm H}'}{r_{\rm H}^2}=2\left(a\sin\theta_0-b\sin\varphi_0-c\cos\varphi_0-\frac{r_{\rm H}'}{r_{\rm H}^2}\right), \tag{10}$$

$$C=\lim_{r\to r_{\rm H},\upsilon\to\upsilon_0,\theta\to\theta_0,\varphi\to\varphi_0}\left(2g^{13}-2g^{33}r_{{\rm H}\varphi}'\right)=-2h_{\rm H}-\frac{2r_{{\rm H}\varphi}'}{r_{\rm H}^2\sin^2\theta_0}$$
$$=2\left(c\sin\varphi_0\cot\theta_0-b\cos\varphi_0\cot\theta_0-\frac{r_{{\rm H}\varphi}'}{r_{\rm H}^2\sin^2\theta_0}\right), \tag{11}$$

$$D=\lim_{r\to r_{\rm H},\upsilon\to\upsilon_0,\theta\to\theta_0,\varphi\to\varphi_0}\left[-\left(\frac{\partial g^{11}}{\partial r}+\frac{\partial g^{22}}{\partial r}r_{\rm H}'^2+\frac{\partial g^{33}}{\partial r}r_{{\rm H}\varphi}'^2\right)-(g^{22}r_{\rm H}''+g^{33}r_{{\rm H}\varphi}'')\right.$$
$$\left.-\frac{2}{r}(g^{10}\dot r_{\rm H}+g^{12}r_{\rm H}'+g^{13}r_{{\rm H}\varphi}')-g^{22}\cot\theta r_{\rm H}'+\left(\frac{2}{r}g^{11}+\frac{\partial g^{11}}{\partial r}+\frac{\partial g^{12}}{\partial\theta}+\frac{\partial g^{13}}{\partial\varphi}+g^{12}\cot\theta\right)\right], \tag{12}$$

$$\omega_0=\lim_{r\to r_{\rm H},\upsilon\to\upsilon_0,\theta\to\theta_0,\varphi\to\varphi_0}[eA_1g^{10}\dot r_{\rm H}+e(A_1g^{12}+A_2g^{22})r_{\rm H}'+e(A_1g^{13}+A_3g^{33})r_{{\rm H}\varphi}'$$
$$-e(A_0g^{10}+A_1g^{11}+A_2g^{12}+A_3g^{13})]. \tag{13}$$

When r approaches $r_{\rm H}$, the solution of Eq. (9) can be written as[17]

$$\Phi=R(r_*)\,{\rm e}^{-{\rm i}\omega\upsilon_*+{\rm i}k_\theta\theta_*+{\rm i}k_\varphi\varphi_*}. \tag{14}$$

We can obtain

$$\frac{\partial^2 R(r_*)}{\partial r_*^2} + \{D - \mathrm{i}[2(\omega-\omega_0) - Bk_\theta - Ck_\varphi]\}\frac{\partial R(r_*)}{\partial r_*} = 0, \tag{15}$$

and its solution is

$$\begin{aligned} R_\omega^{\rm in} &= \mathrm{e}^{-\mathrm{i}\omega\upsilon_*}, \\ R_\omega^{\rm out} &= \mathrm{e}^{-\mathrm{i}\omega\upsilon_*}\cdot\mathrm{e}^{-Dr_*+\mathrm{i}[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]r_*}. \end{aligned} \tag{16}$$

The ingoing and outgoing wave at the horizon $r_{\rm H}$ are respectively

$$\begin{aligned} \Phi_\omega^{\rm in} &= \mathrm{e}^{-\mathrm{i}\omega\upsilon_*}\cdot\mathrm{e}^{\mathrm{i}k_\theta\theta_*+\mathrm{i}k_\varphi\varphi_*}, \\ \Phi_\omega^{\rm out} &= \mathrm{e}^{-\mathrm{i}\omega\upsilon_*}\cdot\mathrm{e}^{-Dr_*+\mathrm{i}[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]r_*} \\ &\quad\cdot\mathrm{e}^{\mathrm{i}k_\theta\theta_*+\mathrm{i}k_\varphi\varphi_*} \\ &= \mathrm{e}^{-\mathrm{i}\omega\upsilon_*}\cdot\mathrm{e}^{-Dr+\mathrm{i}[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]r} \\ &\quad\cdot\left(\frac{r-r_{\rm H}}{r_{\rm H}}\right)^{-D/2\kappa+\mathrm{i}[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]/2\kappa} \\ &\quad\cdot\mathrm{e}^{\mathrm{i}k_\theta\theta_*+\mathrm{i}k_\varphi\varphi_*}. \end{aligned} \tag{17}$$

It is obvious that the outgoing wave is not analytical at the horizon $r_{\rm H}$. Extending the outgoing wave from outside to inside of the horizon analytically through the negative half complex plane, we can obtain

$$\begin{aligned} &\Phi_\omega^{\rm out}(r<r_{\rm H}) \\ &= \mathrm{e}^{-\mathrm{i}\omega\upsilon_*}\cdot\mathrm{e}^{-Dr+\mathrm{i}[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]r} \\ &\quad\cdot\left[\left(\frac{r_{\rm H}-r}{r_{\rm H}}\right)\mathrm{e}^{-\mathrm{i}\pi}\right]^{-D/2\kappa+\mathrm{i}[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]/2\kappa} \\ &\quad\cdot\mathrm{e}^{\mathrm{i}k_\theta\theta_*+\mathrm{i}k_\varphi\varphi_*} \\ &= \mathrm{e}^{-\mathrm{i}\omega\upsilon_*}\cdot\mathrm{e}^{-Dr_*+\mathrm{i}[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]r_*} \\ &\quad\cdot\mathrm{e}^{\mathrm{i}k_\theta\theta_*+\mathrm{i}k_\varphi\varphi_*+\mathrm{i}D\pi/2\kappa} \\ &\quad\cdot\mathrm{e}^{\pi[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]/2\kappa}. \end{aligned} \tag{18}$$

The scattering probability of outgoing wave at the horizon is

$$\left|\frac{\Phi_\omega^{\rm out}(r>r_{\rm H})}{\Phi_\omega^{\rm out}(r<r_{\rm H})}\right|^2 = \mathrm{e}^{-\pi[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]/\kappa}. \tag{19}$$

According to the explanation of Sannan, the outgoing wave has black-body spectrum

$$N_\omega = \frac{1}{\mathrm{e}^{[(\omega-\omega_0)-Bk_\theta/2-Ck_\varphi/2]/\kappa_{\rm B}T}-1}, \tag{20}$$

$$T = \frac{\kappa}{2\pi\kappa_{\rm B}}. \tag{21}$$

4. The quantum nonthermal radiation of the black hole

The Hamilton–Jacobi equation of the particle in the arbitrarily accelerating charged spacetime is

$$\begin{aligned} &2g^{10}\left(\frac{\partial S}{\partial\upsilon}\frac{\partial S}{\partial r} - eA_1\frac{\partial S}{\partial\upsilon} - eA_0\frac{\partial S}{\partial r} + e^2A_0A_1\right) \\ &+ g^{22}\left[\left(\frac{\partial S}{\partial\theta}\right)^2 - 2eA_2\frac{\partial S}{\partial\theta} + e^2A_2^2\right] \\ &+ g^{33}\left[\left(\frac{\partial S}{\partial\varphi}\right)^2 - 2eA_3\frac{\partial S}{\partial\varphi} + e^2A_3^2\right] \\ &+ g^{11}\left[\left(\frac{\partial S}{\partial r}\right)^2 - 2eA_1\frac{\partial S}{\partial r} + e^2A_1^2\right] \\ &+ 2g^{12}\left(\frac{\partial S}{\partial r}\frac{\partial S}{\partial\theta} - eA_2\frac{\partial S}{\partial r} - eA_1\frac{\partial S}{\partial\theta} + e^2A_1A_2\right) \\ &+ 2g^{13}\left(\frac{\partial S}{\partial r}\frac{\partial S}{\partial\varphi} - eA_3\frac{\partial S}{\partial r} - eA_1\frac{\partial S}{\partial\varphi} + e^2A_1A_3\right) \\ &- \mu^2 = 0, \end{aligned} \tag{22}$$

where S is Hamilton master function.

According to the tortoise coordinate transformation of Eq. (6), the Hamilton–Jacobi equation (22) can be rewritten as

$$\frac{A'}{2\kappa r_{\rm H}^2(r-r_{\rm H})}\left(\frac{\partial S}{\partial r_*}\right)^2 + \frac{B'}{r_{\rm H}}\frac{\partial S}{\partial r_*} + 2\kappa(r-r_{\rm H})C' = 0, \tag{23}$$

where

$$\begin{aligned} A' &= g^{11}r_{\rm H}^2[1+2\kappa(r-r_{\rm H})]^2 + g^{22}r^2r_{\rm H}'^2 \\ &\quad + g^{33}r^2r_{{\rm H}\varphi}'^2 - 2rr_{\rm H}[1+2\kappa(r-r_{\rm H})] \\ &\quad\times\{g^{10}\dot r_{\rm H} + g^{12}r_{\rm H}' + g^{13}r_{{\rm H}\varphi}'\}, \\ B' &= 2er[A_1(g^{10}\dot r_{\rm H} + g^{12}r_{\rm H}' + g^{13}r_{{\rm H}\varphi}') \\ &\quad + g^{22}A_2r_{\rm H}' + g^{33}A_3r_{{\rm H}\varphi}'] \\ &\quad - 2r(g^{22}lr_{\rm H}' + g^{33}jr_{{\rm H}\varphi}') - 2[1+2\kappa(r-r_{\rm H})] \\ &\quad\times\{g^{10}r_{\rm H}(\omega+eA_0) + g^{11}eA_1r_{\rm H} \\ &\quad + g^{12}r_{\rm H}(-l+eA_2) + g^{13}r_{\rm H}(-j+eA_3)\}, \\ C' &= 2g^{10}eA_1(\omega+eA_0) + 2g^{12}eA_1(-l+eA_2) \\ &\quad + 2g^{13}eA_1(-j+eA_3) \\ &\quad + e^2(g^{11}A_1^2 + g^{22}A_2^2 + g^{33}A_3^2) \\ &\quad - 2e(g^{22}A_2l + g^{33}A_3j) + g^{22}l^2 + g^{33}j^2 - \mu^2, \\ \omega &= -\frac{\partial S}{\partial\upsilon_*},\quad l = \frac{\partial S}{\partial\theta_*},\quad j = \frac{\partial S}{\partial\varphi_*} \end{aligned}$$

with parameters ω, l and j. The parameter ω is the energy of the particle, l and j are the angular momenta of the particle.

The solution of Eq. (23) is

$$\frac{\partial S}{\partial r_*} = \frac{\kappa r_{\rm H}(r-r_{\rm H})}{A'}(-B' \pm \sqrt{B'^2 - 4A'C'}), \tag{24}$$

where

$$\begin{aligned} B' &= B_1 - 2g^{10}r_{\rm H}[1+2\kappa(r-r_{\rm H})]\omega, \\ C' &= C_1 + 2g^{10}eA_1\omega. \end{aligned}$$

Because S and $\partial S/\partial r_*$ are real numbers, the energy ω of the particle in the curved spacetime described by Eq. (1) satisfies

$$4(g^{10})^2 r_{\rm H}^2 \left[1+2\kappa\left(r-r_{\rm H}\right)\right]^2 \omega^2 - \{4B_1 g^{10} r_{\rm H}\left[1+2\kappa\left(r-r_{\rm H}\right)\right] + 8g^{10} eA' A_1\}\omega + B_1^2 - 4A'C_1 \geq 0, \tag{25}$$

from which we have

$$\omega^{\pm} = \frac{B_1 r_{\rm H}\left[1+2\kappa\left(r-r_{\rm H}\right)\right] + 2eA'A_1 \pm 2\sqrt{B_1 r_{\rm H} eA'A_1\left[1+2\kappa\left(r-r_{\rm H}\right)\right] + e^2A'^2A_1^2 + r_{\rm H}^2 A'C_1\left[1+2\kappa\left(r-r_{\rm H}\right)\right]^2}}{2g^{10}r_{\rm H}^2\left[1+2\kappa\left(r-r_{\rm H}\right)\right]^2}, \tag{26}$$

where

$$\begin{cases} \omega \geq \omega^+, \\ \omega \leq \omega^-. \end{cases} \tag{27}$$

The states of particle's energy for $\omega^- \leq \omega \leq \omega^+$ are located in the forbidden region, its width is

$$\Delta\omega = \omega^+ - \omega^- = \frac{2\sqrt{B_1 r_{\rm H} eA'A_1\left[1+2\kappa\left(r-r_{\rm H}\right)\right] + e^2A'^2A_1^2 + r_{\rm H}^2 A'C_1\left[1+2\kappa\left(r-r_{\rm H}\right)\right]^2}}{g^{10}r_{\rm H}^2\left[1+2\kappa\left(r-r_{\rm H}\right)\right]^2}. \tag{28}$$

Both equations (26) and (27) describe the distribution characteristics of the particle energy levels of the curved spacetime described by Eq. (1). From Eqs. (26) and (3), we can obtain

$$\begin{aligned} &\lim_{r\to r_{\rm H},\upsilon\to\upsilon_0,\theta\to\theta_0,\varphi\to\varphi_0} \omega^+ \\ &= \lim_{r\to r_{\rm H},\upsilon\to\upsilon_0,\theta\to\theta_0,\varphi\to\varphi_0} \omega^- \\ &= \omega_0 \\ &= \left.\frac{B_1}{2g^{10}r_{\rm H}}\right|_{r\to r_{\rm H}}, \end{aligned} \tag{29}$$

$$\lim_{r\to r_{\rm H},\upsilon\to\upsilon_0,\theta\to\theta_0,\varphi\to\varphi_0} \Delta\omega = 0, \tag{30}$$

which shows that there exists a crossing of the particle's energy level near the event horizon, which causes the particle to have negative energy levels, but its energy is larger than minimum positive energy, it goes through a forbidden region by quantum tunnel effect and radiates out. This is the nonthermal radiation caused by the black hole's movement and is not connected to temperature, which can be called the Starobinsky–Unruh process.

5. Conclusion and discussion

Using a new tortoise coordinate transformation to solve the Klein–Gordon equation, the event horizon surface gravity κ has been obtained by taking the limitation of the coefficient of $\partial^2\Phi/\partial r_*^2$ in the Klein–Gordon equation, and the Hawking radiation has been studied. It is found that Hawking temperature T not only depends on time but also varies with angles. In addition, by discussing Hamilton–Jacobi equation in the curved spacetime, we have studied the quantum nonthermal radiation characteristics of black holes near the event horizon, the results show that there exists a crossing of a particle's energy level near the event horizon. For some negative energy particles, their energy can be larger than the energy of the nearby positive energy levels. The negative energy particle can then travel across the forbidden region because of the quantum tunneling effect and becomes a positive energy particle that can move out. In summary, we can obtain a more reasonable result if we use the improved tortoise coordinate transformation Eq. (6).

Acknowledgements

The authors would like to thank Professor Zhao Zheng for his helpful discussions.

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