Undetermination of the relation between network synchronizability and betweenness centrality*

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Betweenness centrality is taken as a sensible indicator of the synchronizability of complex networks. To test whether betweenness centrality is a proper measure of the synchronizability in specific realizations of random networks, this paper adds edges to the networks and then evaluates the changes of betweenness centrality and network synchronizability. It finds that the two quantities vary independently.

Keywords: synchronization, chaotic oscillator, complex networks, fluctuation

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The synchronization of nonlinear oscillators in complex networks has attracted increasing attention recently.[1] The master stability function (MSF) approach, in particular, is widely used for studying the effect of network topology on the stability of the synchronized state of identical chaotic oscillators in various networks.[2−7] In the framework of MSF, the synchronizability of a network is related to the spectral properties of the underlying network structure, i.e., the ratio of the largest to the smallest non-zero eigenvalues of the Laplacian matrix of the network. One remarkable finding is that the measure of the betweenness centrality may be considered as an indicator of network synchronizability. It was firstly found that the betweenness centrality closely follows the eigenratio when the heterogeneity of the network is changed.[3] To understand the relation, the synchronizability was associated with communication traffic, which tends become overloaded at the node with the maximum betweenness. Then, it was confirmed that betweenness centrality is a good indicator for network synchronizability.[4] In some recent studies on enhancing network synchronizability by weighting the edges, the betweenness–synchronizability relation was used to guide the weighting operation.[8] However, subjective extensions of numerical observations to the claimed relation between the network synchronizability and various topological characteristics may be misinterpreted,[1] and in fact an example was given in Ref. [9] showing an inconsistency between betweenness centrality and network synchronizability.

To further dispel the myth, in the present work we test the betweenness indicator of the network synchronizability from a different perspective. Existing works basically study the relation between the mean values of betweenness centrality and synchronizability that are averaged over an ensemble of network realizations. This relation has been generally considered as a statistical relation. However, for real networked systems, we often consider a specific network rather than an ensemble of them. Is the relation correct for a specific network? For the statistical study of networked systems, the question makes no sense, because macroscopic quantities are defined by statistical average over ensembles and fluctuations always occur in each specific sample. Notice that, in the study of networks, only a part of quantities are defined by statistical average, some quantities are defined for specific networks and their statistical properties largely remain unknown. In the past, both network synchronizability and betweenness centrality are defined for specific networks, while the ensemble average of these quantities remains undefined. Thus, the study of the betweenness–synchronizability relation for spe-

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cific networks are practical, as carried out in Ref. [10]. For this reason, in this paper we study the relation between network synchronizability and betweenness centrality only for specific networks; more precisely, random networks. Furthermore, we address the question that when the relation is not correct for specific networks, whether or not one can explain the deviation by a trivial fluctuation. In this study, we aim to provide more insight into the understanding of the betweenness–synchronizability relation.

In doing so, we add some new edges to specific realizations of a network model and then numerically investigate the variations of both the betweenness and the synchronizability, and finally examine their relations. Throughout this study, we consider the Erdös and Rényi (ER) random network model for the reason that most topological characteristics of this kind of networks are robust against random edge addition.

An ER random network is generated by starting with N isolated nodes and then connecting every couple of nodes with a probability $0 < p < 1$. The topology of the network is represented by the adjacency matrix $\{a_{ij}\}$ whose entry a_{ij} is equal to 1 if i and j are connected, and zero otherwise. The node betweenness centrality B_i is a measure of node i in the network, defined as the number of shortest paths traversing through node i.[11,12] It is usually referred to as load. Following Ref. [3], we normalize the load on each node by the total load of the network and denote the maximum normalized load in the network by B.

Consider an ER random network of N identical dynamical nodes, generated as above but each evolves following $\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i)$. The network is in the following form:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} L_{ij} \boldsymbol{H}(\boldsymbol{x}_j), \quad i = 1, \ldots, N, \quad (1)$$

where $\boldsymbol{H}(\boldsymbol{x})$ is the output function, σ is the coupling strength, and $L = \{L_{ij}\}$ is the Laplacian matrix defined by $L_{ij} = -a_{ij}$ if $i \neq j$, and $L_{ii} = \sum_j a_{ij}$. It is well known that the eigenvalues of L can be arranged as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$.

The linear stability of the synchronized state $\boldsymbol{x}_i \to \boldsymbol{x}_s, \forall i$, is determined by the corresponding variational equations. Using the eigenvectors of L, we can diagonalize the variational equations into N blocks of the form $\dot{\zeta} = [J\boldsymbol{F}(\boldsymbol{x}_s) - \nu J\boldsymbol{H}(\boldsymbol{x}_s)]\zeta$, where ζ represents different modes of the perturbation from the synchronized state, with $\nu = \sigma\lambda_i$ for the i-th block of the Jacobian matrix J, $i = 1, 2, \ldots, N$. If the largest Lyapunov exponent $\Lambda(\sigma\lambda_i)$ of this equation is negative for all $i = 2, \ldots, N$, then the synchronized state is locally stable. It was found[13−15] that for a large class of chaotic oscillatory systems, there exists a single parametric interval (ν_1, ν_2) on which $\Lambda(\nu) < 0$. In this case, there is a value of the coupling strength σ for which the synchronized state is linearly stable if and only if $\lambda_N/\lambda_2 = \beta$, where β is a constant depending on $\boldsymbol{F}(\cdot), \boldsymbol{x}_s$, and $\boldsymbol{H}(\cdot)$. Thus, the network's ability to synchronize is determined by a single quantity, $R \equiv \lambda_N/\lambda_2$ and the smaller the ratio, the easier for the network to synchronize.[3,13,16]

Before studying specific network realizations, we show a relation of the average eigenratio $\overline{R}$ to the average maximum load $\overline{B}$, where the bar denotes the average over a set of realizations of the ER random networks. Figure 1 shows the average eigenratio $\overline{R}$ and the average maximum load $\overline{B}$ as functions of the average degree $\langle k \rangle$ of some simulated ER random networks. As $\langle k \rangle$ increases, both $\overline{B}$ and $\overline{R}$ decrease, and $\overline{B}$ closely follows $\overline{R}$. This is consistent with the common understanding of the relation between the synchronizability and the betweenness centrality.

However, when we add an edge to a specific realization of the ER network model, the relation between eigenratio R and maximum load B can become dramatically different. In simulation, we add the new edge between two randomly chosen nodes, avoiding multiple connections. Then, the variations of eigenratio R and maximum load B induced by the new edge are recorded. The variations are denoted by $\mathrm{d}B$ and $\mathrm{d}R$, respectively. If B is proportional to R, the points $(\mathrm{d}R, \mathrm{d}B)$ form a curve with deviation attributed to the random topology. However, in Fig. 2, each data point $(\mathrm{d}R, \mathrm{d}B)$ represents the result of a numerical experiment of adding an edge to the original network. We test five realizations of the ER model with $N = 1000$ and $p = 0.01$. For each realization of the ER model, 100 times of simulations of adding one edge to the original network were performed. Figure 2 shows that the data points $(\mathrm{d}R, \mathrm{d}B)$ are randomly distributed around the origin of the coordinates plane. The result shows that when the maximum load B increases, the eigenratio R does not necessarily increase; and when B decreases, R does not necessarily decrease. The simulation disproves the hypothesis that lower maximum load B indicates smaller eigenratio R. Therefore, the betweenness centrality is not a proper indictor for synchronizability at least in such specific

realizations of random networks.

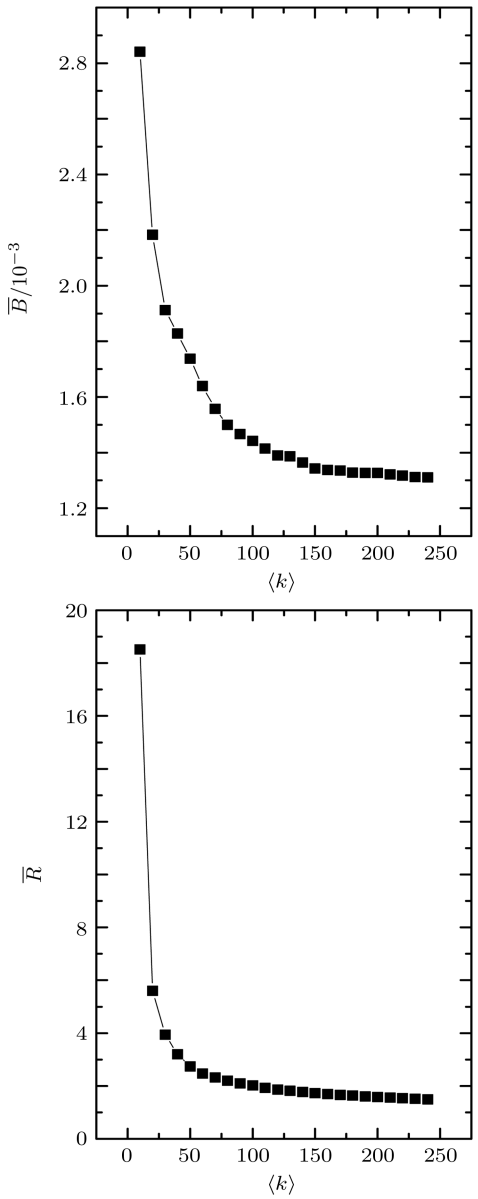


Fig. 1. Both maximum load $\overline{B}$ and eigenratio $\overline{R}$ decrease as the average degree $\langle k\rangle$ of the networks increases. The maximum load and eigenratio were averaged over 20 realizations of the ER model.

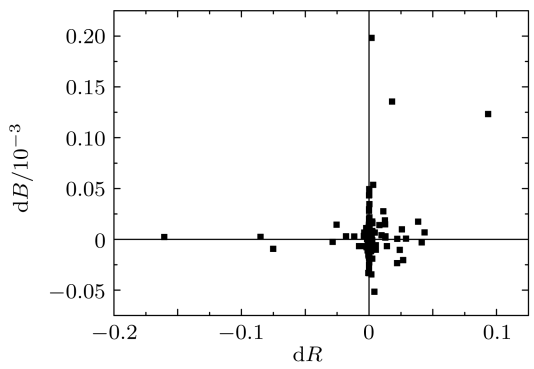


Fig. 2. The variation of maximum load B versus the variation of eigenratio R. Each point represents one result of the numerical experiment of adding one edge to a network.

Then, we carried out a case study on the trajectory of data points (R, B) generated in the case that multi-edges were sequentially added to a given network. Starting with a realization of the ER model with $N = 1000$ and $p = 0.01$, we sequentially added a total of 100 edges to it. The resultant trajectory of (R, B) is shown in Fig. 3. One can see that the points (R, B) randomly distribute in the plane as the edges are gradually added to the network. We remark that if the data points move horizontally or vertically in a step, then the eigenratio R or the maximum load B varies individually while the other remains unchanged. When the motions are not parallel to any axis, the variations of B and R are also independent each other in those steps which have similar lengths and arbitrary directions. In Fig. 3, one can see that there are some horizontal and vertical motions of obviously large displacement besides abundant random steps of small lengths.

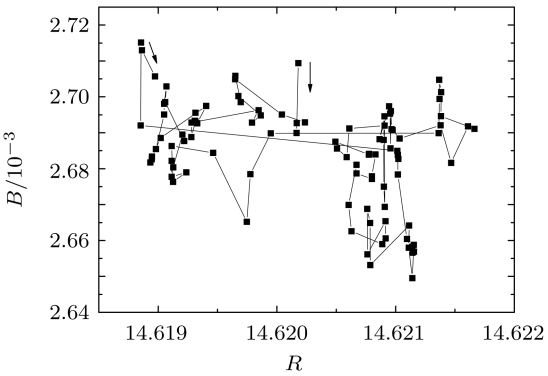


Fig. 3. The location changes of data points (R, B) in the variable plane as edges were sequentially added into a realization of the ER model. Totally one hundred edges were added.

Another example of a longer trajectory is shown in Fig. 4. It is obvious that mostly the decrease of the eigenratio is due to the horizontal motion of the points (R, B). This means that the decrease of R is essentially independent of B when the connections of the network are increased. On the other hand, there exist vertical motions in the trajectory. This means that the maximum load B can either increase or decrease independently while the eigenratio R is fixed.

It is notable that another kind of motion of the trajectory exists, besides the independent changes of R and B. In the insert of Fig. 4, a part of the trajectory is magnified. One can see that the eigenratio R increases with the maximum load B in some steps. This kind of motion is consistent with the hypothesis that the larger the maximum load B, the lower the network synchronizability. The claimed relation

of eigenratio R to maximum load B is correct in these special cases.

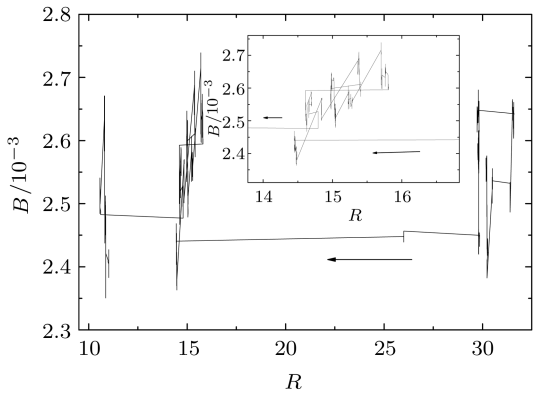


Fig. 4. The trajectory of data points (R, B) in the variable plane. Two thousands of edges were added. Inset: the magnification of a part of the trajectory.

In Ref. [4], it is shown that removing the node with the largest load induces a dramatic change of the eigenratio. The result suggests that the node with the largest load is important for the synchronizability. However, the variation of maximum load B has not been provided in Ref. [4], so the relation of the maximum load B to the eigenratio R is not clear there. Here, we investigate the variations of B and R when a new edge is attached to the node with the largest load. We randomly selected a realization of the ER model and built an edge to connect the node with the largest load to a randomly chosen node in the network. Using the same realization, we performed 100 times of simulation of adding one edge to the original network. The variations of eigenratio and maximum load $(\mathrm{d}R, \mathrm{d}B)$ are plotted in Fig. 5.

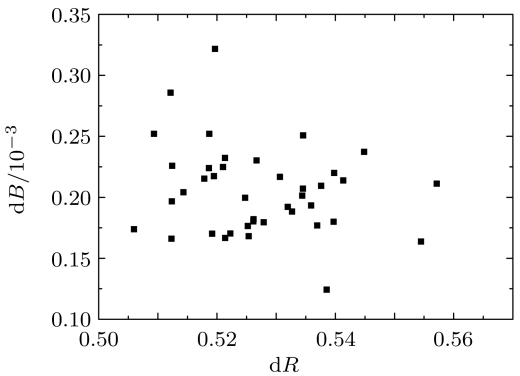


Fig. 5. The variation of the maximum load versus the variation of the eigenratio. Data come from 100 numerical experiments of adding one edge to a specific realization of the ER model.

In this simulation, when a randomly chosen node was connected to the node with the largest load, both maximum load B and eigenratio R increase. The result suggests a possible edge-adding effect due to the co-increase of B and R in the trajectory shown in Fig. 4.

The co-increase of maximum load B and eigenratio R induced by attaching a new edge to the node with the largest load, however, is not a universal property but instead, dependent on the realization of the ER model. In our simulations, for each realization of the ER model, we averaged the variations of the maximum load, $\mathrm{d}B$, and of the eigenratio, $\mathrm{d}R$, denoted by $\mathrm{d}B'$ and $\mathrm{d}R'$, respectively, over 100 experiments where the new edge is attached to the node with the largest load in the original network. We then calculated the averaged variations for 20 different network realizations. The average $\mathrm{d}B'$ and $\mathrm{d}R'$ are different from the average results shown in Fig. 2, because each quantity here is averaged over the numerical experiments performed on the same original network, rather than on an ensemble of network realizations. So, each of these average variations represents the property of a specific network realization. The corresponding data points $(\mathrm{d}R', \mathrm{d}B')$ are shown in Fig. 6.

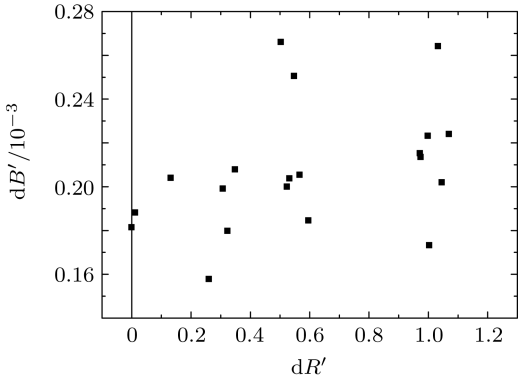


Fig. 6. The mean variation of maximum load $\mathrm{d}B'$ versus the mean variation of eigenratio $\mathrm{d}R'$. The average was taken over 100 numerical experiments for each specific network realization. Each point represents one ($\mathrm{d}R'$, $\mathrm{d}B'$) value of a network realization. Totally 20 different realizations of the ER model are presented.

The plot shows that in one part of the network realizations, the eigenratio R varies only slightly. This means that only the maximum load B is increased by adding a new edge in each of these realizations. Thus, independent changes corresponding to the horizontal motions in the variable plane (R, B) also exist when edges are attached to the node with the largest load. So, the co-increase of the maximum load B and the eigenratio R induced by connecting new edges to the node with the largest load is not necessarily a universal relation either. Therefore, the effect of the nodes with largest loads on the synchronizability is undetermined

for general random networks. Similar dependence of network properties on specific networks was also found in scale-free networks,[10] where it was shown that the largest eigenvalue of the Laplacian matrix of a scale-free network depends on the specific realization and cannot be well reflected by the average value of an ensemble of networks.

In summary, we have tested the relation of the betweenness centrality to the synchronizability in some specific random networks. We added edges to networks to induce changes of both betweenness centrality and synchronizability. We found that, contrary to the averaged quantities over an ensemble of networks, in a specific network the variations of betweenness centrality and synchronizability are essentially mutually independent. When multi-edges were sequentially added to a network, the resultant network exhibits a random-walk behaviour rather than a monotonic change in the phase space of betweenness-centrality and eigenratio variables. We observed clear independent decreases of the two quantities. Besides the existence of some steps showing the independence of betweenness centrality and eigenratio, their co-increase can also be identified from their value trajectories in the phase space, where both betweenness centrality and eigenratio increase by a large amplitude in some same steps. The co-increase relation seems to be the result of the new edge attachment to the node with largest load. The importance of the nodes with largest loads was claimed in Ref. [4], but we found that the co-increase induced by the nodes with largest loads does not appear in some network realizations. Our results actually suggest that in some network realizations, the nodes with largest loads are not effectively affecting the network synchronizability. Another notable observation, different from the well-known fluctuation, is that a relatively large decrease of either the betweenness or the eigenratio always occurs when the other is fixed.

Our main finding is that there is undetermination in the relation between the betweenness centrality and the network synchronizability, at least for general random networks. This undetermination suggests that information traffic over a network may not provide a proper explanation about the effect of the network topology on its synchronizability.

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