New synchronization analysis for complex networks with variable delays*

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This paper deals with the issue of synchronization of delayed complex networks. Differing from previous results, the delay interval $[0, d(t)]$ is divided into some variable subintervals by employing a new method of weighting delays. Thus, new synchronization criteria for complex networks with time-varying delays are derived by applying this weighting-delay method and introducing some free weighting matrices. The obtained results have proved to be less conservative than previous results. The sufficient conditions of asymptotical synchronization are derived in the form of linear matrix inequality, which are easy to verify. Finally, several simulation examples are provided to show the effectiveness of the proposed results.

Keywords: complex networks, synchronization, time-varying delay, weighting delay, linear matrix inequality

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1. Introduction

Complex network has become a focal research topic and attracted a great deal of attention in various fields in recent years. Many systems in nature can be modeled as complex networks, which are composed of a large number of highly interconnected dynamical units. Such systems include the Internet, which is a network of routers or domains; the World Wide Web, which is a network of web sites; the brain, which is a network of neurons; and an organization, which is a network of people; and so on (Refs. [1]–[7] and references therein).

Among all the issues concerning collective motion of complex networks, it has been found out that the general structure of the interaction network may play a crucial role in the emergence of synchronization phenomena in various fields, such as physics, technology and life sciences. Since there are so many interesting synchronization phenomena in the natural world, more and more researches have shown their ubiquity in many systems. For example, crickets chirping in synchrony and heart cells beating in rhythm,[8] complex networks of chaotic oscillators,[9,10] the coupled systems exhibiting spatiotemporal chaos,[11] and the array of coupled neural networks.[12−15]

On the other hand, time delays occur commonly in complex networks because of the network traffic congestions as well as the finite speed of signal transmission over the links.[16−19] The synchronization problem for various types of networks with delayed coupling has been extensively studied.[13−16] By using the definition of matrix measure, the complex dynamical networks with different time delays were studied in Ref. [20]. Tang *et al.*[21] proposed an adaptive synchronization of two complex networks with nonidentical topological structures. Lu and Qin[22] gave some cluster synchronization criteria in complex networks with coupling delays. By weighted path-lengths, Meng *et al.*[23] gave a sufficient condition for the synchronization of directed networks. In Ref. [24], the synchronization of complex network family with linearly coupled time-delayed systems was investigated. However, the time delays in their models are constant and the obtained results are independent of time delay. In Refs. [25] and [26], the synchronization of general complex dynamical networks with time-varying delays were proposed, but it is worth while noting that the delay interval $[0, d(t)]$ in their papers was usually considered as a single interval. For complex networks with time-varying delay $d(t)$, Fei *et*

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al.[27] considered networks with interval time-varying delays $[h_1, h_2]$, and divided the h_1 into m sub-intervals with the same sizes, but the method in Ref. [27] is unsuitable when h_1 is very small. Thus, how to utilize the information in $[0, d(t)]$ in order to further obtain less conservative synchronization for complex networks with time-varying delays remains an unsolved problem and a challenge.

In this paper, we shall study the synchronization problem of a general complex network with time-varying delays by a new weighting-delay-based method. Our proposed approach is to divide the delay interval $[0, d(t)]$ into smaller variable subintervals and study the synchronization based on these subintervals. Different from Ref. [27] in which the constant delay part was divided into the fixed subintervals with the same sizes, in this paper a dynamic mode is provided, in which the divided delay interval sizes are unfixed, i.e., the point or points can be chosen arbitrarily in the interval $[0, d(t)]$. Let $\alpha = \rho d(t)$ or $\alpha_i = \rho_i d(t)$ where $\rho \in (0,1)$ or $\rho_i \in (0,1)$ and $\rho_1, \rho_2, \ldots, \rho_\gamma$ is a parameter sequence satisfying $0 < \rho_1 < \rho_2 < \cdots < \rho_\gamma < 1$. By using the weighting-delay method and introducing new free weighting matrices, we can acquire some sufficient conditions and the obtained results prove to be less conservative than the existing results. Finally, our main results are illustrated through some numerical simulation examples.

Notations: R^n is the n-dimensional Euclidean space; $R^{m\times n}$ denotes a set of $m \times n$ real matrices. The notation $X > 0$ ($X < 0$), for $X \in R^{n\times n}$, means that the matrix X is real symmetric positive-definite (negative-definite). Symbol $*$ denotes the entry implied by symmetry. Matrix dimensions, if not explicitly stated, are assumed to be compatible with algebraic operations.

2. Problem formulation

Consider a delayed complex dynamical network consisting of N coupled nodes, in which each node is an n-dimensional dynamical subsystem

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} g_{ij} A x_j(t - d(t)), \quad (i = 1, 2, \ldots, N), \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in R^n$ is the state vector of the i-th node; $f(x_i(t)) \in R^n$ is a continuously differentiable vector function; constant $c > 0$ represents the coupling strength; $A = (a_{ij})_{n\times n}$ is an inner-coupling matrix; $G = (g_{ij})_{N\times N}$ represents the outer-coupling connections; time delay $d(t)$ is the time-varying differentiable function which satisfies

$$0 \le d(t) \le h, \quad \dot{d}(t) \le \mu < 1.$$

In the following discussion, some elementary situations are introduced, which plays an important role in proving the main result.

Throughout this paper, the following assumption is needed.

Assumption (H1) The outer-coupling configuration matrices of the complex network satisfy

$$\begin{cases} g_{ij} = g_{ji} \ge 0, & i \ne j, \\ g_{ii} = -\sum\limits_{j=1, j\ne i}^{n} g_{ij}, & i, j = 1, 2, \ldots, N. \end{cases}$$

Next, we give some useful definitions and lemmas.

Definition 1 Delayed dynamical network (1) is said to achieve asymptotic synchronization if

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t), \quad s \to \infty, \tag{2}$$

where $s(t)$ is a solution of an isolate node, satisfying $\dot{s}(t) = f(s(t))$.

Lemma 1[17] Let X and Y be arbitrary n-dimensional real vectors and K be an $n \times n$ positive-semidefinite matrix. $P \in R^{n\times n}$ is an arbitrary real matrix. Then, the following matrix inequality holds:

$$2X^{\mathrm{T}}PY \le X^{\mathrm{T}}PK^{-1}P^{\mathrm{T}}X + Y^{\mathrm{T}}KY.$$

Lemma 2[19] Consider delayed dynamical network (1), the eigenvalues of outer coupling matrix G are denoted by

$$0 = \lambda_1 > \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_N,$$

if the following $N - 1$ of n-dimensional time-varying delayed differential equations are asymptotically stable about their zero solution:

$$\dot{w}_k(t) = J(t)w_k(t) + c\lambda_k A w_k(t - d(t)), \tag{3}$$

where $J(t)$ is the Jacobian of $f(x(t))$ at $s(t)$, then synchronized states in expression (2) are asymptotically stable.

Lemma 3[28] The following inequalities:

$$\begin{cases} \Delta + \alpha X_1 < 0, \\ \Delta + \alpha X_2 < 0, \end{cases} \tag{4}$$

are equivalent to the following condition:

$$\Delta + \tau X_1 + (\alpha - \tau)X_2 < 0, \tag{5}$$

where X_1, X_2 and Δ are constant matrices with appropriate dimensions, variable $\tau \in [0, \alpha] \in R$ and $\alpha > 0$.

3. Main results

In this section, we are in the position to present our main results for synchronization criteria of complex networks with time-varying delays.

Theorem 1 Under H1, for given scalars $0 < \rho < 1, h > 0, \mu > 0$, if there exist positive-definite symmetric matrices P_k, Q_k, R_k, Y_k, M_k, S_{1k}, S_{2k} and any real matrices H_{1k}, H_{2k}, X_i $(i = 1, 2, \ldots, 8)$ such that the following LMIs hold for all $2 \le k \le N$:

$$\begin{bmatrix} \Omega_k + \psi + \psi^{\mathrm{T}} & hZ_{3k} & h(1-\rho)Z_{4k} \\ * & -hS_{1k} & 0 \\ * & * & -h(1-\rho)S_{2k} \end{bmatrix} < 0, \tag{6}$$

$$\begin{bmatrix} \Omega_k + \psi + \psi^{\mathrm{T}} & \rho hZ_{1k} & (1-\rho)hZ_{2k} & (1-\rho)hZ_{4k} \\ * & -\rho hS_{1k} & 0 & 0 \\ * & * & -(1-\rho)hS_{1k} & 0 \\ * & * & * & -(1-\rho)hS_{2k} \end{bmatrix} < 0, \tag{7}$$

where

$$\Omega_k = \begin{bmatrix} \Xi_{11} & 0 & c\lambda_k P_k A + c\lambda_k H_{1k} A & 0 & 0 & J(t)^{\mathrm{T}} H_{2k}^{\mathrm{T}} - H_{1k} \\ * & -(1-\rho\mu)R_k & 0 & 0 & 0 & 0 \\ * & * & -(1-\mu)Q_k & 0 & 0 & c\lambda_k A^{\mathrm{T}} H_{2k}^{\mathrm{T}} \\ * & * & * & -M_k & 0 & 0 \\ * & * & * & * & -Y_k & 0 \\ * & * & * & * & * & \Xi_{66} \end{bmatrix},$$

$$\Xi_{11} = J^{\mathrm{T}}(t)P_k + P_k J(t) + Q_k + R_k + Y_k + M_k + H_{1k}J(t) + J(t)^{\mathrm{T}} H_{1k}^{\mathrm{T}},$$
$$\Xi_{66} = hS_{1k} + h(1-\rho)S_{2k} - H_{2k}^{\mathrm{T}} - H_{2k},$$

$$\psi = \begin{bmatrix} Z_{1k} & -Z_{1k} + Z_{2k} & -Z_{2k} + Z_{3k} & Z_{4k} & -Z_{3k} - Z_{4k} & 0 \end{bmatrix},$$

$$Z_{1k} = \begin{bmatrix} X_1 & X_2 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$
$$Z_{2k} = \begin{bmatrix} 0 & X_3 & X_4 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$
$$Z_{3k} = \begin{bmatrix} 0 & 0 & X_5 & 0 & X_6 & 0 \end{bmatrix}^{\mathrm{T}},$$
$$Z_{4k} = \begin{bmatrix} 0 & 0 & 0 & X_7 & X_8 & 0 \end{bmatrix}^{\mathrm{T}},$$

then the asymptotic synchronization in expression (1) is achieved.

Proof We introduce the following new Lyapunov functional candidated for the complex networks:

$$\begin{aligned} V_k(t) = {} & w_k^{\mathrm{T}}(t)P_k w_k(t) + \int_{t-d(t)}^{t} w_k^{\mathrm{T}}(s)Q_k w_k(s)\mathrm{d}s + \int_{t-\rho h}^{t} w_k^{\mathrm{T}}(s)M_k w_k(s)\mathrm{d}s \\ & + \int_{t-\rho d(t)}^{t} w_k^{\mathrm{T}}(s)R_k w_k(s)\mathrm{d}s + \int_{t-h}^{t} w_k^{\mathrm{T}}(s)Y_k w_k(s)\mathrm{d}s + \int_{t-h}^{t}\int_{\theta}^{t} \dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s\mathrm{d}\theta \\ & + \int_{t-h}^{t-\rho h}\int_{\theta}^{t} \dot{w}_k^{\mathrm{T}}(s)S_{2k}\dot{w}_k(s)\mathrm{d}s\mathrm{d}\theta. \end{aligned} \tag{8}$$

The derivative of $V_k(t)$ is written as

$$\begin{aligned} \dot{V}_k(t) = {} & 2w_k^{\mathrm{T}}(t)P_k\left[J(t)w_k(t) + c\lambda_k A w_k(t-d(t))\right] + w_k^{\mathrm{T}}(t)Q_k w_k(t) \\ & - (1-\dot{d}(t))w_k^{\mathrm{T}}(t-d(t))Q_k w_k(t-d(t)) + w_k^{\mathrm{T}}(t)M_k w_k(t) - w_k^{\mathrm{T}}(t-\rho h)M_k w_k(t-\rho h) \\ & + w_k^{\mathrm{T}}(t)R_k w_k(t) - (1-\rho\dot{d}(t))w_k^{\mathrm{T}}(t-\rho d(t))R_k w_k(t-\rho d(t)) \end{aligned}$$

$$+ w_k^{\mathrm{T}}(t)Y_k w_k(t) - w_k^{\mathrm{T}}(t-h)Y_k w_k(t-h) + h\dot{w}_k^{\mathrm{T}}(t)S_{1k}\dot{w}_k(t)$$
$$- \int_{t-h}^{t} \dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s + h(1-\rho)\dot{w}_k^{\mathrm{T}}(t)S_{2k}\dot{w}_k(t) - \int_{t-h}^{t-\rho h} \dot{w}_k^{\mathrm{T}}(s)S_{2k}\dot{w}_k(s)\mathrm{d}s. \tag{9}$$

According to Leibniz–Newton formula, the following equations hold for arbitrary matrices Z_{1k}, Z_{2k}, Z_{3k} and Z_{4k} with appropriate dimensions:

$$2\zeta_k^{\mathrm{T}}(t)Z_{1k}\left[w_k(t) - w_k(t-\rho d(t)) - \int_{t-\rho d(t)}^{t} \dot{w}_k(s)\mathrm{d}s\right] = 0, \tag{10}$$

$$2\zeta_k^{\mathrm{T}}(t)Z_{2k}\left[w_k(t-\rho d(t)) - w_k(t-d(t)) - \int_{t-d(t)}^{t-\rho d(t)} \dot{w}_k(s)\mathrm{d}s\right] = 0, \tag{11}$$

$$2\zeta_k^{\mathrm{T}}(t)Z_{3k}\left[w_k(t-d(t)) - w_k(t-h) - \int_{t-h}^{t-d(t)} \dot{w}_k(s)\mathrm{d}s\right] = 0, \tag{12}$$

$$2\zeta_k^{\mathrm{T}}(t)Z_{4k}\left[w_k(t-\rho h) - w_k(t-h) - \int_{t-h}^{t-\rho h} \dot{w}_k(s)\mathrm{d}s\right] = 0, \tag{13}$$

where $\zeta_k^{\mathrm{T}}(t) = [w_k^{\mathrm{T}}(t), w_k^{\mathrm{T}}(t-\rho d(t)), w_k^{\mathrm{T}}(t-d(t)), w_k^{\mathrm{T}}(t-\rho h), w_k^{\mathrm{T}}(t-h), \dot{w}_k^{\mathrm{T}}(t)]$.

It is easy to obtain the following inequalities by using Lemma 1:

$$-2\zeta_k^{\mathrm{T}}(t)Z_{1k}\int_{t-\rho d(t)}^{t} \dot{w}_k(s)\mathrm{d}s \le \int_{t-\rho d(t)}^{t} \dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s + \rho d(t)\zeta_k^{\mathrm{T}}(t)Z_{1k}S_{1k}^{-1}Z_{1k}^{\mathrm{T}}\zeta_k(t), \tag{14}$$

$$-2\zeta_k^{\mathrm{T}}(t)Z_{2k}\int_{t-d(t)}^{t-\rho d(t)} \dot{w}_k(s)\mathrm{d}s \le \int_{t-d(t)}^{t-\rho d(t)} \dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s + (1-\rho)d(t)\zeta_k^{\mathrm{T}}(t)Z_{2k}S_{1k}^{-1}Z_{2k}^{\mathrm{T}}\zeta_k(t), \tag{15}$$

$$-2\zeta_k^{\mathrm{T}}(t)Z_{3k}\int_{t-h}^{t-d(t)} \dot{w}_k(s)\mathrm{d}s \le \int_{t-h}^{t-d(t)} \dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s + (h-d(t))\zeta_k^{\mathrm{T}}(t)Z_{3k}S_{1k}^{-1}Z_{3k}^{\mathrm{T}}\zeta_k(t), \tag{16}$$

$$-2\zeta_k^{\mathrm{T}}(t)Z_{4k}\int_{t-h}^{t-\rho h} \dot{w}_k(s)\mathrm{d}s \le \int_{t-h}^{t-\rho h} \dot{w}_k^{\mathrm{T}}(s)S_{2k}\dot{w}_k(s)\mathrm{d}s + (h-\rho h)\zeta_k^{\mathrm{T}}(t)Z_{4k}S_{2k}^{-1}Z_{4k}^{\mathrm{T}}\zeta_k(t). \tag{17}$$

To obtain less conservative criteria, we introduce the following equalities for arbitrary real matrices H_{1k} and H_{2k} with compatible dimensions:

$$0 = 2[w_k^{\mathrm{T}}(t)H_{1k} + \dot{w}_k^{\mathrm{T}}(t)H_{2k}][-\dot{w}_k(t) + J(t)w_k(t) + c\lambda_k A w_k(t-d(t))], \tag{18}$$

combing expressions (9)–(18), we obtain

$$\dot{V}_k(t) \le \zeta_k^{\mathrm{T}}(t)(\Omega + \psi + \psi^{\mathrm{T}} + \rho d(t)Z_{1k}S_{1k}^{-1}Z_{1k}^{\mathrm{T}} + (1-\rho)d(t)Z_{2k}S_{1k}^{-1}Z_{2k}^{\mathrm{T}}$$
$$+ (h-d(t))Z_{3k}S_{1k}^{-1}Z_{3k}^{\mathrm{T}} + (h-\rho h)Z_{4k}S_{2k}^{-1}Z_{4k}^{\mathrm{T}})\zeta_k(t). \tag{19}$$

Note that $0 \le d(t) \le h$, by making use of Lemma 3, inequality

$$\Omega + \psi + \psi^{\mathrm{T}} + \rho d(t)Z_{1k}S_{1k}^{-1}Z_{1k}^{\mathrm{T}} + (1-\rho)d(t)Z_{2k}S_{1k}^{-1}Z_{2k}^{\mathrm{T}} + (h-d(t))Z_{3k}S_{1k}^{-1}Z_{3k}^{\mathrm{T}} + (h-\rho h)Z_{4k}S_{2k}^{-1}Z_{4k}^{\mathrm{T}} < 0$$

holds if and only if inequalities

$$\Omega + \psi + \psi^{\mathrm{T}} + hZ_{3k}S_{1k}^{-1}Z_{3k}^{\mathrm{T}} + (h-\rho h)Z_{4k}S_{2k}^{-1}Z_{4k}^{\mathrm{T}} < 0 \tag{20}$$

and

$$\Omega + \psi + \psi^{\mathrm{T}} + \rho hZ_{1k}S_{1k}^{-1}Z_{1k}^{\mathrm{T}} + (1-\rho)hZ_{2k}S_{1k}^{-1}Z_{2k}^{\mathrm{T}} + (h-\rho h)Z_{4k}S_{2k}^{-1}Z_{4k}^{\mathrm{T}} < 0 \tag{21}$$

are true. From the well-known Schur complement, the proof is completed.

Remark 1 Obviously, when weighting-delay parameter ρ is set at different values, the synchronization results in Theorem 1 are also changed. Properly choosing the value of parameter ρ can lead to less conservative stability results compared with previous results. This viewpoint is demonstrated in Table 1.

Table 1. Allowable upper bound of h for different values of μ and ρ.

methods	coupling strength c	0.3	0.4	0.5
Ref. [25]	$\mu = 0$	1.345	0.950	0.731
Theorem 1	$\mu = 0$, $\rho = 0.1, 0.3, 0.5, 0.7, 0.9$	1.935	1.130	0.813
Ref. [26]	$\mu = 0.5$	1.275	0.854	0.646
Theorem 1	$\mu = 0.5, \rho = 0.1$	1.378	0.913	0.686
Theorem 1	$\mu = 0.5, \rho = 0.3$	1.377	0.912	0.686
Theorem 1	$\mu = 0.5, \rho = 0.5$	1.377	0.912	0.686
Theorem 1	$\mu = 0.5, \rho = 0.7$	1.377	0.912	0.686
Theorem 1	$\mu = 0.5, \rho = 0.9$	1.378	0.913	0.686
Ref. [26]	$\mu = 0.9$	0.970	0.711	0.562
Theorem 1	$\mu = 0.9, \rho = 0.1$	1.271	0.896	0.681
Theorem 1	$\mu = 0.9, \rho = 0.3$	1.265	0.894	0.681
Theorem 1	$\mu = 0.9, \rho = 0.5$	1.258	0.892	0.681
Theorem 1	$\mu = 0.9, \rho = 0.7$	1.262	0.894	0.681
Theorem 1	$\mu = 0.9, \rho = 0.9$	1.275	0.895	0.681

In the previous description, the time-varying delay is only divided into two subintervals $[0, \rho d(t)]$ and $[\rho d(t), d(t)]$. Naturally, the idea of introducing more subintervals should be considered. In this viewpoint, the delay interval $[0, d(t)]$ is divided into $\gamma+1$ dynamical sub-intervals, i.e., $[0, \rho_1 d(t)]$, $[\rho_1 d(t), \rho_2 d(t)]$, ..., $[\rho_\gamma d(t), d(t)]$, where $0 < \rho_1 < \rho_2 < \cdots < \rho_\gamma < 1$. That is to say, there is a parameter sequence $(\rho_1, \rho_2, \ldots, \rho_\gamma)$ satisfying the following conditions:

$$0 < \rho_1 d(t) < \rho_2 d(t) < \cdots < \rho_\gamma d(t) < d(t),$$
$$0 \leqslant \rho_i \dot{d}(t) \leqslant \rho_i \mu,$$

where $\rho_i \in (0,\ 1)$, $i = 1, 2, \ldots, \gamma$, with γ being a positive integer and $\gamma \geqslant 2$. Then, a stability criterion with multiple weighting-delays can be proposed.

Theorem 2 Dynamical system (1) is asymptotically synchronized if there exist parameters ρ_i satisfying

$$0 = \rho_0 < \rho_1 < \rho_2 < \cdots < \rho_\gamma < \rho_{\gamma+1} = 1,$$

symmetric matrices $P_k = P_k^{\mathrm{T}} > 0$, $Y_k = Y_k^{\mathrm{T}} > 0$, $S_{1k} = S_{1k}^{\mathrm{T}} > 0$, $R_{(i+1)k} = R_{(i+1)k}^{\mathrm{T}} > 0$, $i = 0, 1, 2, \ldots, \gamma$, where γ is a positive integer and arbitrary real matrices H_{1k} and H_{2k}, such that for all $2 \leq k \leq N$ the following matrix inequality holds

$$\tilde{\Omega}_k = \begin{bmatrix} \Xi_1 & \frac{1}{\rho_1 h}S_{1k} & 0 & \cdots & 0 & c\lambda_k P_k A + c\lambda_k H_{1k} A & 0 & J^{\mathrm{T}}(t)H_{2k}^{\mathrm{T}} - H_{1k} \\ * & \Xi_2 & \frac{1}{(\rho_2-\rho_1)h}S_{1k} & \cdots & 0 & 0 & 0 & 0 \\ * & * & \Xi_3 & \ddots & \vdots & 0 & 0 & 0 \\ * & * & * & \ddots & \frac{1}{(\rho_\gamma-\rho_{\gamma-1})h}S_{1k} & \vdots & \vdots & \vdots \\ * & * & * & * & \Xi_{\gamma+1} & \frac{1}{(1-\rho_\gamma)h}S_{1k} & 0 & 0 \\ * & * & * & * & * & \Xi_{\gamma+2} & \frac{1}{h}S_{1k} & c\lambda_k A^{\mathrm{T}} H_{2k}^{\mathrm{T}} \\ * & * & * & * & * & * & -Y_k - \frac{1}{h}S_{1k} & 0 \\ * & * & * & * & * & * & * & \Xi_{\gamma+4} \end{bmatrix} < 0, \quad (22)$$

with

$$\Xi_1 = J^{\mathrm{T}}(t)P_k + P_k J(t) + R_{1k} + Y_k + H_{1k}J(t) + J(t)^{\mathrm{T}}H_{1k}^{\mathrm{T}} - \frac{1}{\rho_1 h}S_{1k},$$
$$\Xi_2 = -(1-\rho_1\mu)R_{1k} + R_{2k} - \frac{1}{\rho_1 h}S_{1k} - \frac{1}{(\rho_2-\rho_1)h}S_{1k},$$
$$\Xi_3 = -(1-\rho_2\mu)R_{2k} + R_{3k} - \frac{1}{(\rho_2-\rho_1)h}S_{1k} - \frac{1}{(\rho_3-\rho_2)h}S_{1k},$$
$$\vdots$$
$$\Xi_{\gamma+1} = -(1-\rho_\gamma\mu)R_{\gamma k} + R_{(\gamma+1)k} - \frac{1}{(\rho_\gamma-\rho_{\gamma-1})h}S_{1k} - \frac{1}{(1-\rho_\gamma)h}S_{1k},$$
$$\Xi_{\gamma+2} = -(1-\mu)R_{(\gamma+1)k} - \frac{1}{(1-\rho_\gamma)h}S_{1k} - \frac{1}{h}S_{1k},$$
$$\Xi_{\gamma+4} = hS_{1k} - H_{2k} - H_{2k}^{\mathrm{T}}.$$

Proof Consider the following Lyapunov–Krasovkii functional

$$V_k(t) = w_k^{\mathrm{T}}(t)P_k w_k(t) + \sum_{i=0}^{\gamma}\int_{t-\rho_{i+1}d(t)}^{t-\rho_i d(t)} w_k^{\mathrm{T}}(s)R_{(i+1)k}w_k(s)\mathrm{d}s + \int_{t-h}^{t} w_k^{\mathrm{T}}(s)Y_k w_k(s)\mathrm{d}s$$

$$+\int_{t-h}^{t}\int_{\theta}^{t}\dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s\mathrm{d}\theta, \tag{23}$$

the derivatives of $V_k(t)$ are given as

$$\begin{aligned}\dot{V}_k(t) \le{}& 2w_k^{\mathrm{T}}(t)P_k\left[J(t)w_k(t)+c\lambda_k Aw_k(t-d(t))\right]+\sum_{i=0}^{\gamma}[w_k^{\mathrm{T}}(t-\rho_i d(t))R_{(i+1)k}w_k(t-\rho_i d(t))\\ &-(1-\rho_{i+1}\mu)w_k^{\mathrm{T}}(t-\rho_{i+1}d(t))R_{(i+1)k}w_k(t-\rho_{i+1}d(t))]+w_k^{\mathrm{T}}(t)Y_kw_k(t)\\ &-w_k^{\mathrm{T}}(t-h)Y_kw_k(t-h)+h\dot{w}_k^{\mathrm{T}}(t)S_{1k}\dot{w}_k(t)-\int_{t-h}^{t}\dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s.\end{aligned} \tag{24}$$

By Jensen's inequality, we can easily obtain

$$\begin{aligned}-\int_{t-h}^{t}\dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s ={}& -\sum_{i=0}^{\gamma}\int_{t-\rho_{i+1}d(t)}^{t-\rho_i d(t)}\dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s-\int_{t-h}^{t-d(t)}\dot{w}_k^{\mathrm{T}}(s)S_{1k}\dot{w}_k(s)\mathrm{d}s\\ \le{}& -\sum_{i=0}^{\gamma}\frac{1}{h(\rho_{i+1}-\rho_i)}[(w_k(t-\rho_i d(t))-w_k(t-\rho_{i+1}d(t)))^{\mathrm{T}}\\ &\times S_{1k}(w_k(t-\rho_i d(t))-w_k(t-\rho_{i+1}d(t))]\\ &-\frac{1}{h}(w_k(t-d(t))-w_k(t-h))^{\mathrm{T}}S_{1k}(w_k(t-d(t))-w_k(t-h)).\end{aligned} \tag{25}$$

Combining Eq. (18) with inequalities (24) and (25), we have

$$\dot{V}_k(t)\le\zeta_k^{\mathrm{T}}(t)\tilde{\Omega}_k\zeta_k(t), \tag{26}$$

where $\tilde{\Omega}_k$ is defined in inequality (22) and $\zeta_k^{\mathrm{T}}(t)=[w_k^{\mathrm{T}}(t)\ \ w_k^{\mathrm{T}}(t-\rho_1 d(t))\ \ w_k^{\mathrm{T}}(t-\rho_2 d(t))\ \cdots\ w_k^{\mathrm{T}}(t-\rho_\gamma d(t))\ \ w_k^{\mathrm{T}}(t-d(t))\ \ w_k^{\mathrm{T}}(t-h)\ \ \dot{w}_k^{\mathrm{T}}(t)]$. It is obvious to see that if $\Omega_k<0$, the asymptotic synchronization in Definition 1 is achieved.

Remark 2 For synchronization criteria in Ref. [27], the delay is assumed to vary over an interval where the lower and the upper bounds are known. By dividing the interval time-varying delay into a constant part and a time-varying part and using a delay-partitioning approach, a new criterion is obtained. But, when $h_1=0$ or very small, the method in Ref. [27] is unapplicable. In Theorem 2, by appropriately choosing the number of weighting delays and the value of weighting-delay parameter sequence, the less conservative stability results can be obtained.

4. Numerical examples

In this section, some numerical examples are given to demonstrate the usefulness of the obtained results.

Example 1 Consider a lower-dimensional network model with 5 nodes, where each node is a simple three-demensional stable linear system described as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}-x_1\\ -2x_2\\ -3x_3\end{bmatrix}, \tag{27}$$

which is asymptotically stable at $s(t)=0$ and its Jacobian is $J=\mathrm{diag}(-1,-2,-3)$. We assume that the inner-coupling matrix is $A=\mathrm{diag}(1,1,1)$ and the outer-coupling matrix is defined as

$$G=\begin{bmatrix}-2&1&0&0&1\\ 1&-3&1&1&0\\ 0&1&-2&1&0\\ 0&1&1&-3&1\\ 1&0&0&1&-2\end{bmatrix}. \tag{28}$$

Here we give two detailed computational results among different methods. In Table 1, the delay interval $[0,d(t)]$ is divided into two subintervals: $[0,\rho d(t)]$ and $[\rho d(t),d(t)]$, where $\rho d(t)$ is a weighting delay and $0<\rho<1$. In Table 2, the delay interval $[0,d(t)]$ is divided into $\gamma+1$ dynamical sub-intervals $[0,\rho_1 d(t)]$, $[\rho_1 d(t),\rho_2 d(t)],\ldots,$ $[\rho_\gamma d(t),d(t)]$ and $0<\rho_1<\rho_2<\cdots<\rho_\gamma<1$. For simplicity, we choose $\gamma=2$ in this example. From these tables, it is clear that the results in this paper are less conservative than the existing results and the weighting-delay-independent stability criteria can be proposed. To illustrate the efficiency

of our method, we plot the synchronization errors between the states of nodes in Fig. 1. In this simulation, $e_j(t) = x_{1j}(t) - x_{ij}$ for $i = 2, 3, 4, 5, j = 1, 2, 3$, with $c = 0.3$, $h = 0.8$ and $\mu = 0.5$, the initial values are randomly chosen from $[-1, 1]$.

Table 2. Allowable upper bound of h for different values of μ and ρ_i $(i = 1, 2)$ and $\gamma = 2$.

methods	coupling strength c	0.3	0.4	0.5
Ref. [25]	any μ	1.345	0.950	0.731
[19], [26]	$\mu = 0$	1.935	1.130	0.813
Theorem 2	$\mu = 0, \rho_1 = 0.1, \rho_2 = 0.9$	2.118	1.222	0.872
Ref. [26]	$\mu = 0.5$	1.275	0.854	0.646
Theorem 2	$\mu = 0.5, \rho_1 = 0.1, \rho_2 = 0.2$	1.302	0.877	0.665
Ref. [26]	$\mu = 0.9$	0.970	0.711	0.562
Theorem 2	$\mu = 0.9, \rho_1 = 0.1, \rho_2 = 0.8$	1.004	0.728	0.570

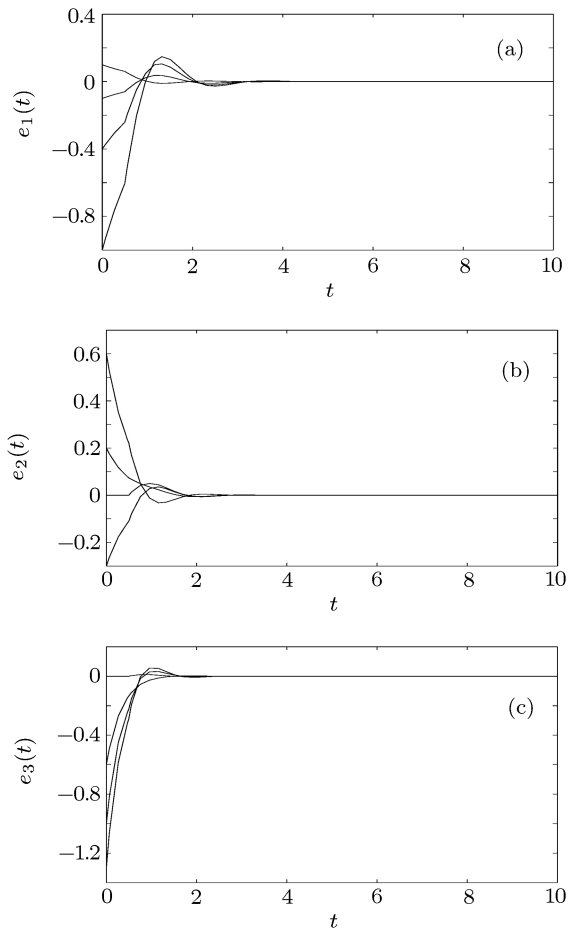


Fig. 1. Synchronization errors for networks with time-varying delays: (a) $e_1(t)$, (b) $e_2(t)$ and (c) $e_3(t)$.

Example 2 In order to further illustrate the effectiveness of our results, we choose a higher-dimensional network model with 100 nodes, where each node is a simple three-demensional stable linear system described as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -x_1 \\ -2x_2 \\ -3x_3 \end{bmatrix}, \tag{29}$$

which is asymptotically stable at $s(t) = 0$ and its Jacobian is $J = \mathrm{diag}(-1, -2, -3)$. We assume that the inner-coupling matrix is $A = \mathrm{diag}(1, 1, 1)$ and the outer-coupling matrix $G = (g_{ij})_{N\times N}, (N = 100)$ defined as

$$G = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{100\times 100}. \tag{30}$$

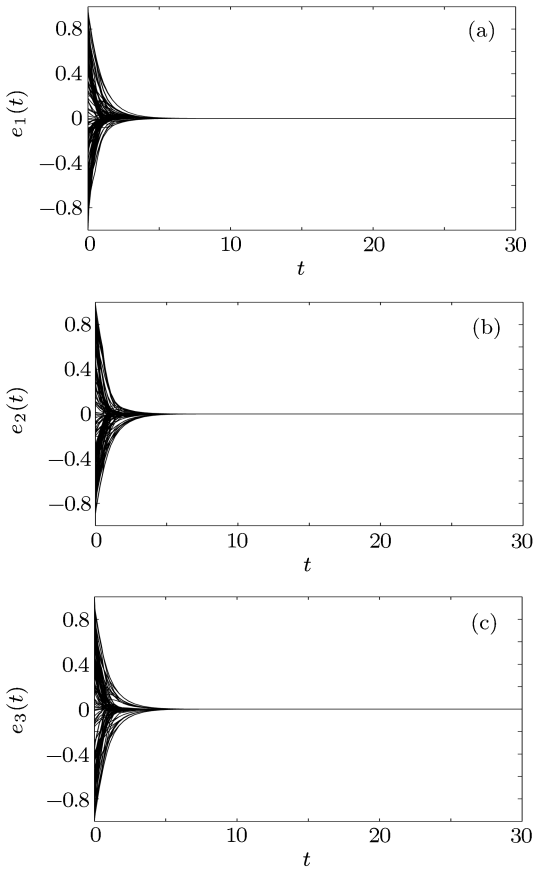


Fig. 2. Synchronization errors for networks with 100 nodes: (a) $e_1(t)$, (b) $e_2(t)$ and (c) $e_3(t)$.

In this simulation, we plot the synchronization error nodes in Fig. 2. For simplicity, synchronization

error function is described as

$$e_j(t) = x_{ij}(t) - \frac{1}{100}\sum_{k=1}^{100} x_{kj} \tag{31}$$

with $i = 1, 2, \ldots, 100$; $j = 1, 2, 3$, $c = 0.3$, $h = 0.8$, $\mu = 0.5$ and the initial values are randomly chosen from $[-1, 1]$.

5. Conclusion

In this paper, with a new method of weighting delays, we have investigated the synchronization of a general complex network. By dividing the variation interval of the time delay into some subintervals and introducing free weighting matrix, we have derived two new sufficient conditions for the addressed problem. Since more information is involved, the developed conditions are shown to be less conservative than the previous ones and have better convergence property. Furthermore, the effectiveness of these synchronization criteria is demonstrated by numerical simulations.

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