# Ion-acoustic waves in plasma of warm ions and isothermal electrons using time-fractional KdV equation

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The ion-acoustic solitary wave in collisionless unmagnetized plasma consisting of warm ions-fluid and isothermal electrons is studied using the time fractional KdV equation. The reductive perturbation method has been employed to derive the Korteweg-de Vries equation for small but finite amplitude ion-acoustic wave in warm plasma. The Lagrangian of the time fractional KdV equation is used in a similar form to the Lagrangian of the regular KdV equation with fractional derivative for the time differentiation. The variation of the functional of this Lagrangian leads to the Euler–Lagrange equation that gives the time fractional KdV equation. The variational-iteration method is used to solve the derived time fractional KdV equation. The calculations of the solution are carried out for different values of the time fractional order. These calculations show that the time fractional can be used to modulate the electrostatic potential wave instead of adding a higher order dissipation term to the KdV equation. The results of the present investigation may be applicable to some plasma environments, such as the ionosphere plasma.

**Keywords:** ion-acoustic waves, Euler–Lagrange equation, Riemann–Liouvulle fractional derivative, fractional KdV equation, variational-iteration method

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## 1. Introduction

The ion-acoustic solitary wave is one of the fundamental nonlinear wave phenomena appearing in fluid dynamics[1] and plasma physics.[2,3] There are different methods of studying nonlinear systems in plasma physics. Washimi and Taniti[4] were the first to use reductive perturbation method to study the propagation of a slow modulation of quasi-monochromatic waves through plasma. Since then many authors[5,6] have paid much attention to the subject. The evolution of small but finite-amplitude solitary waves, studied by means of the Korteweg-de Vries (KdV) equation, is of considerable interest in plasma dynamics.

It was shown that such fractional derivatives in the Lagrangian describe nonconservative forces. So, if a Lagrangian is constructed using non-integer order derivatives, the resulting equation of motion can be nonconservative.[7] Because most of classical processes observed in the physical world are nonconservative, it is important to be able to use the fractional calculus to describe such cases. Riewe[7,8] constructed a method using a Lagrangian that leads to a Euler–Lagrange equation which, in some sense is equivalent to the desired equation of motion. Further study of the fractional Euler–Lagrange can be found in the work of Agrawal,[9,10] Baleanu and coworkers,[11−13] Tarasov and Zaslavsky,[14,15] Fujioka,[16] Wu and He[17] and others. During the last few decades, fractional calculus has been applied to almost every field of science, engineering and mathematics. Some of the areas where fractional calculus has been applied include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics and control theory.[18−20]

To the authors' knowledge, the problem of time fractional KdV (TFKdV) equation in collisionless plasma has not been addressed in the literature before. So, our motivation here is to study the effects of time fractional parameter on the electrostatic structures for a system of collisionless plasma consisting of a mixture of warm ion-fluid and isothermal electrons. We expect that the inclusion of time fractional parameter will change the properties as well as the regime of existence of solitons.

Several methods have been used to solve fractional differential equations such as, the Laplace transform method, the Fourier transform method, the iteration method and the operational method.[21,22] Re-

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cently, there have been various papers dealing with the existence and multiplicity of solution of nonlinear fractional differential equation by using techniques of nonlinear analysis.[23−30] In the present paper, the resulting TFKdV equation will be solved using a variational-iteration method (VIM) firstly used by He.[31]

The rest of the present paper is organized as follows. In Section 2 the formulation of the TFKdV equation is described using the variational Euler–Lagrange method. In Section 3, VIM is discussed, and the resulting TFKdV equation is solved approximately using VIM. In Section 5 the calculation results and their discussion are presented.

## 2. Basic equations and KdV equation

Consider collisionless ionization-free unmagnetized plasma consisting of a mixture of warm ions-fluid and isothermal electrons. The basic equations, which describe the system in dimensionless variables, are represented by[3]

$$\frac{\partial}{\partial t}n(x,t)+\frac{\partial}{\partial x}[n(x,t)u(x,t)]=0, \tag{1a}$$

$$\frac{\partial}{\partial t}u(x,t)+u(x,t)\frac{\partial}{\partial x}u(x,t)+\frac{\sigma}{n(x,t)}\frac{\partial}{\partial x}p(x,t)+\frac{\partial}{\partial x}\phi(x,t)=0, \tag{1b}$$

$$\frac{\partial}{\partial t}p(x,t)+u(x,t)\frac{\partial}{\partial x}p(x,t)+3p(x,t)\frac{\partial}{\partial x}u(x,t)=0, \tag{1c}$$

$$\frac{\partial^2}{\partial x^2}\phi(x,t)+n(x,t)-n_{\rm e}(x,t)=0, \tag{1d}$$

$$n_{\rm e}(x,t)-\exp[\phi(x,t)]=0. \tag{1e}$$

In these earlier equations, $n(x,t)$ and $n_{\rm e}(x,t)$ are the densities of ions and electrons, respectively, $u(x,t)$ is the ions flow velocity, $p(x,t)$ is the ions pressure, $\phi(x,t)$ is the electrostatic potential, $x$ is the space coordinate and $t$ is the time variable. $\sigma=T_{\rm i}/T_{\rm e}\ll 1$ is the ratio of ion temperature $T_{\rm i}$ to electron temperature $T_{\rm e}$. All of these quantities are dimensionless and normalized in terms of the following characteristic quantities: $n(x,t)$ and $n_{\rm e}(x,t)$ are normalized by $n_0$, the unperturbed electron density; $u(x,t)$ is normalized by the sound velocity $\sqrt{k_{\rm B}T_{\rm e}/m_{\rm i}}$, $p(x,t)$ and $\phi(x,t)$ are normalized by $n_0k_{\rm B}T_{\rm i}$ and $k_{\rm B}T_{\rm e}/e$, respectively; $t$ and $x$ are normalized by the inverse of the plasma frequency $\omega_{\rm pi}^{-1}=(\sqrt{4\pi\,e^2n_0/m_{\rm i}})^{-1}$ and the electron Debye length $\lambda_D=\sqrt{k_{\rm B}T_{\rm e}/(4\pi\,e^2n_0)}$, respectively; $k_{\rm B}$ is the Boltzmann's constant; $m_{\rm i}$ is the mass of plasma ion. According to the general method of reductive perturbation theory,[4] stretched variables are introduced as

$$\tau=\varepsilon^{3/2}t,\quad \xi=\varepsilon^{1/2}(x-vt), \tag{2}$$

where $\varepsilon$ represents the amplitude of the perturbation and $v$ is a normalized phase velocity of the wave. All the physical quantities appearing in Eq. (1) are expanded as power series in $\varepsilon$ about the equilibrium value as

$$n(\xi,\tau)=1+\varepsilon n_1(\xi,\tau)+\varepsilon^2n_2(\xi,\tau)+\cdots, \tag{3a}$$

$$u(\xi,\tau)=\varepsilon u_1(\xi,\tau)+\varepsilon^2u_2(\xi,\tau)+\cdots, \tag{3b}$$

$$p(\xi,\tau)=1+\varepsilon p_1(\xi,\tau)+\varepsilon^2p_2(\xi,\tau)+\cdots, \tag{3c}$$

$$\phi(\xi,\tau)=\varepsilon\phi_1(\xi,\tau)+\varepsilon^2\phi_2(\xi,\tau)+\cdots. \tag{3d}$$

The boundary conditions are imposed as $|\xi|\to\infty$, $n=n_{\rm e}=p=1$ and $u=\phi=0$. Substituting Eqs. (2) and (3) into the equations in expression (1) and equating the coefficients of like powers of $\varepsilon$ lead to a set of equations. From the lowest, second-order equations in $\varepsilon$ by eliminating the second order perturbed quantities $n_2(\xi,\tau)$, $u_2(\xi,\tau)$, $p_2(\xi,\tau)$ and $\phi_2(\xi,\tau)$, a KdV equation for the first-order perturbed potential is obtained as

$$\frac{\partial}{\partial\tau}\phi_1(\xi,\tau)+A\phi_1(\xi,\tau)\frac{\partial}{\partial\xi}\phi_1(\xi,\tau)+B\frac{\partial^3}{\partial\xi^3}\phi_1(\xi,\tau)=0, \tag{4a}$$

where

$$A=\frac{6\sigma+1}{\sqrt{3\sigma+1}}\quad\text{and}\quad B=\frac{1}{2\sqrt{3\sigma+1}}. \tag{4b}$$

In Eq. (4a), $\phi_1(\xi,\tau)$ is the field variable, $\xi\in R$ is a space coordinate in the propagation direction of the field and $\tau\in T\ (=[0,T_0])$ is a time variable.

The resulting KdV equation (4a) can be converted into TFKdV equation as follows:[28]

Using a potential function $V(\xi,\tau)$ where $\phi_1(\xi,\tau)=V_\xi(\xi,\tau)$ gives the potential equation of the regular KdV equation (4a) in the form

$$V_{\xi\tau}(\xi,\tau)+AV_\xi(\xi,\tau)v_{\xi\xi}(\xi,\tau)+BV_{\xi\xi\xi\xi}(\xi,\tau)=0, \tag{5}$$

where the subscripts denote the partial differentiation of the function with respect to the parameter.

The Lagrangian of the regular KdV equation (4a) can be defined using the semi-inverse method[32,33] as follows:

The functional of the potential equation (5) can be represented by

$$J(V)=\int_R{\rm d}\xi\int_T{\rm d}\tau\{V(\xi,\tau)[c_1V_{\xi\tau}(\xi,\tau)$$

$$+ c_2 A V_\xi(\xi,\tau) v_{\xi\xi}(\xi,\tau) + c_3 B V_{\xi\xi\xi\xi}(\xi,\tau)]\}, \quad (6)$$

where $c_1$, $c_2$ and $c_3$ are constants to be determined. Integrating by parts and taking $V_\tau|_R = V_\xi|_R = V_\xi|_T = 0$ lead to

$$J(V) = \int_R \mathrm{d}\xi \int_T \mathrm{d}\tau \{V(\xi,\tau)[-c_1 V_\xi(\xi,\tau) V_\tau(\xi,\tau) - \frac{1}{2} c_2 A V_\xi^3(\xi,\tau) + c_3 B V_{\xi\xi}^2(\xi,\tau)]\}. \quad (7)$$

The unknown constants $c_i$ $(i = 1, 2, 3)$ can be determined by taking the variation of the functional (7) to make it optimal. Taking the variation of this functional, integrating each term by parts and make the variation optimum give the following relation:

$$2c_1 V_{\xi\tau}(\xi,\tau) + 3c_2 A V_\xi(\xi,\tau) V_{\xi\xi}(\xi,\tau) + 2c_3 B V_{\xi\xi\xi\xi}(\xi,\tau) = 0. \quad (8)$$

As this equation must be equal to Eq. (5), the unknown constants are given as

$$c_1 = 1/2, \quad c_2 = 1/3 \ \text{ and } \ c_3 = 1/2. \quad (9)$$

Therefore, the functional given by Eq. (7) gives the Lagrangian of the regular KdV equation as

$$L(V_\tau, V_\xi, V_{\xi\xi}) = -\frac{1}{2} V_\xi(\xi,\tau) V_\tau(\xi,\tau) - \frac{1}{6} A V_\xi^3(\xi,\tau) + \frac{1}{2} B V_{\xi\xi}^2(\xi,\tau). \quad (10)$$

Similar to this form, the Lagrangian of the time-fractional version of the KdV equation can be written in the form

$$F({}_0D_\tau^\alpha V, V_\xi, V_{\xi\xi}) = -\frac{1}{2}[{}_0D_\tau^\alpha V(\xi,\tau)] V_\xi(\xi,\tau) - \frac{1}{6} A V_\xi^3(\xi,\tau) + \frac{1}{2} B V_{\xi\xi}^2(\xi,\tau), \quad 0 \le \alpha < 1, \quad (11)$$

where the fractional derivative is represented, using the left Riemann–Liouville fractional derivative definition as[21,22]

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(k-\alpha)} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[ \int_a^t \mathrm{d}\tau (t-\tau)^{k-\alpha-1} f(\tau) \right], \quad k-1 \le \alpha \le k, \quad t \in [a, b]. \quad (12)$$

The functional of the TFKdV equation can be represented in the form

$$J(V) = \int_R \mathrm{d}\xi \int_T \mathrm{d}\tau \, F({}_0D_\tau^\alpha V, V_\xi, V_{\xi\xi}), \quad (13)$$

where the time-fractional Lagrangian $F({}_0D_\tau^\alpha V, V_\xi, V_{\xi\xi})$ is defined by Eq. (11).

Following Agrawal's method,[9,10] the variation of functional (13) with respect to $V(\xi,\tau)$ leads to

$$\delta J(V) = \int_R \mathrm{d}\xi \int_T \mathrm{d}\tau \left\{ \frac{\partial F}{\partial\, {}_0D_\tau^\alpha V} \delta\, {}_0D_\tau^\alpha V + \frac{\partial F}{\partial V_\xi} \delta V_\xi + \frac{\partial F}{\partial V_{\xi\xi}} \delta V_{\xi\xi} \right\}. \quad (14)$$

The formula for fractional integration by part reads[21,22]

$$\int_a^b \mathrm{d}t f(t) {}_aD_l^\alpha g(t) = \int_a^b \mathrm{d}t g(t) {}_tD_b^\alpha f(t), \quad f(t),\, g(t) \in [a,\ b], \quad (15)$$

where ${}_tD_b^\alpha$, the right Riemann–Liouville fractional derivative, is defined by[21,22]

$${}_tD_b^\alpha f(t) = \frac{(-1)^k}{\Gamma(k-\alpha)} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[ \int_t^b \mathrm{d}\tau (\tau - t)^{k-\alpha-1} f(\tau) \right], \quad k-1 \le \alpha \le k, \quad t \in [a, b]. \quad (16)$$

Integrating the right-hand side of Eq. (14) by part using formula (15) leads to

$$\delta J(V) = \int_R \mathrm{d}\xi \int_T \mathrm{d}\tau \left[ {}_\tau D_{T_0}^\alpha \left( \frac{\partial F}{\partial_0 D_\tau^\alpha V} \right) - \frac{\partial}{\partial \xi} \left( \frac{\partial F}{\partial V_\xi} \right) + \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial F}{\partial V_{\xi\xi}} \right) \right] \delta V, \quad (17)$$

where it is assumed that $\delta V|_T = \delta V|_R = \delta V_\xi|_R = 0$.

Optimizing this variation of the functional $J(V)$, i.e, $\delta J(V) = 0$, gives the Euler–Lagrange equation for the TFKdV equation in the form

$${}_\tau D_{T_0}^\alpha \left( \frac{\partial F}{\partial_0 D_\tau^\alpha V} \right) - \frac{\partial}{\partial \xi} \left( \frac{\partial F}{\partial V_\xi} \right) + \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial F}{\partial V_{\xi\xi}} \right) = 0. \quad (18)$$

Applying the Lagrangian of the TFKdV equation (11) to this Euler–Lagrange formula (18) gives

$$-\frac{1}{2}\, {}_\tau D_{T_0}^\alpha V_\xi(\xi,\tau) + \frac{1}{2}\, {}_0D_\tau^\alpha V_\xi(\xi,\tau) + A V_\xi(\xi,\tau) V_{\xi\xi}(\xi,\tau) + B V_{\xi\xi\xi\xi}(\xi,\tau) = 0. \quad (19)$$

Substituting for the potential function, $V_\xi(\xi,\tau) = \phi_1(\xi,\tau) = \Phi(\xi,\tau)$, gives the TFKdV equation for the state function $\Phi(\xi,\tau)$ in the form

$$\frac{1}{2}[\, {}_0D_\tau^\alpha \Phi(\xi,\tau) - {}_\tau D_{T_0}^\alpha \Phi(\xi,\tau)] + A \Phi(\xi,\tau) \Phi_\xi(\xi,\tau) + B \Phi_{\xi\xi\xi}(\xi,\tau) = 0, \quad (20)$$

where the fractional derivatives ${}_0D_\tau^\alpha$ and ${}_\tau D_{T_0}^\alpha$ are, respectively the left and the right Riemann–Liouville fractional derivatives and are defined by Eqs. (12) and (16).

The TFKdV equation represented in Eq. (20) can be rewritten as the following formula:

$$ {}_0^R D_\tau^\alpha \Phi(\xi,\tau) + A\Phi(\xi,\tau)\Phi_\xi(\xi,\tau) + B\Phi_{\xi\xi\xi}(\xi,\tau) = 0, \tag{21} $$

where the fractional operator ${}_0^R D_\tau^\alpha$ is called Riesz fractional derivative and can be represented by[21,22]

$$ \begin{aligned} {}_0^R D_t^\alpha f(t) &= \frac{1}{2}[{}_0 D_t^\alpha f(t) + (-1)_t^k D_{T_0}^\alpha f(t)] \\ &= \frac{1}{2}\frac{1}{\Gamma(k-\alpha)}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\int_\alpha^t \mathrm{d}\tau |t-\tau|^{k-\alpha-1} f(\tau)\right], \\ & k-1 \le \alpha \le k, \quad t \in [a,b]. \end{aligned} \tag{22} $$

The nonlinear fractional differential equations have been solved using different techniques.[23–30] In this paper, a variational-iteration method, first given by He[31] is used to solve the TFKdV equation that is formulated using Euler–Lagrange variational technique.

## 3. Variational-iteration method

Variational-iteration method[31] has been used successfully to solve different types of integer nonlinear differential equations.[34,35] Also, VIM is used to solve linear and nonlinear fractional differential equations.[23–30] This VIM is used in this paper to solve the formulated TFKdV equation.

The basic features of the VIM are as follows: the solution of the linear mathematical problem or the initial (boundary) condition of the nonlinear problem is used as the initial approximation or trail function. A more highly accurate approximation can be obtained using iteration correction functional. Consider a nonlinear partial differential equation that consists of a linear part $\hat{L}U(x,t)$, a nonlinear part $\hat{N}U(x,t)$ and a free term $f(x,t)$ represented as

$$ \hat{L}U(x,t) + \hat{N}U(x,t) = f(x,t), \tag{23} $$

where $\hat{L}$ is the linear operator and $\hat{N}$ is the nonlinear operator. According to the VIM, the $(n+1)$-th approximation solution of Eq. (23) can be given by the iteration correction functional as[31]

$$ \begin{aligned} U_{n+1}(x,t) = U_n(x,t) + \int_0^t \mathrm{d}\tau\lambda(\tau)[\hat{L}U_n(x,\tau) \\ + \hat{N}\tilde{U}_n(x,\tau) - f(x,\tau)], \quad n \ge 0, \end{aligned} \tag{24} $$

where $\lambda(\tau)$ is a Lagrangian multiplier and $\tilde{U}_n(x,\tau)$ is considered as a restricted variation function, i.e, $\delta\tilde{U}_n(x,\tau) = 0$. The extreme variation of correction functional (24) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration can be used as the solution of the linear part of Eq. (23) or the initial value $U(x,0)$. As $n$ tends to infinity, the iteration leads to the exact solution of Eq. (23), i.e.

$$ U(x,t) = \lim_{n\to\infty} U_n(x,t). \tag{25} $$

## 4. Solution of TFKdV equation

The FKdV equation in space and time has been solved using the VIM by Momani *et al.*[25] The TFKdV equation represented by Eq. (21) will be solved using the VIM but in slightly different manner from that of Momani *et al.* The iteration correction functional (24) is used to solve Eq. (21) as follows:[28]

Acting from the left on Eq. (21) by the fractional operator ${}_0^R D_\tau^{1-\alpha}$ leads to

$$ \begin{aligned} \frac{\partial}{\partial\tau}\Phi(\xi,\tau) = {}_0^R D_\tau^{\alpha-1}\Phi(\xi,\tau)|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} - {}_0^R D_\tau^{1-\alpha}\left[A\Phi(\xi,\tau)\frac{\partial}{\partial\xi}\Phi(\xi,\tau) + B\frac{\partial^3}{\partial\xi^3}\Phi(\xi,\tau)\right], \\ 0 \le \alpha \le 1, \quad \tau \in [0,T_0], \end{aligned} \tag{26} $$

where the following fractional derivative property is used:[21,22]

$$ {}_a^R D_b^\alpha [{}_a^R D_b^\beta f(t)] = {}_a^R D_b^{\alpha+\beta} f(t) - \sum_{j=1}^k {}_a^R D_b^{\beta-j} f(t)|_{t=a}\frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \quad k-1 \le \beta < k. \tag{27} $$

As $\alpha < 1$, the Riesz fractional derivative ${}_0^R D_\tau^{\alpha-1}$ is considered as Riesz fractional integral ${}_0^R I_\tau^{1-\alpha}$ that is defined as[21,22]

$$ {}_0^R I_\tau^\alpha f(t) = \frac{1}{2}[{}_0 I_\tau^\alpha f(t) +_\tau I_b^\alpha f(t)] = \frac{1}{2}\frac{1}{\Gamma(\alpha)}\int_a^b \mathrm{d}\tau |t-\tau|^{\alpha-1} f(\tau), \quad \alpha > 0, \tag{28} $$

where ${}_0 I_\tau^\alpha f(t)$ and ${}_\tau I_b^\alpha f(t)$ are the left and the right Riemann–Liouvulle fractional integrals, respectively.[21,22]

The iterative correction functional of Eq. (26) is given as

$$\Phi_{n+1}(\xi,\tau) = \Phi_n(\xi,\tau) + \int_0^\tau \mathrm{d}\tau' \lambda(\tau') \Big\{ \frac{\partial}{\partial\tau'} \Phi_n(\xi,\tau') - {}_0^R I_{\tau'}^{1-\alpha} \Phi_n(\xi,\tau') \Big|_{\tau'=0} \frac{\tau'^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^R D_\tau'^{1-\alpha} \Big[ A\tilde{\Phi}_n(\xi,\tau') \frac{\partial}{\partial\xi} \tilde{\Phi}_n(\xi,\tau') + B \frac{\partial^3}{\partial\xi^3} \tilde{\Phi}_n(\xi,\tau') \Big] \Big\}, \quad n \geq 0, \tag{29}$$

where function $\tilde{\Phi}_n(\xi,\tau)$ is considered as a restricted variation function, i.e. $\delta\tilde{\Phi}(\xi,\tau) = 0$. The extreme variation of Eq. (29) using the restricted variation function leads to

$$\begin{aligned} \delta\Phi_n + 1(\xi,\tau) &= \delta\Phi_n(\xi,\tau) + \int_0^\tau \mathrm{d}\tau' \lambda(\tau') \delta \frac{\partial}{\partial\tau'} \Phi_n(\xi,\tau') \\ &= \delta\Phi_n(\xi,\tau) + \lambda(\tau)\delta\Phi_n(\xi,\tau) - \int_0^\tau \mathrm{d}\tau' \frac{\partial}{\partial\tau'} \lambda(\tau')\delta\Phi_n(\xi,\tau') = 0. \end{aligned}$$

This relation leads to the stationary conditions $1 + \lambda(\tau) = 0$ and $\frac{\partial}{\partial\tau'}\lambda(\tau') = 0$, which leads to the Lagrangian multiplier as $\lambda(\tau') = -1$. Therefore, the correction functional (29) is given in the form

$$\Phi_{n+1}(\xi,\tau) = \Phi_n(\xi,\tau) - \int_0^\tau \mathrm{d}\tau' \Big\{ \frac{\partial}{\partial\tau'} \Phi_n(\xi,\tau') - {}_0^R I_\tau'^{1-\alpha} \Phi_n(\xi,\tau') \Big|_{\tau'=0} \frac{\tau'^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^R D_\tau'^{1-\alpha} \Big[ A\Phi_n(\xi,\tau') \frac{\partial}{\partial\xi} \Phi_n(\xi,\tau') + B \frac{\partial^3}{\partial\xi^3} \Phi_n(\xi,\tau') \Big] \Big\}, \quad n \geq 0. \tag{30}$$

In physics, if $\tau$ denotes the time-variable, the right Riemann–Liouville fractional derivative is interpreted as a future state of the process. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development.[9] Therefore, the right-derivative is equal to zero in the following calculations.

The zero order correction of the solution can be taken as the initial value of the state variable, which is taken in this case as

$$\Phi_0(\xi,\tau) = \Phi(\xi,0) = A_0 \operatorname{sech}^2(c\xi), \tag{31}$$

where $A_0 = 3v/A$ and $c = \frac{1}{2}\sqrt{v/B}$ are constants.

Substituting this zero order approximation into Eq. (30) and using the definition of the fractional derivative (22) lead to the first order approximation as

$$\begin{aligned} \Phi_1(\xi,\tau) = {} & A_0 \operatorname{sech}^2(c\xi) + 2A_0 c \sinh(c\xi) \operatorname{sech}^3(c\xi) \\ & \times [4c^2B + (A_0A - 12c^2B) \operatorname{sech}^2(c\xi)] \\ & \times \frac{\tau^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{32}$$

Substituting this equation into Eq. (30), using definition (22) and the Maple package lead to the second order approximation in the form

$$\begin{aligned} \Phi_2(\xi,\tau) = {} & A_0 \operatorname{sech}^2(c\xi) + 2A_0 c \sinh(c\xi) \operatorname{sech}(c\xi)^3 [4c^2B + (A_0A - 12c^2B) \operatorname{sech}(c\xi)^2] \frac{\tau^\alpha}{\Gamma(\alpha+1)} \\ & + 2A_0c^2 \operatorname{sech}(c\xi)^2 [32c^4B^2 + 16c^2B(5A_0A - 63c^2B) \operatorname{sech}(c\xi)^2 \\ & + 2(3A_0^2A^2 - 176A_0c^2AB + 1680c^4B^2) \operatorname{sech}(c\xi)^4 \\ & - 7(A_0^2A^2 - 42A_0c^2AB + 360c^4B^2) \operatorname{sech}(c\xi)^6] \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} \\ & + 4A_0^2c^3 \sinh(c\xi) \operatorname{sech}(c\xi)^5 [32c^4B^2 + 24c^2B(A_0A - 14c^2B) \operatorname{sech}(c\xi)^2 \\ & + 4(A_0^2A^2 - 32A_0c^2AB + 240c^4B^2) \operatorname{sech}(c\xi)^4 \\ & - 5(A_0^2A^2 - 24A_0c^2AB + 144c^4B^2) \operatorname{sech}(c\xi)^6] \frac{\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2} \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \tag{33}$$

The higher order approximations can be calculated using the Maple or the Mathematica package to the appropriate order where the infinite approximation leads to the exact solution.

## 5. Results and discussion

For a small but finite amplitude of ion-acoustic solitary wave in unmagnetized collisionless plasma consisting of warm ion-fluid and isothermal electrons, we have obtained the Korteweg-de Vries equation by using the reductive perturbation method.[4] The TFKdV equation has been derived using Agrawal's method.[9,10] The time fractional operator in the TFKdV equation is derived in terms of the Riesz differential operator.[28] He's variational-iteration method[31] is used to solve the derived TFKdV equation.

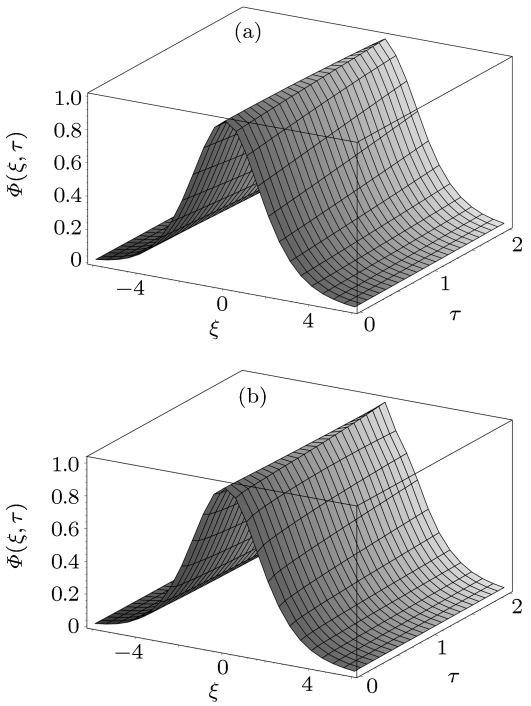



**Fig. 1.** Profiles for electrostatic potential $\Phi(\xi, \tau)$ against position $\xi$ and time $\tau$ at $\sigma = 0.05$ and $v = 0.4$ for (a) $\alpha = 1.0$ and (b) $\alpha = 0.5$.

Our motivations are to study the effects of the ratio of ion temperature to the electron temperature $\sigma$ and the time fractional order $\alpha$ on the existence of solitary waves in warm plasma. In Fig. 1 profiles of the solution of the TFKdV equation each as a solitary pulse are obtained for two values of the fractional order $\alpha$ =1 and $\alpha$ =1/2. This shows that the present system admits a solitary wave solution for any time fractional order approximation.

Figure 2 shows that both the amplitude and the width of the solitary wave decrease with the increase of temperature ratio $\sigma$. The effects of the time fractional order $\alpha$ on the soliton amplitude and width are shown in Figs. 3 and 4.

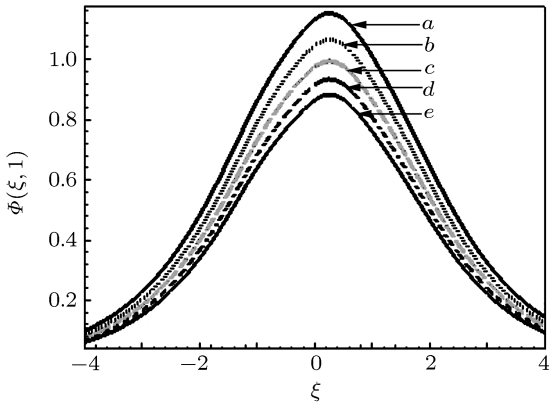



**Fig. 2.** Curves for electrostatic potential $\Phi(\xi, \tau)$ against position $\xi$ at $\tau = 1$, $v = 0.4$ and $\alpha = 0.5$ for different values of temperature ratio $\sigma$: curve $a$ for $\sigma = 0.01$, curve $b$ for $\sigma = 0.03$, curve $c$ for $\sigma = 0.05$, curve $d$ for $\sigma = 0.07$ and curve $e$ for $\sigma = 0.09$.

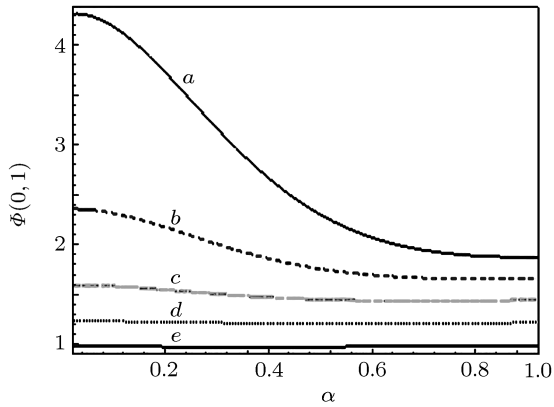



**Fig. 3.** Curves for electrostatic potential amplitude $\Phi(0, \tau)$ against fractional order $\alpha$ at $\tau = 1$ and $\sigma = 0.05$ for different values of velocity $v$: curve $a$ for $v = 0.40$, curve $b$ for $v = 0.50$, curve $c$ for $v = 0.60$, curve $d$ for $v = 0.70$ and curve $e$ for $v = 0.80$.

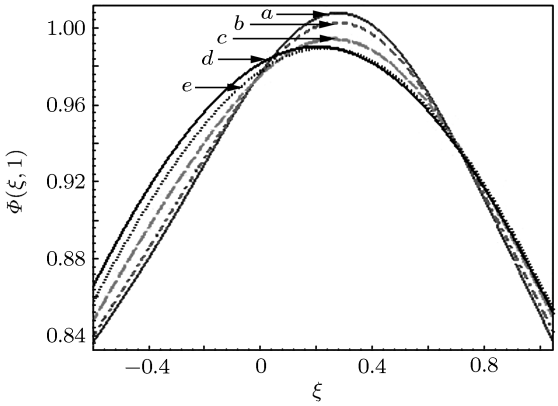



**Fig. 4.** Curves for electrostatic potential $\Phi(\xi, \tau)$ against position $\xi$ at $\tau = 1$, $v = 0.4$ and $\sigma = 0.05$ for different values of the fractional order $\alpha$: curve $a$ for $\alpha = 1$, curve $b$ for $\alpha = 3/4$, curve $c$ for $\alpha = 1/2$, curve $d$ for $\alpha = 1/3$ and curve $e$ for $\alpha = 1/4$.

These figures show that the increase of the fractional parameter $\alpha$ causes the amplitude of the electrostatic potential solitary wave to decrease while leading the width of the wave to increase. This feature leads to the fact that the time fractional differentiation can be used to modulate the electrostatic potential wave instead of adding a higher order dissipation term to the KdV equation.[5,6]

In summery, it has been found that the amplitude of the ion-acoustic wave and the parametric regime where the solitons can exist are sensitive to the temperature ratio $\sigma$ and time fractional order $\alpha$. Moreover, the time fractional order $\alpha$ plays a role of higher order perturbation in modulating the soliton shape. The application of our model might be interesting particularly in some plasma environments, such as the ionosphere plasma.

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