Pseudo analytical solution to time periodic stiffness systems*

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An analytical form of state transition matrix for a system of equations with time periodic stiffness is derived in order to solve the free response and also allow for the determination of system stability and bifurcation. A pseudo-closed form complete solution for parametrically excited systems subjected to inhomogeneous generalized forcing is developed, based on the Fourier expansion of periodic matrices and the substitution of matrix exponential terms via Lagrange–Sylvester theorem. A Mathieu type of equation with large amplitude is presented to demonstrate the method of formulating state transition matrix and Floquet multipliers. A two-degree-of-freedom system with irregular time periodic stiffness characterized by spiral bevel gear mesh vibration is presented to find forced response in stability and instability. The obtained results are presented and discussed.

Keywords: parametric excitation, time periodic stiffness, stability, response

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1. Introduction

Dynamical systems with time periodic stiffness in real-world engineering problems, such as gear mesh vibrations, yield-unloading in elastoplastic structures and cutting-tool flutters in mechanical process, can be characterized as parametrically excited equations in abundant physical background and various mathematical forms.[1−3] It has been demonstrated that Floquet theorem is practicable to a system of equations with large amplitude parameters and the state transition matrix is a key function in evaluation of system response, stability and bifurcation.[4,5] Butcher *et al.*[6−8] have introduced a hybrid formulation for the analysis of system of equations with periodic coefficients via Chebyshev polynomials. A parametrically excited oscillator with strong nonlinearity characterized by a periodic parameter with large amplitude has been studied for static bifurcation by Tang, Fu *et al.*[9,10] The pioneering work in real-world engineering problems can be traced to Singh, Lim, Parker *et al.*[11−15] Dynamics of gears pair with considering the effects of multiform time-varying mesh stiffness in spur gear, helical gear, planetary gear and hypoid geared systems has been presented in their work. Another type of parametrically excited system with time periodic stiffness derived from driving system of heavy and complex rotary machines has been developed by Dong, Zhao and Wang *et al.*[16−22]

In the present paper, we examine a pseudo-closed form complete solution depending on state transition matrix, for parametrically excited systems involve external generalized forcing. State transition matrix is considered, afterwards focused on time periodic stiffness systems, to solve for the complete response and also allow for determination of stability and bifurcation. After the techniques have been introduced, a Mathieu type of equation and a two-degree-of-freedom system with irregular time periodic parameter are presented.

2. Forced response of parametrically excited system

Consider a linear system of equations with periodic coefficients

$$\dot{\boldsymbol{d}}(t) = \boldsymbol{A}(t)\,\boldsymbol{d}(t) + \boldsymbol{f}(t), \tag{1}$$

where $\boldsymbol{A}(t)$ is a $2N \times 2N$ continuous matrix with a period of T, $\boldsymbol{d}(t) = [\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)]^{\mathrm{T}}$, $\boldsymbol{f}(t) = [\boldsymbol{F}(t), 0]^{\mathrm{T}}$, $\boldsymbol{x}$ and $\boldsymbol{F}$ are the degree of freedom and the generalized forcing respectively.

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Under the definition of state transition matrix, it can be shown that the complete solution can be written as

$$\begin{aligned}&\boldsymbol{d}(t)\\&=\boldsymbol{\Phi}(t,0)\,\boldsymbol{d}(0)+\int_0^t\boldsymbol{\Phi}(t,\tau)\,\boldsymbol{f}(\tau)\,\mathrm{d}\tau\\&=\boldsymbol{\Phi}(t,0)\,\boldsymbol{d}(0)+\boldsymbol{\Phi}(t,0)\int_0^t\boldsymbol{\Phi}^{-1}(\tau,0)\,\boldsymbol{f}(\tau)\,\mathrm{d}\tau.\end{aligned}\quad(2)$$

The first term on the right-hand side is, of course, the homogeneous solution of the system of equations and the second term is the inhomogeneous forced response.

Based on the Floquet theorem, the state transition matrix evaluated after one period T or the discrete transition matrix $\boldsymbol{\Phi}(T,0)$ can be written as

$$\boldsymbol{\Phi}(T,0)=\boldsymbol{C}=\mathrm{e}^{\boldsymbol{\Gamma}T},\quad(3)$$

where $\boldsymbol{\Gamma}$ is a constant matrix. It is then convenient to define the periodic matrix

$$\boldsymbol{P}(t)=\boldsymbol{\Phi}(t,0)\,\mathrm{e}^{-\boldsymbol{\Gamma}t}.\quad(4)$$

Then the general transition matrix and its inverse can be written as

$$\boldsymbol{\Phi}(t,0)=\boldsymbol{P}(t)\,\mathrm{e}^{\boldsymbol{\Gamma}t},\quad(5a)$$

$$\boldsymbol{\Phi}^{-1}(\tau,0)=\mathrm{e}^{-\boldsymbol{\Gamma}\tau}\boldsymbol{P}^{-1}(\tau).\quad(5b)$$

On the assumption that the input force can be represented as a cosine waveform, the forcing function is

$$\boldsymbol{f}(\tau)=\frac{1}{2}\boldsymbol{r}\left(\mathrm{e}^{\mathrm{j}\omega_\mathrm{f}\tau}+\mathrm{e}^{-\mathrm{j}\omega_\mathrm{f}\tau}\right),\quad(6)$$

where $\boldsymbol{r}$ is a column vector representing the amplitude of the forcing, and ω_f is the forcing frequency. The derivation of the general forced solution will be demonstrated using Eq. (6) thus a cosine forcing function is assumed to be used. Actually, an arbitrary forcing function can be modeled using the Fourier series approximation. Now substituting Eqs. (5) and (6) into the forced response term in Eq. (2) yields

$$\begin{aligned}\boldsymbol{d}_\mathrm{F}(t)=&\frac{1}{2}\boldsymbol{P}(t)\,\mathrm{e}^{\boldsymbol{\Gamma}t}\int_0^t\mathrm{e}^{-\boldsymbol{\Gamma}\tau}\boldsymbol{P}^{-1}(\tau)\boldsymbol{r}\\&\times\left(\mathrm{e}^{\mathrm{j}\omega_\mathrm{f}\tau}+\mathrm{e}^{-\mathrm{j}\omega_\mathrm{f}\tau}\right)\mathrm{d}\tau.\end{aligned}\quad(7)$$

For matrix exponential functions in Eqs. (4) and (7), there is a very important transformation to be made through the use of Lagrange–Sylvester theorem. This is achieved by making the substitution

$$\mathrm{e}^{\pm\boldsymbol{\Gamma}\tau}=\boldsymbol{V}\mathrm{diag}\left[\mathrm{e}^{\pm\mu_i\tau}\right]\boldsymbol{V}^{-1},\quad i=1,2,\ldots,2N,\quad(8a)$$

where μ are the eigenvalues of $\boldsymbol{\Gamma}$ and the matrix $\boldsymbol{V}$ is comprised of eigenvectors of $\boldsymbol{\Gamma}$.

Another way to deal with the matrix exponential functions can be performed for a larger system by using Sylvester interpolation polynomial

$$\mathrm{e}^{\pm\boldsymbol{\Gamma}\tau}=\sum_{l=1}^{v}\mathrm{e}^{\pm\mu_l\tau}\boldsymbol{Z}_l,\quad(8b)$$

with

$$\boldsymbol{Z}_l=\frac{\prod\limits_{s=1,s\neq l}^{v}(\mu_s\boldsymbol{I}-\boldsymbol{\Gamma})}{\prod\limits_{s=1,s\neq l}^{v}(\mu_s-\mu_l)},\quad(8c)$$

where v is equal to the dimension of $\boldsymbol{\Gamma}$ and $\boldsymbol{I}$ represents the identity matrix.

In addition, the periodic matrix $\boldsymbol{P}(t)$ possesses continuous derivatives and is nonsingular. Therefore, $\boldsymbol{P}^{-1}(t)$ is also periodic, nonsingular and possesses continuous derivatives. These will allow the use of Fourier series in the development of the pseudo-closed form of the forced solution. Writing $\boldsymbol{P}(t)$ and $\boldsymbol{P}^{-1}(t)$ matrices in the Fourier series form yields

$$\boldsymbol{P}(t)=\sum_{h=-\infty}^{+\infty}\boldsymbol{P}_h\,\mathrm{e}^{\mathrm{j}h\omega_\mathrm{p}t},\quad(9a)$$

$$\boldsymbol{P}^{-1}(\tau)=\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_k\,\mathrm{e}^{\mathrm{j}k\omega_\mathrm{p}\tau},\quad(9b)$$

where ω_p is the pump frequency, which can be written in terms of the period of parametric excitation as $\omega_\mathrm{p}=2\pi/T$. As a general rule for determining h and k, the highest frequency component of the Fourier coefficient representation should be at a minimum, twice the highest frequency component possessed by the parametric function. Besides, for a parametric function containing an infinite number of frequency components or with unknown frequency compositions, discrete spectral analysis can be performed first to ascertain the frequencies and their magnitudes. Then appropriate values of h and k can be chosen for Eq. (9).

Now that all the useful transformations have been introduced, the final form of the forced solution can begin to take shape. First, substituting Eqs. (8) and (9b) for the terms inside the integration in Eq. (7) yields

$$\begin{aligned}&\boldsymbol{d}_\mathrm{F}(t)\\&=\frac{1}{2}\boldsymbol{P}(t)\,\mathrm{e}^{\boldsymbol{\Gamma}t}\int_0^t\boldsymbol{V}\mathrm{diag}\left[\mathrm{e}^{-\mu_i\tau}\right]\boldsymbol{V}^{-1}\\&\quad\times\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_k\,\mathrm{e}^{\mathrm{j}k\omega_\mathrm{p}\tau}\boldsymbol{r}\left(\mathrm{e}^{\mathrm{j}\omega_\mathrm{f}\tau}+\mathrm{e}^{-\mathrm{j}\omega_\mathrm{f}\tau}\right)\mathrm{d}\tau.\end{aligned}\quad(10)$$

Combining sums and moving the appropriate terms from the inside of the integral, then performing the integration, the forced response becomes

$$\begin{aligned}\boldsymbol{d}_{\mathrm{F}}(t) &= \frac{1}{2}\boldsymbol{P}(t)\,\mathrm{e}^{\boldsymbol{\Gamma}t}\int_0^t \boldsymbol{V}\mathrm{diag}\left[\mathrm{e}^{(-\mu_i+\mathrm{j}(k\omega_{\mathrm{p}}+\omega_{\mathrm{f}}))\tau}+\mathrm{e}^{(-\mu_i+\mathrm{j}(k\omega_{\mathrm{p}}-\omega_{\mathrm{f}}))\tau}\right]\boldsymbol{V}^{-1}\mathrm{d}\tau\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_k\boldsymbol{r}\\ &= \frac{1}{2}\boldsymbol{P}(t)\,\mathrm{e}^{\boldsymbol{\Gamma}t}\boldsymbol{V}\mathrm{diag}\left[\frac{\mathrm{e}^{(-\mu_i+\mathrm{j}(k\omega_{\mathrm{p}}+\omega_{\mathrm{f}}))t}-1}{-\mu_i+\mathrm{j}\,(k\omega_{\mathrm{p}}+\omega_{\mathrm{f}})}\right]\boldsymbol{V}^{-1}\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_k\boldsymbol{r}\\ &\quad+\frac{1}{2}\boldsymbol{P}(t)\,\mathrm{e}^{\boldsymbol{\Gamma}t}\boldsymbol{V}\mathrm{diag}\left[\frac{\mathrm{e}^{(-\mu_i+\mathrm{j}(k\omega_{\mathrm{p}}-\omega_{\mathrm{f}}))t}-1}{-\mu_i+\mathrm{j}\,(k\omega_{\mathrm{p}}-\omega_{\mathrm{f}})}\right]\boldsymbol{V}^{-1}\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_k\boldsymbol{r}. \end{aligned}\tag{11}$$

At this point, substituting Eqs. (8) and (9a) for the terms outside the summations in Eq. (11), then making algebraic operations, we have

$$\begin{aligned}\boldsymbol{d}_{\mathrm{F}}(t) &= \frac{1}{2}\sum_{h=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_h\boldsymbol{V}\mathrm{diag}\left[\frac{\mathrm{e}^{\mathrm{j}((h+k)\omega_{\mathrm{p}}+\omega_{\mathrm{f}})t}}{-\mu_i+\mathrm{j}\,(k\omega_{\mathrm{p}}+\omega_{\mathrm{f}})}\right]\boldsymbol{V}^{-1}\boldsymbol{P}_k\boldsymbol{r}\\ &\quad+\frac{1}{2}\sum_{h=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_h\boldsymbol{V}\mathrm{diag}\left[\frac{\mathrm{e}^{\mathrm{j}((h+k)\omega_{\mathrm{p}}-\omega_{\mathrm{f}})t}}{-\mu_i+\mathrm{j}\,(k\omega_{\mathrm{p}}-\omega_{\mathrm{f}})}\right]\boldsymbol{V}^{-1}\boldsymbol{P}_k\boldsymbol{r}\\ &\quad-\frac{1}{2}\sum_{h=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_h\boldsymbol{V}\mathrm{diag}\left[\frac{\mathrm{e}^{(\mu_i+\mathrm{j}h\omega_{\mathrm{p}})t}}{-\mu_i+\mathrm{j}\,(k\omega_{\mathrm{p}}+\omega_{\mathrm{f}})}\right]\boldsymbol{V}^{-1}\boldsymbol{P}_k\boldsymbol{r}\\ &\quad-\frac{1}{2}\sum_{h=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_h\boldsymbol{V}\mathrm{diag}\left[\frac{\mathrm{e}^{(\mu_i+\mathrm{j}h\omega_{\mathrm{p}})t}}{-\mu_i+\mathrm{j}\,(k\omega_{\mathrm{p}}-\omega_{\mathrm{f}})}\right]\boldsymbol{V}^{-1}\boldsymbol{P}_k\boldsymbol{r}. \end{aligned}\tag{12}$$

Equation (12) is the general form of forced response of parametrically excited systems with using series expansions, which is actually comprised of two sets of complex conjugate matrices. Combining complex conjugate terms yields

$$\begin{aligned}\boldsymbol{d}_{\mathrm{F}}(t) &= \mathrm{Re}\left(\sum_{h=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_h\boldsymbol{V}\mathrm{diag}\left[\frac{\mathrm{e}^{\mathrm{j}((h+k)\omega_{\mathrm{p}}+\omega_{\mathrm{f}})t}}{-\mu_i+\mathrm{j}\,(k\omega_{\mathrm{p}}+\omega_{\mathrm{f}})}\right]\boldsymbol{V}^{-1}\boldsymbol{P}_k\boldsymbol{r}\right)\\ &\quad-\mathrm{Re}\left(\sum_{h=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\boldsymbol{P}_h\boldsymbol{V}\mathrm{diag}\left[\frac{\mathrm{e}^{(\mu_i+\mathrm{j}h\omega_{\mathrm{p}})t}}{-\mu_i+\mathrm{j}\,(k\omega_{\mathrm{p}}+\omega_{\mathrm{f}})}\right]\boldsymbol{V}^{-1}\boldsymbol{P}_k\boldsymbol{r}\right). \end{aligned}\tag{13}$$

The first term on the right-hand side represents the steady portion of the forced response and the second term is the transient part. If the system of equations is stable, the eigenvalues of the matrix $\boldsymbol{\Gamma}$ or the Floquet exponent μ values possess all negative real parts, then the transient part of the forced response will approach zero with time increasing, so will do the homogeneous portion or free response of the complete solution. In contrast, if any Floquet exponent has a positive real part, the transient portion of the forced solution will diverge to infinity, which shows the instability.

An additional point should be mentioned after Eq. (13) taking shape. As a set of Floquet multipliers approaches the real axis with magnitude near unity, the corresponding Floquet exponents will approach a minimum. Simultaneously, each term in the summation response formula involves a division by a term including the Floquet exponents. Therefore slow convergence of the response solution may occur near saddle-node bifurcation indicated by positive real Floquet multipliers, especially for some parametric excitations containing high frequencies.

3. Computation of state transition matrix

Time periodic stiffness is one example of parametric excitations in a mechanical system and time-varying mass or inertia is another example. In this paper the focus will be the derivation of state transition matrix for a parametrically excited system with time-varying stiffness.

The methods of determining the state transition matrix can be divided into two categories. One common approach utilizes numerical time integration to approximate the state transition matrix over a given time by solving the homogeneous solution for a lin-

early independent set of initial conditions. Each integration pass generates one column of the state transition matrix, thus numerical error is unavoidable. The other technique estimates the state transition matrix by dividing the time period into smaller subintervals and treating the system as time-invariant over each subinterval. The system of equations including time periodic stiffness is written as

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}(t)\,\dot{\boldsymbol{x}} + \boldsymbol{K}(t)\,\boldsymbol{x} = 0. \tag{14}$$

Let the number of intervals for the period of T be n, whose value should be at least twice the highest frequency component possessed by the parametric function, as the general rule for determining h and k.

The system of equations can now be solved over each subinterval as a system possesses constant parameters. The assumed solution form over subinterval $(t_1, 0)$ is written as

$$\begin{aligned} q_j(t_1) = {} & \mathrm{e}^{-\zeta_j\omega_j t}\left(u_j\cos\omega_{\mathrm{d}_j}t + v_j\sin\omega_{\mathrm{d}_j}t\right), \\ & j = 1, 2, \ldots, N, \end{aligned} \tag{15}$$

where ω_{d_j}, ω_j and ζ_j are damped natural frequencies, undamped natural frequencies and damping coefficients respectively with a relationship expressed by $\omega_{\mathrm{d}_j} = \sqrt{1-\zeta_j^2}\omega_j$. Then the displacement portion and the velocity portion of the response can be written in terms of the normal modes matrix $\boldsymbol{U}$ as

$$\begin{aligned} \boldsymbol{x}(t_1) = {} & \sum_{j=1}^{N}\boldsymbol{U}^j\left(u_j\cos\omega_{\mathrm{d}_j}t + v_j\sin\omega_{\mathrm{d}_j}t\right) \\ & \times\ \mathrm{e}^{-\zeta_j\omega_j t}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \dot{\boldsymbol{x}}(t_1) = {} & \sum_{j=1}^{N}\boldsymbol{U}^j\left(-u_j\omega_{\mathrm{d}_j}\sin\omega_{\mathrm{d}_j}t + v_j\omega_{\mathrm{d}_j}\cos\omega_{\mathrm{d}_j}t\right) \\ & \times\ \mathrm{e}^{-\zeta_j\omega_j t} \\ & -\sum_{j=1}^{N}\boldsymbol{U}^j\left(u_j\cos\omega_{\mathrm{d}_j}t + v_j\sin\omega_{\mathrm{d}_j}t\right)\zeta_j\omega_j \\ & \times\ \mathrm{e}^{-\zeta_j\omega_j t}. \end{aligned} \tag{17}$$

In addition, the homogenous solution can be written as a function of state transition matrix as

$$\left[\boldsymbol{x}(t_1), \dot{\boldsymbol{x}}(t_1)\right]^{\mathrm{T}} = \boldsymbol{\Phi}(t_1, 0)\left[\boldsymbol{x}(0), \dot{\boldsymbol{x}}(0)\right]^{\mathrm{T}}. \tag{18}$$

By substituting Eqs. (16) and (17) into Eq. (18), the resulting analytical form of $\boldsymbol{\Phi}(t_1, 0)$ is

$$\boldsymbol{\Phi}(t_1, 0) = \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{bmatrix}, \tag{19}$$

where

$$\begin{aligned} \boldsymbol{\Phi}_{11} = {} & \boldsymbol{U}\left[\mathrm{e}^{-\zeta_j\omega_j t}\cos\omega_{\mathrm{d}_j}t\right]\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M} \\ & + \boldsymbol{U}\left[\mathrm{e}^{-\zeta_j\omega_j t}\sin\omega_{\mathrm{d}_j}t\right]\left[\omega_{\mathrm{d}_j}\right]^{-1}\left[\zeta_j\omega_j\right]\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M}, \\ \boldsymbol{\Phi}_{12} = {} & \boldsymbol{U}\left[\mathrm{e}^{-\zeta_j\omega_j t}\sin\omega_{\mathrm{d}_j}t\right]\left[\omega_{\mathrm{d}_j}\right]^{-1}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M}, \\ \boldsymbol{\Phi}_{21} = {} & \boldsymbol{U}\left[-\mathrm{e}^{-\zeta_j\omega_j t}\omega_{\mathrm{d}_j}\sin\omega_{\mathrm{d}_j}t\right]\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M} \\ & + \boldsymbol{U}\left[-\zeta_j\omega_j\,\mathrm{e}^{-\zeta_j\omega_j t}\cos\omega_{\mathrm{d}_j}t\right]\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M} \\ & + \boldsymbol{U}\left[\mathrm{e}^{-\zeta_j\omega_j t}\omega_{\mathrm{d}_j}\cos\omega_{\mathrm{d}_j}t\right]\left[\omega_{\mathrm{d}_j}\right]^{-1} \\ & \times\ \left[\zeta_j\omega_j\right]\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M} + \boldsymbol{U}\left[-\zeta_j\omega_j\,\mathrm{e}^{-\zeta_j\omega_j t}\sin\omega_{\mathrm{d}_j}t\right] \\ & \times\ \left[\omega_{\mathrm{d}_j}\right]^{-1}\left[\zeta_j\omega_j\right]\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M}, \\ \boldsymbol{\Phi}_{22} = {} & \boldsymbol{U}\left[\mathrm{e}^{-\zeta_j\omega_j t}\omega_{\mathrm{d}_j}\cos\omega_{\mathrm{d}_j}t\right]\left[\omega_{\mathrm{d}_j}\right]^{-1}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M} \\ & + \boldsymbol{U}\left[-\zeta_j\omega_j\,\mathrm{e}^{-\zeta_j\omega_j t}\sin\omega_{\mathrm{d}_j}t\right] \\ & \times\ \left[\omega_{\mathrm{d}_j}\right]^{-1}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M}. \end{aligned}$$

Ultimately, equation (19) allows for the calculation of the state transition matrix over each interval, then all the state transition matrices over intervals can be combined through matrix multiplication to arrive at the discrete transition matrix, then its eigenvalues or Floquet multipliers containing information necessary to determine system stability and bifurcation can be found. After the transition matrix for each interval is found, analytical solution of the discrete transition matrix $\boldsymbol{\Phi}(T, 0)$ exists by the combining operation written as

$$\boldsymbol{\Phi}(T, 0) = \boldsymbol{\Phi}(T, t_{n-1})\cdots\boldsymbol{\Phi}(t_2, t_1)\,\boldsymbol{\Phi}(t_1, 0). \tag{20}$$

Now we turn back to Eq. (13) representing the forced response of parametrically excited system. By computing state transition matrices over the terminal time of each interval ($\boldsymbol{\Phi}(t_1, 0)$, $\boldsymbol{\Phi}(t_2, 0)$, …, $\boldsymbol{\Phi}(T, 0)$) according to Eq. (20), the periodic $\boldsymbol{P}(t)$ and $\boldsymbol{P}^{-1}(t)$ matrices can be achieved using Eqs. (4) and (8) without relying on matrix exponential functions, then the calculation of Fourier coefficients matrices $\boldsymbol{P}_h$ and $\boldsymbol{P}_k$ needed in Eq. (13) is feasible after giving the Fourier expansion terms. According to Eq. (3), the discrete transition matrix $\boldsymbol{\Phi}(T, 0)$ and $\boldsymbol{\Gamma}$ share the same eigenvectors and the eigenvalues of $\boldsymbol{\Gamma}$ are related to Floquet multiplier λ values by the relationship $\mu = \ln(\lambda)/T$. Therefore, the values of Floquet exponent μ and the eigenvector matrix $\boldsymbol{V}$ needed in Eq. (13) can be found by evaluating the eigenvalues and eigenvectors of $\boldsymbol{\Phi}(T, 0)$. Finally, all variables needed to calculate the forced response are found and equation (13) turns into a pseudo-closed form of solution for parametrically excited systems subjected to inhomogeneous generalized forcing.

4. Mathieu type of equation

A Mathieu type of equation can be written as

$$\ddot{x} + (\delta + 2\varepsilon \cos 2t)\, x = 0. \tag{21}$$

Dividing the period π into 32 intervals while using either the average or mid-point value of the periodic parameter over each subinterval, we compute the discrete transition matrix while the range of parameters was set to be $\delta = (0, 5)$ and $\varepsilon = (0, 2)$. Figure 1 shows the plots of the magnitudes of Floquet multipliers greater than unity. Figure 1(a) depicts that a region of instability emanates from location on $\delta = 1$, in which the unstable Floquet multiplier exceeds the unit circle on the negative real axis. This period doubling bifurcation can be well demonstrated that the pump frequency $\omega_{\rm p}$ is actually twice the system natural frequency ω_i at the very transition point of the instability. Figure 1(b) depicts that another region of instability originates from $\delta = 4$ where the pump frequency $\omega_{\rm p}$ is half the system natural frequency ω_i. Accordingly, the unstable Floquet multiplier in this region exceeds the unit circle on the positive real axis, with saddle-node bifurcation forecasted

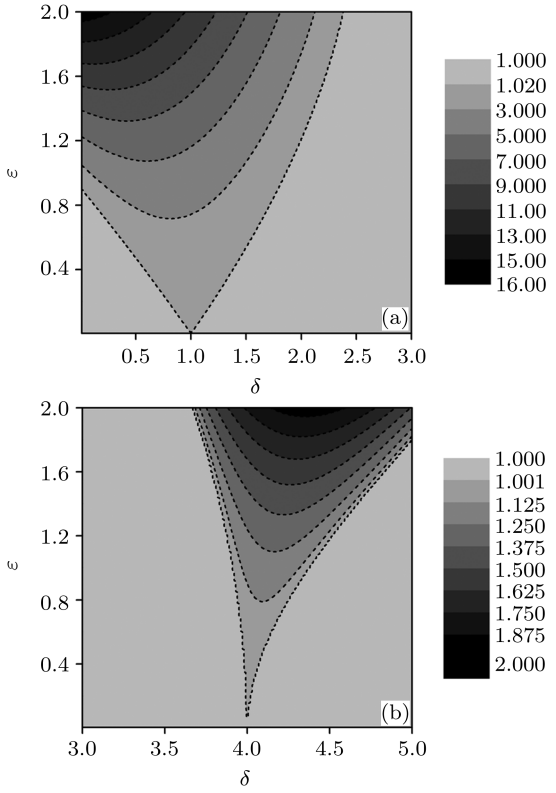


Fig. 1. Instabilities of Mathieu equation with large amplitude emanating from $\delta = 1$ (a) and $\delta = 4$ (b).

5. Two-degree-of-freedom system with irregular periodic stiffness

For spur and helical gear dynamics, parametrically excited vibrations with time varying mesh stiffness could be well characterized by Hill–Meissner and Mathieu type of equations. Unfortunately, numerous real-world engineering problems do not fit well to these narrowly defined models, for example, time-varying mesh stiffness of spiral bevel gear in aero engines and helicopter transmission systems is irregular, due to the influence of both gear design parameters and machine tool settings.

Including the irregular parameter δ with a period of π as shown in Fig. 2, a two-degree-of-freedom system is given as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2+\delta & -\delta \\ -\delta & 2+\delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix}. \tag{22}$$

Dividing the period π into 80 intervals while using either the average or mid-point value of the periodic parameter over each subinterval, stable solutions are found and shown in Fig. 3. Figure 3 shows that using

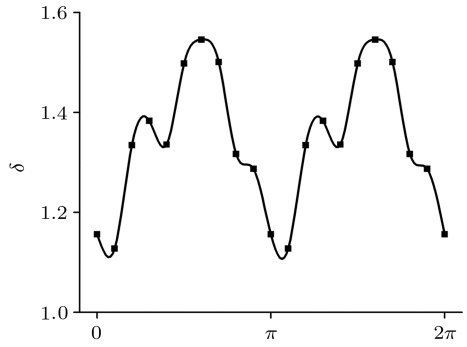


Fig. 2. Irregular time periodic parameter.

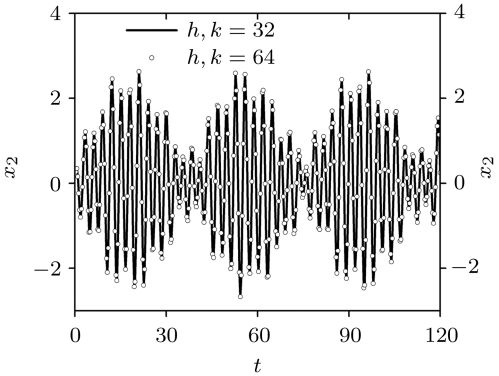


Fig. 3. Stable solutions for irregular periodic system, showing time history of x_2.

of a thirty-two and a sixty-four term Fourier expansion of the periodic matrices provides a good approximation of the time history of x_2.

Substituting δ into the parametric matrix for the right-bottom term, unstable Floquet multiplier exceeds the unit circle on the positive real axis. Unstable solutions in saddle-node bifurcation are shown in Fig. 4. Figure 4(a) shows the time history of x_2 and figure 4(b) exhibits the phase trajectory of limit cycle for x_1.

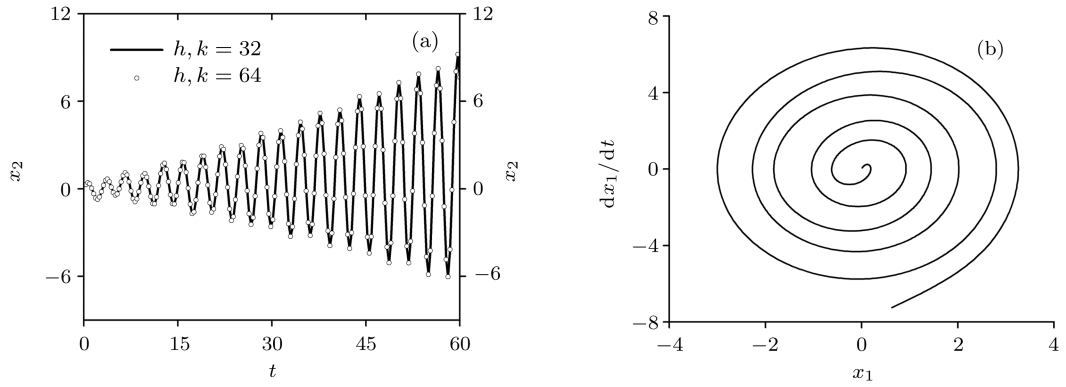


Fig. 4. Unstable solutions for irregular periodic system, showing (a) time history of x_2 and (b) phase diagram of x_1.

6. Conclusion

In conclusion, techniques are presented that can be used to solve for both free and forced response of time periodic system of equations. Derivation based on Fourier series expansion and Lagrange–Sylvester theorem has been developed to avoid the limitation of the forced response in terms of periodic functions and matrices for large order, parametrically excited systems. The determination of system stability and bifurcation along with the calculation of complete response hinge on the successful evaluation of the state transition matrix composed of linearly independent solutions of the homogenous form of time periodic stiffness systems. From this research, a method of evaluating the stability and response of mechanical systems with time periodic stiffness is provided and it is especially practicable in vibrations with irregular time-varying mesh stiffness characterized by spiral bevel geared system in aerospace power machines.

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