Stability of trapped Bose–Einstein condensates in one-dimensional tilted optical lattice potential*

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Using the direct perturbation technique, this paper obtains a general perturbed solution of the Bose–Einstein condensates trapped in one-dimensional tilted optical lattice potential. We also gave out two necessary and sufficient conditions for boundedness of the perturbed solution. Theoretical analytical results and the corresponding numerical results show that the perturbed solution of the Bose–Einstein condensate system is unbounded in general and indicate that the Bose–Einstein condensates are Lyapunov-unstable. However, when the conditions for boundedness of the perturbed solution are satisfied, then the Bose–Einstein condensates are Lyapunov-stable.

Keywords: Bose–Einstein condensate, chaos, stability, tilted optical lattice potential

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1. Introduction

In recent years, the Bose–Einstein condensate (BEC) has become a topic of great interest in the field of physics,[1−3] many new and important results have been reported. Some of them adopted magnetic and optical trapping potentials to discuss the macroscopic quantum interference and phase coherence,[4−6] the atomic tunneling between two trapped BECs,[7−9] the Josephson effects[10] and the formation and propagation of matter-wave solitons.[11−13] The experimental realization of optical lattices[14−16] set up a new perspective in the study of trapped BEC and arouses more and more interest, e.g., the experimental observation and theoretical explanation of the nonlinear dynamics for a BEC in a periodic optical lattice[17,18] and the applications in quantum computation and quantum information.[19,20] The optical lattice potential induced by the AC Stark effect of interfering laser beams has often been used to confine ultra-cold atoms[21−26] and its intensity can be modulated from very weak to much stronger in a real experiment.[14] This provided the facility for controlling the condensate and made it possible to employ the perturbation technique. Recently, experiments with BECs in a Wannier–Stark (WS) system[27] attracted much attention and many important theoretical works have been reported. These works included coherent pulse output from BECs in WS system,[28] the study of Bloch oscillations,[29] WS ladders and collective tunneling effects.[30] However, these theories are based on a numerical calculation method. In this paper, we give a quantitative analytical method for discussing the stability of the BECs in one-dimensional tilted optical lattice potential. This theory gives complements and extensions to previous numerical simulations.[31]

2. Chaotic solutions of the BEC system

It is well known that the mean-field theory is a successful one used widely to describe the BEC. In this theory, the dynamical behaviours are governed by the Gross–Pitaevskii equation (GPE),[32] which provides us with a nonlinear quantum system. In this case, we consider a quasi-one-dimensional BEC, the order parameter was identified with a one-body macroscopic condensate wave function, which obeys the one-dimensional nonlinear Schrödinger equation, namely the one-dimensional GPE[33]

$$\mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + [V_{\mathrm{trap}}(x) + g_0|\psi(x,t)|^2]\psi(x,t), \quad (1)$$

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where $g_0 = 4\pi\hbar^2 a/m$ is the interatomic scattering pseudopotential with a and m being the atomic scattering length and mass, respectively. The dynamics of 1D, tilted, periodic trap potential has often been considered in recent literature, both for free atoms and for BECs.[34–37] In this paper, we consider that the trap potential $V_{\rm trap}(x)$ in Eq. (1) is a tilted optical lattice potential[31]

$$V_{\rm trap}(x) = V_0 \cos(kx) + Fx, \tag{2}$$

where V_0 is the amplitude of optical potential, k the wave vector of the laser light and F the inertial force (the limit $|F| \to 0$ as $x \to \infty$). Due to the inertial force (such as the gravitational force) appears, a "tilted" potential is produced (see Fig. 1), and the force accelerates the atoms in the x direction and leads to the atoms tunnelling out of the traps. In this case, the eigenstates of the linear part of the Hamiltonian

$$H_0 = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V_0 \cos(kx) + Fx \tag{3}$$

are the so-called WS states.[30,31]

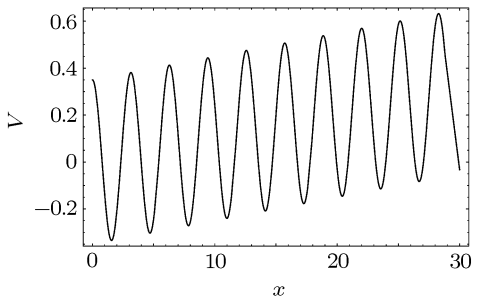


Fig. 1. Plot the tilted potential with the parameters $V_0 = 0.35$, $F = 0.01$, $k = 2$.

In order to obtain a simple description and a better understanding of the BEC dynamics, we consider only the solution of Eq. (1) with the trivial phase. In this case, we employ the time-dependent variational wave function[32,38]

$$\psi(x,t) = \phi(x) \exp\left(-\mathrm{i}\,\frac{\mu}{\hbar}t\right), \tag{4}$$

where μ is the chemical potential and $\phi(x)$ is real and normalized to the total number of particles, $\int \phi^2(x)\mathrm{d}x = N_0 = N$. In the trivial phase case, the phase $\theta(x) = 0$. Substituting Eqs. (2) and (4) into Eq. (1), we find that the GPE (1) becomes a stationary state GPE

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} + [V_0\cos(kx) + Fx]\phi(x) + g_0\phi^3(x) = \mu\phi(x). \tag{5}$$

Equation (5) can also be written in the following form:

$$-\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} + [\alpha\cos(kx) + \beta x]\phi(x) + \gamma\phi^3(x) = \Gamma\phi(x), \tag{6}$$

where $\alpha = 2mV_0/\hbar^2$, $\beta = 2mF/\hbar^2$, $\gamma = 2mg_0/\hbar^2$, $\Gamma = 2m\mu/\hbar^2$. In order to solve the nonlinear Eq. (6), we expand $\phi(x)$ to the first order

$$\phi(x) = \phi_0(x) + \phi_1(x), \quad |\phi_1(x)| \ll |\phi_0(x)|. \tag{7}$$

Inserting Eq. (7) into Eq. (6), we can obtain the zero-order equation

$$\frac{\mathrm{d}^2\phi_0(x)}{\mathrm{d}x^2} + \Gamma\phi_0(x) - \gamma\phi_0^3(x) = 0, \tag{8}$$

and the first-order equation

$$\frac{\mathrm{d}^2\phi_1(x)}{\mathrm{d}x^2} + \Gamma\phi_1(x) - 3\gamma\phi_0^2(x)\phi_1(x) = \varepsilon(x), \tag{9}$$

$$\varepsilon(x) \approx [\alpha\cos(kx) + \beta x]\phi_0(x). \tag{10}$$

Equation (8) is an unperturbed Duffing equation, which has a well-known analytic solution near the homoclinic orbit[39,40]

$$\phi_0(x) = \sqrt{2\Gamma\gamma^{-1}}\,\mathrm{sech}\,\xi, \quad \xi = \sqrt{\Gamma}x - C,$$
$$C = \sqrt{\Gamma}x_0 - Ar\,\mathrm{sech}\,\frac{\phi_0(x_0)}{\sqrt{2\Gamma\gamma^{-1}}}, \tag{11}$$

where C is the initial constants and x_0 is the initial position. In Eq. (10), we assume that the parameters α and β are so small that the $\varepsilon(x)$ can be regarded as a perturbation added to the unperturbed zero-order Eq. (8). Combining Eq. (9) with Eq. (11) and making use of the variation of constants,[41] we easily construct the general solution of Eq. (9) in the integral form[42]

$$\begin{aligned}\phi_1(x) &= g\int_A^x f\varepsilon(x)\mathrm{d}x - f\int_B^x g\varepsilon(x)\mathrm{d}x \\ &= \frac{1}{\sqrt{\Gamma}}\Bigg[g\int_A^\xi f\left(\alpha\cos\left(k\frac{\xi+C}{\sqrt{\Gamma}}\right)\right. \\ &\quad \left.+\beta\left(\frac{\xi+C}{\sqrt{\Gamma}}\right)\right)\phi_0(\xi)\mathrm{d}\xi \\ &\quad - f\int_B^\xi g\left(\alpha\cos\left(k\frac{\xi+C}{\sqrt{\Gamma}}\right)\right. \\ &\quad \left.+\beta\left(\frac{\xi+C}{\sqrt{\Gamma}}\right)\right)\phi_0(\xi)\mathrm{d}\xi\Bigg],\end{aligned} \tag{12}$$

where A and B are two integration constants, f and g are the two linear independent solutions of Eq. (9) for $\varepsilon(x) = 0$, which are easily shown to be

$$f = \frac{\mathrm{d}\phi_0(x)}{\mathrm{d}x} = -\sqrt{2\Gamma\gamma^{-1}}\,\mathrm{sech}\,\xi\tanh\xi, \tag{13}$$

$$g = f\int f^{-2}\mathrm{d}x = -\frac{\sqrt{2\Gamma\gamma^{-1}}}{2}\Big(\operatorname{sech}\xi - \frac{3}{2}\xi\tanh\xi\operatorname{sech}\xi - \frac{1}{2}\tanh\xi\sinh\xi\Big). \quad (14)$$

The derivative of $\phi_1(x)$ is given by Eq. (12) as

$$\begin{aligned}\frac{\mathrm{d}\phi_1(x)}{\mathrm{d}x} &= \frac{\mathrm{d}g}{\mathrm{d}x}\int_A^x f\varepsilon(x)\mathrm{d}x - \frac{\mathrm{d}f}{\mathrm{d}x}\int_B^x g\varepsilon(x)\mathrm{d}x \\ &= \frac{\mathrm{d}g}{\mathrm{d}\xi}\int_A^\xi f\Big(\alpha\cos\Big(k\frac{\xi+C}{\sqrt{\Gamma}}\Big) + \beta\Big(\frac{\xi+C}{\sqrt{\Gamma}}\Big)\Big)\phi_0(\xi)\mathrm{d}\xi \\ &\quad - \frac{\mathrm{d}f}{\mathrm{d}\xi}\int_B^\xi g\Big(\alpha\cos\Big(k\frac{\xi+C}{\sqrt{\Gamma}}\Big) + \beta\Big(\frac{\xi+C}{\sqrt{\Gamma}}\Big)\Big)\phi_0(\xi)\mathrm{d}\xi. \quad (15)\end{aligned}$$

3. Stability analysis of BECs

We are interested in the stability of the BECs. Combining Eq. (7) and the stable law of Lyapunov, if the conditions

$$\sqrt{\phi_1^2(x_0)+\dot{\phi}_1^2(x_0)} < \eta, \quad \sqrt{\phi_1^2(x)+\dot{\phi}_1^2(x)} < \zeta, \quad \forall x > x_0 \quad (16)$$

are satisfied, then we find that the BECs are Lyapunov-stable. Here η and ζ are any two small positive numbers. From Eq. (16), we know that the stability of the BECs is related to the first corrections $\phi_1(x)$ and $\dot{\phi}_1(x)$. Combining Eqs. (12), (13), and (14) we can see that the corrected solution (12) is unbounded, because of the unboundedness of the function g. In this case, although the $\sqrt{\phi_1^2(x_0)+\dot{\phi}_1^2(x_0)}$ is less than a small positive number η, the $\sqrt{\phi_1^2(x)+\dot{\phi}_1^2(x)}$ tends to the infinity as $x\to\infty$, the stable condition (16) is not satisfied and hence the BECs are Lyapunov-unstable. However, we can prove that if and only if the necessary and sufficient conditions

$$I_\pm = \lim_{\xi\to\pm\infty}\int_A^\xi f\Big[\alpha\cos\Big(k\frac{\xi+C}{\sqrt{\Gamma}}\Big) + \beta\Big(\frac{\xi+C}{\sqrt{\Gamma}}\Big)\Big]\phi_0(\xi)\mathrm{d}\xi = 0 \quad (17)$$

is satisfied, then the first correction will maintain bounded. The necessity of the conditions (17) is obvious because of the divergence of g. Under the conditions (17), applying Eqs. (12), (15) and the l'Hospital rule, we can obtain the small superior limits $|\phi_1|$ and $|\mathrm{d}\phi_1/\mathrm{d}x|$ for $x\to\infty$. This is the proof for the sufficiency of the conditions (17). Under the conditions (17), the stable condition (16) can be satisfied and the BECs are Lyapunov-stable in this case. Substituting Eq. (13) into Eq. (17), we find

$$\begin{aligned}M(x_0) &= I_+ - I_- = \int_{-\infty}^{\infty} f\varepsilon(x)\mathrm{d}x \\ &= -2\sqrt{\Gamma}\gamma^{-1}\int_{-\infty}^{\infty}\operatorname{sech}^2\xi\tanh\xi \\ &\quad\times\Big[\alpha\cos\Big(k\frac{\xi+C}{\sqrt{\Gamma}}\Big) + \beta\Big(\frac{\xi+C}{\sqrt{\Gamma}}\Big)\Big]\mathrm{d}\xi \\ &= \frac{1}{2}\frac{\pi\alpha k^2}{\sqrt{\Gamma}\gamma}\sin\Big(\frac{kC}{\sqrt{\Gamma}}\Big)\operatorname{csch}\Big(\frac{k\pi}{4\sqrt{\Gamma}}\Big) \\ &\quad\times\operatorname{sech}\Big(\frac{k\pi}{4\sqrt{\Gamma}}\Big) - \frac{2\beta}{\gamma} = 0. \quad (18)\end{aligned}$$

The function $M(x_0)$ is just the Melnikov one,[43] which measures the distance between the stable and unstable manifolds in the Poincaré section at x_0. Equation (18) shows that the Melnikov function has a simple zero, which indicates the existence of chaos and chaotic region in parameter space.[44] We call the solution (12) obeying the chaos criterion (18) the "chaotic solution".[45] Due to the chaotic solution, we find that the BEC system is unstable.

In order to explore this analytically insolvable system, we numerically solve Eq. (6) by using "MATHEMATICA". Taking a set of experimental parameters as $\alpha = 0.35$, $\beta = 0.0001$, $\gamma = 0.5$, $\Gamma = 2$ and $k = 2\pi/850$ $(\mathrm{nm})^{-1}$, from Eq. (6) we numerically plot the phase orbit in the equivalent phase space $(\phi, \mathrm{d}\phi/\mathrm{d}x)$ as in Fig. 2(a). From this phase diagram, we see that the phase orbit evolves with the coordinate in a finite region and exhibits the conclusion, which is of the chaotic feature. Accordingly, the spatial evolutions of the solution $\phi(x)$ and the atomic number density $n = |\phi(x)|^2$ are all chaotic (see Figs. 2(b) and 2(c)). In this chaotic state, the BECs are very unstable. However, we can control the instability of BEC system by adjustment the system parameters. For example, when the parameter α is decreased to $\alpha = 0.035$, the phase orbit of the system becomes an absolute periodic orbit, corresponding to the spatial evolutions of the solution and the atomic number density also becomes periodic one as shown in Fig. 3. In this case, the BEC system becomes a periodic stable state.

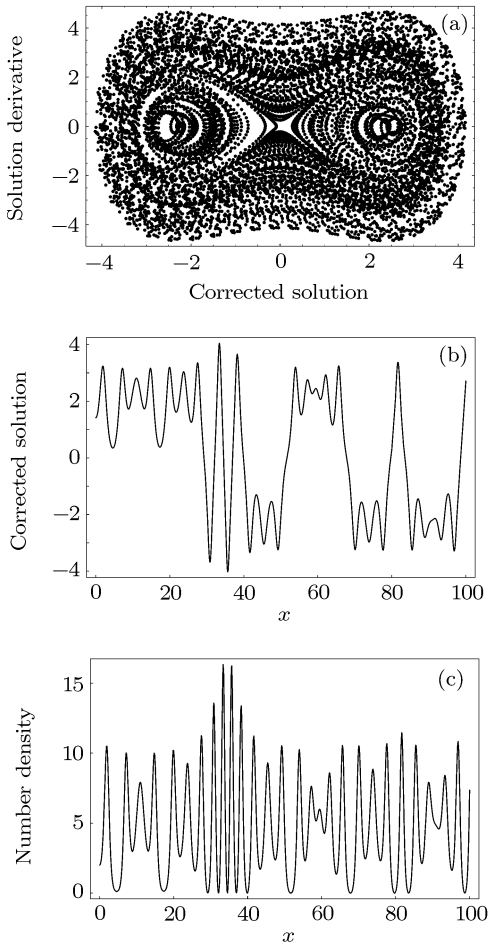


Fig. 2. (a) The phase orbit in the equivalent phase space of the ϕ versus $\mathrm{d}\phi/\mathrm{d}x$; (b) the spatial evolution of the solution $\phi(x)$; and (c) the spatial evolution of the atomic number density $n = |\phi(x)|^2$ with the parameters $\alpha = 0.35$, $\beta = 0.0001$, $\gamma = 0.35$, $\Gamma = 2$ and $k = 2\pi/850$ $(\mathrm{nm})^{-1}$.

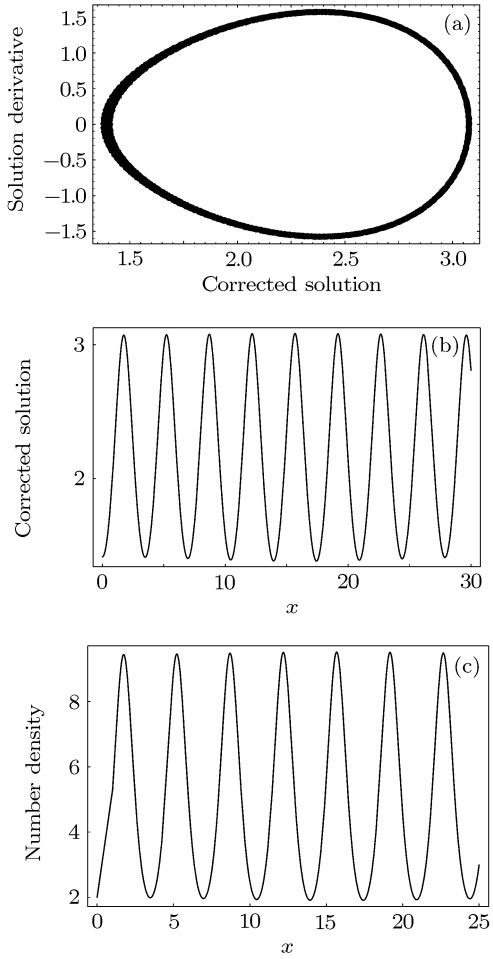


Fig. 3. (a) The phase orbit in the equivalent phase space of the ϕ versus $\mathrm{d}\phi/\mathrm{d}x$; (b) the spatial evolution of the solution $\phi(x)$; and (c) the spatial evolution of the atomic number density $n = |\phi(x)|^2$ with the parameters $\alpha = 0.035$, $\beta = 0.0001$, $\gamma = 0.35$, $\Gamma = 2$ and $k = 2\pi/850$ $(\mathrm{nm})^{-1}$.

Particularly, equation (18) gives the stable region and unstable region in the parameter space

$$F \leq \frac{1}{4}\frac{V_0\pi k^2}{\sqrt{2\mu}}\,\mathrm{csch}\left(\frac{\pi k}{4\sqrt{2\mu}}\right)\mathrm{sech}\left(\frac{\pi k}{4\sqrt{2\mu}}\right), \tag{19}$$

where we set units such that $\hbar = m = 1$. If fix $V_0 = 0.4$, and let wave vector as (a) $k = 1$ and (b) $k = 2$ respectively, then we can plot the chaotic region in parameter F–μ plane as shown in Fig. 4 (above the curve we have the unstable chaotic region and below the curve we have the stable region).

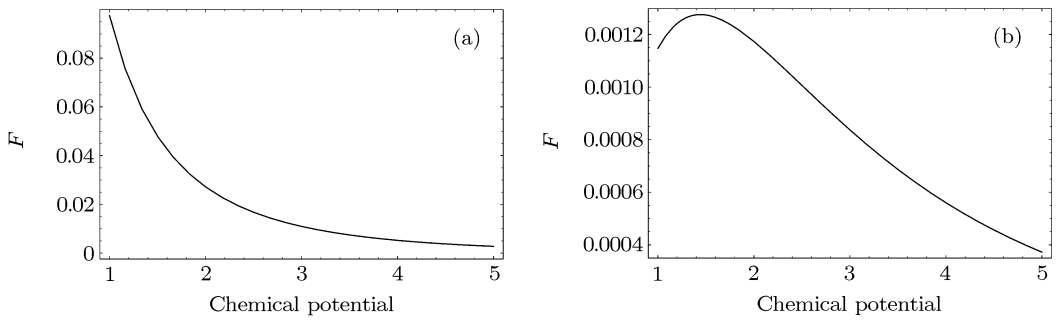


Fig. 4. Plot of the unstable region and the stable region in the parameter F–μ plane with parameters (a) $V_0 = 0.35$, $k = 1$; (b) $V_0 = 0.35$, $k = 2$.

From Fig. 4 we see that the unstable chaotic region is very sensitive for the wave vector k, and due to $k = 2\pi/\lambda_L$ (λ_L is the wave length of the laser light), so we can control the chaotic region by change the wave length of the incident laser light. We can also easily see that the system leaves chaotic region to stable region by adjusting the inertial force F to a minimum from Fig. 4. This shows that the system parameters play an important role in controlling the unstability of the BEC system.

4. Conclusion

In this paper, we have investigated the stability of the BEC system in 1D tilted optical potential. Using the direct perturbation method, we have obtained the chaotic solution of the stationary state GPE and showed that the BECs are unstable in general. However, when the necessary and sufficient condition (17) is satisfied, then the perturbation solution (12) will maintain bounded and hence the BEC system becomes stable. We have found that controlling the chaos and stability of the BEC system may change the wave length of the incident laser light or adjust the system parameters. In this paper, we have studied only the BEC system held in 1D titled optical lattice potential. We shall discuss the BEC systems held in two-dimensional and three-dimensional tilted optical lattices in future works.

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