A quantum search algorithm based on partial adiabatic evolution*

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This paper presents and implements a specified partial adiabatic search algorithm on a quantum circuit. It studies the minimum energy gap between the first excited state and the ground state of the system Hamiltonian and it finds that, in the case of $M = 1$, the algorithm has the same performance as the local adiabatic algorithm. However, the algorithm evolves globally only within a small interval, which implies that it keeps the advantages of global adiabatic algorithms without losing the speedup of the local adiabatic search algorithm.

Keywords: quantum search, adiabatic evolution, quantum circuit

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Quantum adiabatic computation has attracted a lot of attention in the past decades,[1−9] since it was proposed by Farhi *et al.*[10] A typical quantum adiabatic algorithm starts with the ground state of the initial Hamiltonian $H_{\rm i}$, and evolves slowly to the ground state of the final Hamiltonian $H_{\rm f}$. The system that implements the algorithm uses the time-dependent Hamiltonian

$$H(s(t)) = (1 - s(t))H_{\rm i} + s(t)H_{\rm f}. \tag{1}$$

The running time (evolution time) is essentially determined by[7]

$$T \geq \Theta\left(\frac{1}{g_{\min}^2}\right), \tag{2}$$

$$g_{\min} = \min[E(1, s) - E(0, s)], \quad (0 \leq s \leq 1), \tag{3}$$

where $E(0, s)$ and $E(1, s)$ are the two lowest eigenvalues of $H(s)$. By specifying the Hamiltonian $H_{\rm i}$ and $H_{\rm f}$, the adiabatic evolution can be used to solve many computational problems. In Ref. [3], an adiabatic algorithm was designed to solve the Deutsch–Jozsa problem. The algorithm took an exponential time which provided a quadratic speed up over classical algorithms for the problem. A modified algorithm proposed by Wei and Ying[4] improved the performance to constant time. In Ref. [11], quantum adiabatic computation was applied to solve random instances of NP-complete problems.

The adiabatic evolution can also be specified to solve the unstructured search problem which can be described as follows. Suppose there is an unsorted database with $N = 2^n$ items, and there is a nonempty subset S of the items satisfying the condition $C(s) = 1$ for $s \in S$ and $C(s) \neq 1$ for $s \notin S$. The goal is to find one item of S. For simplicity, we suppose S contains only one item throughout this paper. Grover presented an algorithm[12] to solve this problem which is quadratically faster than classical algorithms. Roland and Cerf[13] considered the same problem and designed a search algorithm based on local adiabatic evolution, which performs as good as Grover's algorithm up to a constant factor.

To solve the search problem, $H_{\rm i}$ and $H_{\rm f}$ are specified as

$$H_{\rm i} = 1 - |\Psi\rangle\langle\Psi|, \tag{4}$$

and

$$H_{\rm f} = 1 - |\beta\rangle\langle\beta|, \tag{5}$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle, \tag{6}$$

$|\beta\rangle$ is the marked item (the only one item contained in S). Farhi *et al.*[10] showed that if the adiabatic condition is satisfied globally (that is, $s(t)$ is a linear function), its corresponding algorithm performs no better than the best classical algorithm. However, if the adiabatic condition is satisfied locally,[13] that is,

$$\frac{{\rm d}s}{{\rm d}t} \leq \epsilon g^2(t), \quad \forall t \in [0, T], \tag{7}$$

where $g(t) = E(1, s(t)) - E(0, s(t))$ is the gap between the two lowest eigenvalues, then its corresponding algorithm took a time steps of $O(\sqrt{N})$ to find a marked

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item. To see this, we solve condition (7) to find

$$t(s) = \frac{1}{2\epsilon}\frac{N}{\sqrt{N-1}}\Big(\arctan\sqrt{N-1}(2s-1) + \arctan\sqrt{N-1}\Big) \tag{8}$$

by using

$$g(s) = \sqrt{1-4\left(1-\frac{1}{N}\right)s(1-s)}. \tag{9}$$

This gives

$$T = \frac{1}{\epsilon}\frac{N}{\sqrt{N-1}}\arctan\sqrt{N-1} \tag{10}$$

by taking $s = 1$. We finally obtain

$$T \simeq \frac{\pi}{2\epsilon}\sqrt{N} \tag{11}$$

with the approximation $N \gg 1$.

We can see that, in the local adiabatic evolution, s changes slowly when t is close to $T/2$, and so does $H(s(t))$. This results from the small gap of the two lowest eigenvalues when t is close to $T/2$. It can be imagined that an adiabatic search algorithm may perhaps make use of this property. A research by Tulsi[2] unveils the imagination. In his paper, Tulsi analysed a kind of quantum adiabatic evolution with a one-dimensional projector Hamiltonian, and proposed the partial adiabatic evolution. It was also checked in the paper that Roland and Cerf's results could be obtained as a special case of the partial adiabatic evolution. A specified search algorithm based on the partial adiabatic evolution has been studied in one of our recent papers,[14] which presents better performance than the local adiabatic search algorithm when $|S| > 1$. Here we focus only on the case that $|S| = 1$.

The algorithm executes the four following steps.

(i) The initial state is prepared to be $|\Psi\rangle$;

(ii) At $t = 0$, the Hamiltonian is suddenly changed to $H(s^-)$ without disturbing the state $|\Psi\rangle$;

(iii) The Hamiltonian evolves from $H(s^-)$ to $H(s^+)$ linearly in time over duration T';

(iv) Measure the state of the system.

Repeat these four steps until we find a marked state.

For the algorithm, s^- and s^+ are specified as $s^- = 1/2 - 1/2\sqrt{N}$ and $s^+ = 1/2 + 1/2\sqrt{N}$. Because the algorithm evolves adiabatically only within a small time interval $[s^-, s^+]$, it is called a partial adiabatic search algorithm. After step (ii), the system that implements the algorithm will be still in the state $|\Psi\rangle$. That is, the system state will be the ground state of $H(s^-)$ with probability $|\langle\Psi|E(0,s^-)\rangle|^2$. The adiabatic theorem[15] guarantees that it will be the ground state of $H(s^+)$ with probability $|\langle\Psi|E(0,s^-)\rangle|^2$ after step (iii). Measuring the state of the system will give the ground state of $H_{\rm f} = 1 - |\beta\rangle\langle\beta|$ with probability $P = |\langle\Psi|E(0,s^-)\rangle|^2 \times |\langle\beta|E(0,s^+)\rangle|^2$. The overall time complexity (evolution time) of the algorithm is $T = T'/P$. Here, $T' = \omega/g_{\min}^2$, where $\omega = s^+ - s^- = 1/\sqrt{N}$. It has been shown that $g_{\min} = 1/\sqrt{N}$,[7] but here, we use a different method. That is, we study the problem in a two-dimensional Hilbert space.

Multiplying $\langle x|$ to $|H_s|E(s)\rangle = E(s)|E(s)\rangle$ gives rise to

$$\langle x|H_s|E(s)\rangle = E(s)\langle x|E(s)\rangle. \tag{12}$$

Using $\langle x|\beta\rangle = 1$ and $\langle x|\Psi\rangle = 1/\sqrt{N}$, we have

$$\begin{aligned}
&\langle x|H_s|E(s)\rangle \\
&= \langle x(1-s)(1-|\Psi\rangle\langle\Psi| + s(1-|\beta\rangle\langle\beta|E(s)\rangle) \\
&= \langle x|E(s)\rangle - \frac{1-s}{\sqrt{N}}\langle\Psi|E(s)\rangle - s\langle x|\beta\rangle\langle\beta|E(s)\rangle \\
&= \begin{cases} \langle x|E(s)\rangle - \dfrac{1-s}{\sqrt{N}}\langle\Psi|E(s)\rangle, & \langle x|\beta\rangle = 0, \\ \langle x|E(s)\rangle - \dfrac{1-s}{\sqrt{N}}\langle\Psi|E(s)\rangle & \\ \qquad - \dfrac{s}{\langle\beta|E(s)\rangle}, & \langle x|\beta\rangle \neq 0. \end{cases}
\end{aligned} \tag{13}$$

So, we can obtain

$$\langle x|E(s)\rangle = \begin{cases} \dfrac{1-s}{\sqrt{N}}\dfrac{\langle\Psi|E(s)\rangle}{1-E(s)}, \\ \dfrac{\dfrac{1-s}{\sqrt{N}} + s\langle\beta|E(s)\rangle}{1-E(s)} \end{cases} \tag{14}$$

by using Eqs. (12) and (13). It can be immediately rewritten as

$$\langle x|E(s)\rangle = \begin{cases} \dfrac{\dfrac{1-s}{\sqrt{N}}\left(\sqrt{\dfrac{N-1}{N}}\langle\alpha|E(s)\rangle + \sqrt{\dfrac{1}{N}}\langle\beta|E(s)\rangle\right)}{1-E(s)}, \\ \dfrac{\dfrac{1-s}{\sqrt{N}}\left(\sqrt{\dfrac{N-1}{N}}\langle\alpha|E(s)\rangle + \sqrt{\dfrac{1}{N}}\langle\beta|E(s)\rangle\right) + s\langle\beta|E(s)\rangle}{1-E(s)}. \end{cases} \tag{15}$$

Let

$$A = \frac{(1-s)\sqrt{N-1}}{N}, \tag{16}$$

$$B = \frac{(1-s)}{N}, \tag{17}$$

$$C = B + s = \frac{(1-s)}{N} + s, \tag{18}$$

it can be rewritten as

$$\langle x|E(s)\rangle = \begin{cases} \dfrac{A}{1-E(s)}\langle\alpha|E(s)\rangle + \dfrac{B}{1-E(s)}\langle\beta|E(s)\rangle, & \langle x|\beta\rangle = 0, \\ \dfrac{A}{1-E(s)}\langle\alpha|E(s)\rangle + \dfrac{C}{1-E(s)}\langle\beta|E(s)\rangle, & \langle x|\beta\rangle = 1. \end{cases} \tag{19}$$

So,

$$\begin{aligned} &\langle\alpha|E(s)\rangle \\ &= \sum_{x=0}^{N-1}\langle\alpha|x\rangle\langle x|E(s)\rangle \\ &= \sum_{\langle\alpha|x\rangle\neq 0}\langle\alpha|x\rangle\langle x|E(s)\rangle \\ &= \frac{1}{\sqrt{N-1}}\sum_{\langle\beta|x\rangle=0}\langle\alpha|x\rangle\langle x|E(s)\rangle \\ &= \frac{\sqrt{N-1}}{1-E(s)}(A\langle\alpha|E(s)\rangle) + B\langle\beta|E(s)\rangle). \end{aligned} \tag{20}$$

Similarly,

$$\langle\beta|E(s)\rangle = \frac{1}{1-E(s)}(A\langle\alpha|E(s)\rangle) + C\langle\beta|E(s)\rangle). \tag{21}$$

Let

$$D = \sqrt{N-1}A = \frac{(1-s)(N-1)}{N}, \tag{22}$$

$$F = \sqrt{N-1}B = \frac{(1-s)(N-1)}{N}, \tag{23}$$

$$G = A = \frac{(1-s)(N-1)}{N} = F, \tag{24}$$

$$H = C = \frac{(1-s)}{N} + s. \tag{25}$$

Equations (20) and (21) become

$$\langle\alpha|E(s)\rangle = \frac{1}{1-E(s)}(D\langle\alpha|E(s)\rangle) + F\langle\beta|E(s)\rangle), \tag{26}$$

$$\langle\beta|E(s)\rangle = \frac{1}{1-E(s)}(F\langle\alpha|E(s)\rangle) + H\langle\beta|E(s)\rangle). \tag{27}$$

With these two equations, it is easy to obtain the secular function of H_s defined by Eq. (1) as

$$(1-E(s))^2 - (H+D)(1-E(s)) + DH - F^2 = 0. \tag{28}$$

So,

$$\begin{aligned} E(s)_{0,1} &= \frac{1-H-D\mp\sqrt{(D-H)^2+4F^2}}{2} \\ &= \frac{1\mp\sqrt{(1-2s)^2+\frac{4s(1-s)}{N}}}{2} \end{aligned} \tag{29}$$

together with Eqs. (22)–(25). And then

$$g_s = \sqrt{(1-2s)^2 + \frac{4s(1-s)}{N}}. \tag{30}$$

When $s = 1/2$, we finally obtain

$$g_{\min} = \sqrt{\frac{1}{N}}. \tag{31}$$

So, $T' = O(\sqrt{N})$. It can be proven that P is lower bounded by a constant.[14] Consequently, $T = O(\sqrt{N})$.

It is known that global adiabatic algorithms exhibit inherent robustness against certain types of quantum errors.[1] It is alo known that the system Hamiltonian of the partial adiabatic search algorithm evolves adiabatically (globally) within a small interval and changes suddenly outside of the interval without disturbing the system state. This implies that the partial adiabatic search algorithm keeps all advantages of global adiabatic algorithms without losing the speedup provided by the local adiabatic search algorithm.

In the following, we consider how to implement the partial adiabatic search algorithm on a quantum circuit. We use the same method exposed in Refs. [16] and [17]. We discretize the evolution by cutting the time T into R intervals $\Delta T = T/R$. During each interval $[t_{j-1}, t_j]$ ($t_j = j\Delta T$), we approximate the varying Hamiltonian $H(s(t))$ by the constant one $H(s(t_j)) = (1-s(t_j))H_{\rm i} + s(t_j)H_{\rm f}$. This is equivalent to replacing the Hamiltonian $H(s(t))$ by $H(s'(t))$, where $s'(t)$ is a monotonic function approaching $s(t)$ but varying at times t_j only.

The approximation of $H(s(t))$ by $H(s'(t))$ is such that

$$|||H(s(t)) - H(s'(t))|||_2$$

$$= |s'(t) - s(t)|||H_{\rm i} - H_{\rm f}|||_2$$
$$\leq |s(t+\Delta T) - s(t)| = \frac{\Delta T}{\sqrt{N}T}, \quad (32)$$

with the fact that

$$|||H_{\rm i} - H_{\rm f}|||_2 = \sqrt{1 - \frac{1}{N}} \leq 1. \quad (33)$$

Let $U(T)$ and $U'(T)$ be the unitary transformations induced by the application of the two Hamiltonians from $t = 0$ to $t = T$. This immediately gives

$$|||U(T) - U'(T)|||_2 \leq \sqrt{2\int_0^T \delta(t)\,{\rm d}t}. \quad (34)$$

Using inequalities (32) and (34), we have

$$|||U(T) - U'(T)|||_2 \leq \sqrt{\frac{2\Delta T}{\sqrt{N}}}. \quad (35)$$

For each step, we apply the unitary operation

$$U'_j = {\rm e}^{-{\rm i}H_j\Delta T} = {\rm e}^{-{\rm i}(1-s_j(t)H_i\Delta T - {\rm i}s_j(t)H_{\rm f}\Delta T}. \quad (36)$$

Now, we use the Campbell–Baker–Hausdorff approximation and replace U'_j by

$$U''_j = {\rm e}^{-{\rm i}(1-s_j)H_{\rm i}\Delta T}\,{\rm e}^{-{\rm i}s_jH_{\rm f}\Delta T}. \quad (37)$$

The error introduced at each step by this approximation is

$$|||U'_j - U''_j|||_2 \in O(s_j(1-s_j)\Delta T^2||||[H_{\rm i}, H_{\rm f}]||||_2$$
$$\in O\left(s_j(1-s_j)\frac{\sqrt{N}}{R^2}\right), \quad (38)$$

where we use

$$||||[H_{\rm i}, H_{\rm f}]||||_2 = \left|\left|\left|\frac{1}{\sqrt{N}}(|\Psi\rangle\langle\beta| - |\beta\rangle\langle\Psi|)\right|\right|\right|_2$$
$$= \frac{1}{\sqrt{N}}\sqrt{1 - \frac{1}{N}} \in O\left(\frac{1}{\sqrt{N}}\right). \quad (39)$$

Besides,

$$\left|\left|\left|\prod_j U'_j - \prod_j U''_j\right|\right|\right|_j \leq \sum_j |||U'_j - U''_j|||_2, \quad (40)$$

when

$$|||U'_j - U''_j|||_2 \ll 1. \quad (41)$$

Consequently, with relations (35), (38) and (40) we can see that the number of steps $R = O(\sqrt{N})$ is required in the previous approximation.

In conclusion, a specified quantum adiabatic search algorithm based on partial adiabatic evolution is presented in this paper. It is shown that the algorithm takes $O(\sqrt{N})$ time steps to find the marked item. That is, the algorithm keeps all advantages of global adiabatic algorithms without losing the speedup provided by the local adiabatic search algorithm. We also study how to implement the algorithm on a quantum circuit.

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