Fidelity susceptibility and geometric phase in critical phenomenon*

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Motivated by recent developments in quantum fidelity and fidelity susceptibility, we study relations among Lie algebra, fidelity susceptibility and quantum phase transition for a two-state system and the Lipkin–Meshkov–Glick model. We obtain the fidelity susceptibilities for $SU(2)$ and $SU(1,1)$ algebraic structure models. From this relation, the validity of the fidelity susceptibility to signal for the quantum phase transition is also verified in these two systems. At the same time, we obtain the geometric phases in these two systems by calculating the fidelity susceptibility. In addition, the new method of calculating fidelity susceptibility is used to explore the two-dimensional XXZ model and the Bose–Einstein condensate (BEC).

Keywords: fidelity susceptibility, geometric phase, quantum phase transition

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1. Introduction

Fidelity, one of the most intriguing features of quantum information science,[1] has been widely studied in recent years.[2−14] The quantum phase transition (QPT), driven purely by quantum fluctuations and occurring at zero temperature, is believed to be an important concept in condensed matter physics.[15] In a quantum many-body system, the QPT is driven purely by the quantum fluctuation in ground state. The wave function of the ground state can have an abrupt change as the system varies across the phase transition point. Therefore, an approach to quantum phase transition based on the quantum-information concept of fidelity has been put forward.[2] However, the calculation of the ground-state fidelity is tedious except that for a few specific models.[2,16] Recently, a neater and simpler formalism of fidelity to critical phenomenon was introduced,[17] for the so-called fidelity susceptibility, to signal the whole QPT. The main advantage of this approach lies in the fact that the fidelity is a purely Hilbert-space geometrical quantity and no prior knowledge of the structure of the considered system is required for its use. As shown in Ref. [17], the fidelity susceptibility is intrinsically related to the dynamic structure factor of the driving Hamiltonian that is evaluated though the scheme based on some numerical techniques including the exact diagonalization and the density matrix renormalization group. On the other hand, Zhang and Tian[18] employed Lie algebra to evaluate the fidelity susceptibility, and show its high efficiency.

In the present paper, we investigate, in a general framework, how the differential form of the fidelity susceptibility can be established in terms of the general Lie algebra. As will be seen below, under certain conditions, the differential form is an effective tool in detecting the critical points of the QPT. To demonstrate this, analytic formulas for the fidelity and the fidelity susceptibility are derived for the $SU(2)$ and $SU(1,1)$. By using these formulas, the fidelity and the fidelity susceptibility can be easily calculated for a large class of many-body systems, as long as the Hamiltonian of the system can be rewritten as the form of [see the following]. Applying our general formulas to the two-state system, we can show the fidelity susceptibility of the system in terms of $SU(2)$. At the same time, the geometric phase, which is also an ef-

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fective indicator in detecting the QPT,[19−22] can be obtained by calculating the fidelity susceptibility. On the other hand, according to the general expression of the differential form of the fidelity susceptibility, one can also expect the same results as those from the Lipkin–Meshkov–Glick (LMG) model in terms of $SU(1,\ 1)$. Furthermore, we extend this differential form of fidelity susceptibility to other physics models.

2. Formulism

The general Hamiltonian of quantum many-body systems reads

$$H(\lambda) = H_0 + \lambda H_{\rm I}, \tag{1}$$

where $H_{\rm I}$ is the driving Hamiltonian and λ denotes its strength. The fidelity is the absolute value of the overlap between two ground states $|\Psi_0(\lambda)\rangle$ and $|\Psi_0(\lambda+\delta\lambda)\rangle$, i.e.,[2]

$$F(\lambda, \lambda+\delta\lambda) = |\langle\Psi_0(\lambda)|\Psi_0(\lambda+\delta\lambda)\rangle| \tag{2}$$

with $\delta\lambda$ being a small deviation. Extracting the leading term of the fidelity, the fidelity susceptibility can be obtained as[17]

$$\chi_{\rm F}(\lambda) = \sum_{n\neq 0} \frac{|\langle\Psi_n(\lambda)|H_{\rm I}|\Psi_0(\lambda)\rangle|^2}{[E_n(\lambda)-E_0(\lambda)]^2}, \tag{3}$$

where $E_n(\lambda)$ denote the eigenvalues, and $|\Psi_n(\lambda)\rangle$ $(n = 1,\ 2,\ 3,\ \ldots)$ are the corresponding normalized eigenvectors. These eigenstates comprise a set of orthogonal complete bases in the Hilbert space.

Given a physical system, the Hamiltonian can be written as

$$H = \sum_i \epsilon_i H_i + \sum_\alpha (\lambda_\alpha E_\alpha + \lambda_\alpha^* E_{-\alpha}), \tag{4}$$

where ϵ_i and λ_α (i and $\alpha = 1, 2, 3, \ldots$) are the coupling parameters and $\{H_i, E_\alpha, E_{-\alpha}\}$ are the Cartan–Weyl basis of a semi-simple Lie algebra. Using the unitary operator

$$\hat{U}(\xi_\alpha(\lambda)) = \exp\left[\sum_\alpha (\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha})\right]$$

with ξ_α being parameters, the Hamiltonian can be turned into the diagonal form

$$\hat{U}^\dagger(\xi_\alpha) H \hat{U}(\xi_\alpha) = \sum_i \eta_i H_i. \tag{5}$$

Therefore, the eigenstate of Hamiltonian in Eq. (4) is $|\Psi\rangle = \hat{U}(\xi_\alpha)|{\rm ref}\rangle$, where $|{\rm ref}\rangle$ is the direct product of normalized eigenstates of H_i. Then the absolute value of the overlap between $|\Psi(\lambda)\rangle = \hat{U}(\xi_\alpha(\lambda))|{\rm ref}\rangle$ and $|\Psi(\lambda+\delta\lambda)\rangle = \hat{U}(\xi_\alpha(\lambda+\delta\lambda))|{\rm ref}\rangle$ is

$$F = \left|\langle{\rm ref}|\hat{U}^\dagger(\xi_\alpha(\lambda))\hat{U}(\xi_\alpha(\lambda+\delta\lambda))|{\rm ref}\rangle\right|. \tag{6}$$

According to the general relation between fidelity and fidelity susceptibility, $F = 1 + \frac{1}{2}(\delta\lambda)^2\chi(\lambda) + \cdots$, the expression of the fidelity susceptibility is given by[18]

$$\begin{aligned}\chi(\lambda) =& -\langle{\rm ref}|\left(\hat{U}^\dagger\partial_\lambda\hat{U}\right)^2|{\rm ref}\rangle \\ & -\left|\langle{\rm ref}|\left(\hat{U}^\dagger\partial_\lambda\hat{U}\right)|{\rm ref}\rangle\right|^2.\end{aligned} \tag{7}$$

3. Two-state system

The two-state system is the simplest quantum system which can be calculated exactly.

Furthermore, the two-state system possesses several typical quantum properties. So we study it first. The Hamiltonian of a two-state system can be written as

$$H = -\boldsymbol{B}\cdot\boldsymbol{\sigma}, \tag{8}$$

where $\boldsymbol{B}$ is an external magnetic field and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices.

By means of the generators of the algebra $SU(2)$: $J_+ = \frac{1}{2}\sigma_+$, $J_- = \frac{1}{2}\sigma_-$ and $J_z = \frac{1}{2}\sigma_z$ which satisfy the usual commutation relations $[J_+, J_-] = 2J_z$ and $[J_z, J_\pm] = \pm J_\pm$, we can rewrite H as

$$H = -2B\cos\theta J_z - B\sin\theta{\rm e}^{{\rm i}\,\phi}J_+ - B\sin\theta{\rm e}^{{\rm i}\,\phi}J_-. \tag{9}$$

Introducing the unitary operator

$$\hat{U}(\theta,\phi) = \exp\left(-\frac{\theta}{2}{\rm e}^{-{\rm i}\,\phi}J_+ + \frac{\theta}{2}{\rm e}^{{\rm i}\,\phi}J_-\right),$$

the Hamiltonian in Eq. (8) can be diagonalized as

$$\hat{U}^\dagger(\theta,\phi) H \hat{U}(\theta,\phi) = -BJ_z. \tag{10}$$

Here θ and ϕ can be regarded as adiabatic parameters. For simplicity and without loss of generality, we fix θ first and have

$$\begin{aligned}&\hat{U}^\dagger(\theta,\phi)\partial_\phi\hat{U}(\theta,\phi) \\ &= \frac{\rm i}{2}\sin\theta\left({\rm e}^{-{\rm i}\,\phi}J_+ + {\rm e}^{{\rm i}\,\phi}J_-\right) + 2\,{\rm i}\sin^2\frac{\theta}{2}J_z,\end{aligned}$$

and

$$\left|\langle{\rm ref}|\left(\hat{U}^\dagger(\theta,\phi)\partial_\phi\hat{U}(\theta,\phi)\right)|{\rm ref}\rangle\right|^2 = \sin^4\frac{\theta}{2}.$$

Finally we obtain the fidelity susceptibility of spin-$\frac{1}{2}$ subjected to an external magnetic field as

$$\chi_{\rm F} = \frac{1}{4}\sin^2\theta. \tag{11}$$

Since $\langle {\rm ref}| \left(\hat{U}^{\dagger}(\theta,\phi)\,\partial_{\phi}\hat{U}(\theta,\phi)\right)|{\rm ref}\rangle$ is just the Berry adiabatic connection, which contributes a Pancharatnam–Berry phase[23,24] to the spin as the magnetic field rotates adiabatically around cone direction, we obtain the Berry phase as

$$\begin{aligned}\gamma(\theta,\phi) &= -{\rm i}\int_0^{2\pi} {\rm d}\phi\,\langle {\rm ref}|\left(\hat{U}^{\dagger}(\theta,\phi)\,\partial_{\phi}\hat{U}(\theta,\phi)\right)|{\rm ref}\rangle \\ &= \pm\pi\,(1-\cos\theta)\,. \end{aligned}\tag{12}$$

The results of Eqs. (11) and (12) are in agreement with those in Ref. [25], which verify the reliability of our method.

4. Lipkin–Meshkov–Glick model

The LMG model[26−28] was originally introduced in nuclear physics. It provides a simple description of the tunneling of bosons between two degenerate levels and can thus be used to describe many physical systems, such as two-mode Bose–Einstein condensates (BEC)[29] and Josephson junctions.[30] Recently the entanglement in this model has attracted much interest because of available numerical calculations and plentiful phase diagrams.[31] In the thermodynamic limit, its phase diagram can be simply established by a semiclassical approach.[32] For a finite large number N of particles, it was studied by the $1/N$ expansion in the Holstein–Primakoff single boson representation[33] and by the continuous unitary transformation.[34]

The Hamiltonian of the LMG model can be written as

$$\begin{aligned} H &= -\frac{\lambda}{N}\sum_{i<j}\left(\sigma_x^i\sigma_x^j+\gamma\sigma_y^i\sigma_y^j\right)-h\sum_i\sigma_z^i \\ &= -\frac{2\lambda}{N}\left(S_x^2+\gamma S_y^2\right)-2hS_z+\frac{\lambda}{2}(1+\gamma) \\ &= -\frac{\lambda}{N}(1+\gamma)\left(S^2-S_z^2-\frac{N}{2}\right)-2hS_z \\ &\quad -\frac{\lambda}{2N}(1-\gamma)\left(S_+^2+S_-^2\right), \end{aligned}\tag{13}$$

where the σ_{κ} $(\kappa=x,y,z)$ are the Pauli matrices; $S_{\kappa}=\sum_i\sigma_{\kappa}^i/2$ with $S_{\pm}=S_x\pm{\rm i}\,S_y$; λ and h are the spin–spin interaction strength and the effective external field, respectively; N is the total spin number and $1/N$ ensures that the free energy per spin is finite in the thermodynamical limit. It is understood that H preserves the magnitudes of the total spin and the parity $P=\Pi_i\sigma_z^i$, i.e.,

$$\left[H,S^2\right]=0,\quad [H,P]=0,$$

for all values of the anisotropy parameter γ. Specially, in the isotropic case $\gamma=1$, one has $[H,S_z]=0$, so that H is diagonal in the eigenbasis of S^2 and S_z. We adopt the $1/N$ expansion method corresponding to the large N limit. We first use the Holstein–Primakoff boson representation of the spin operator[33] in the $S=N/2$ subspace, given by

$$\begin{aligned} S_z &= S-a^{\dagger}a=\frac{N}{2}-a^{\dagger}a, \\ S_+ &= S_-^{\dagger}=\left(2S-a^{\dagger}a\right)^{1/2}a \\ &= N^{1/2}\left(1-\frac{a^{\dagger}a}{N}\right)^{1/2}a, \end{aligned}\tag{14}$$

where the standard bosonic creation and annihilation operators satisfy $\left[a,a^{\dagger}\right]=1$. This representation is well adapted to the computation of the low-energy physics with $\left\langle a^{\dagger}a\right\rangle/N\ll 1$.

The next step consists in inserting these expressions into Eq. (13) and expanding the argument of the square roots. Keeping these terms of order $(1/N)^{-1}$, $(1/N)^{-1/2}$ and $(1/N)^0$ yields $(h\geq 1)$

$$\begin{aligned} H &= -hN+(2h-1-\gamma)\,a^{\dagger}a \\ &\quad -\left[(1-\gamma)/2\right]\left(a^{\dagger 2}+a^2\right). \end{aligned}\tag{15}$$

In order to analyse the fidelity susceptibility of the above equation, we transform Eq. (15) into the form of $SU(1,1)$ according to the ideas in Ref. [35]. We first introduce the generators of $SU(1,1)$, namely,

$$K_+=\frac{1}{2}a^{\dagger 2},\quad K_-=\frac{1}{2}a^2,\quad K_z=\frac{1}{4}\left(2a^{\dagger}a+1\right). \tag{16}$$

Substituting Eq. (16) into Eq. (15), one can have

$$\begin{aligned} H &= -hN-\frac{1}{2}(h-\gamma-1)+2\,(2h-\gamma-1)\,K_z \\ &\quad +(\gamma-1)\,K_+ +(\gamma-1)\,K_-, \end{aligned}\tag{17}$$

which consists with Eq. (4). Here we do not show the diagonalization of $SU(1,1)$ algebraic structure model explicitly, but interested readers are recommended to refer to Ref. [18]. We simply quote the main result that is related to the fidelity susceptibility of the $SU(1,1)$ algebra model. After all the procedures, the fidelity susceptibility of $SU(1,1)$ algebraic structure model becomes

$$\chi_{SU(1,1)}=\frac{1}{8}\left[\left(\frac{\partial\theta}{\partial\lambda}\right)^2+\sinh^2\theta\left(\frac{\partial\phi}{\partial\lambda}\right)\right]. \tag{18}$$

Assuming that $\tanh\theta = (1-\gamma)/(2h-1-\gamma)$, the fidelity susceptibility is represented by

$$\chi_{\rm F} = \frac{(1-\gamma)^2}{32\,(1-h)^2\,(h-\gamma)^2},$$

which is in agreement with the result in Ref. [36].

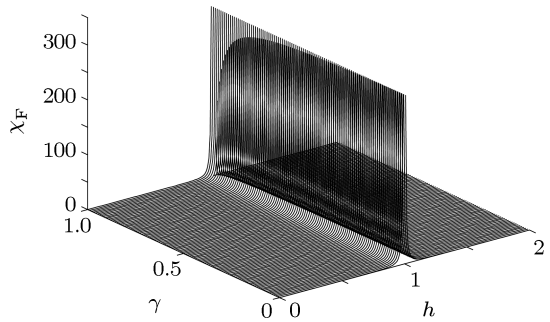


Fig. 1. Fidelity susceptibility as a function of h and γ. The divergent character of $\chi_{\rm F}$ is clearly displayed as $h \to 1$.

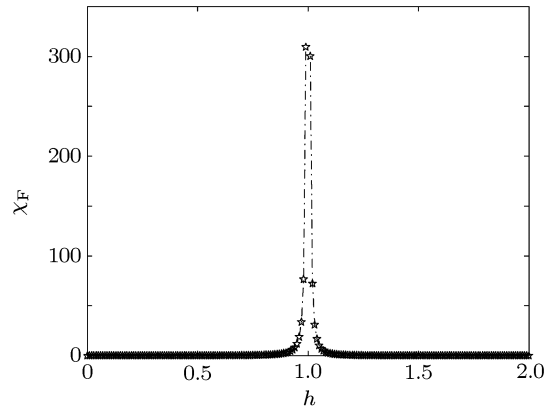


Fig. 2. Fidelity susceptibility as a function of h for $\gamma = 0.5$ in large N limit.

The above derivation is valid only for $h \geq 1$, the calculation for $0 < h < 1$ is actually similar to the above case. When $0 < h < 1$, $\tanh\theta = (h^2-\gamma)/(2-h^2-\gamma)$, the fidelity susceptibility is then obtained accordingly as $\chi_{\rm F} = h^2/8\left(1-h^2\right)^2$. Thus we obtain the fidelity susceptibility of the anisotropic LMG model in the large N limit. As shown in Figs. 1 and 2, it is obvious that $\chi_{\rm F}$ is divergent at h=1 where the LMG model shows that the second-order phase transition is independent of anisotropy γ, which was well described by a mean-field approach.[34] Through the calculation of fidelity susceptibility, we have found the geometric phase. For illustration, we consider the system which has a rotation $g(\phi)$ around the new z direction. The Hamiltonian becomes $H(\phi) = g(\phi)\,Hg^{+}(\phi)$ with $g(\phi) = {\rm e}^{{\rm i}\,\phi S_z}$. Then equation (15) can be written as

$$\begin{aligned} H &= -hN + (2h-1-\gamma)a^{\dagger}a \\ &\quad -[(1-\gamma)/2](a^{\dagger 2}{\rm e}^{-2\,{\rm i}\,\phi} + a^2{\rm e}^{2\,{\rm i}\,\phi}). \end{aligned} \tag{19}$$

The geometric phase of the ground state, accumulated by varying the angle ϕ from 0 to π, is described by $\beta = -{\rm i}\int_0^{\pi} {\rm d}\phi\,\langle g\,|\partial/\partial\phi|\,g\rangle$.[19] Finally we obtain the geometric phase as

$$\begin{aligned} \beta_{\rm g} &= -{\rm i}\int_0^{\pi} {\rm d}\phi\,\langle 1,0\,|U(\theta,\phi)\,\partial_{\phi}U(\theta,\phi)|\,1,0\rangle \\ &= -{\rm i}\int_0^{\pi}\left(-2\,{\rm i}\sinh^2\frac{\theta}{2}\right){\rm d}\phi \\ &= \pi\,(1-\cosh\theta)\,. \end{aligned} \tag{20}$$

The above derivation is also valid only for $h \geq 1$. When $0 < h < 1$, $\tanh\theta = (h^2-\gamma)/(2-h^2-\gamma)$. Therefore we can have the geometric phase of the LMG model in the thermodynamic limit. One can find that $\beta_{\rm g}$ is also divergent at $h = 1$, which is equal to the fidelity susceptibility in indicating the quantum phase transition in Figs. 3 and 4.

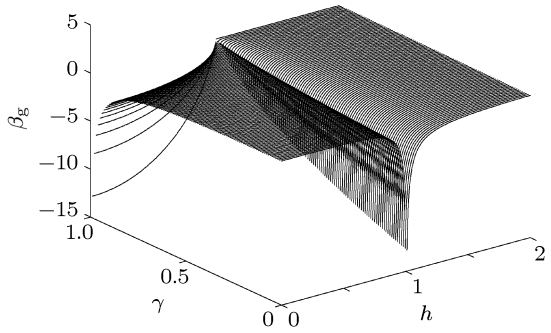


Fig. 3. Geometric phase as a function of h and γ. It is obvious that $\beta_{\rm g}$ is divergent at $h = 1$ and independent of γ.

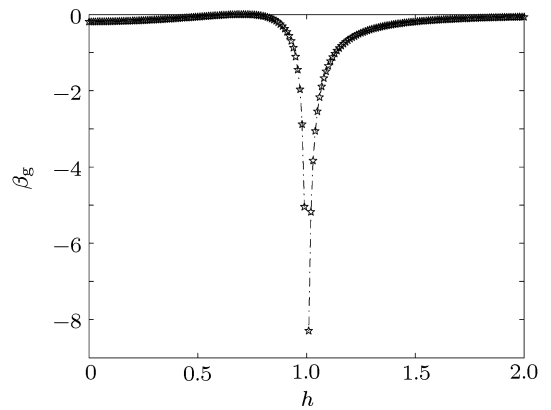


Fig. 4. Geometric phase as a function of h with parameter $\gamma = 0.5$.

5. Other models

In this section, we generalize the new method of calculating fidelity susceptibility to other quantum many-body models. As there exist no exact analytical results of the fidelity susceptibility of the two-dimensional XXZ model and the Bose–Einstein condensate, we expect the following results can induce more wonderful ideas or results.

5.1. Two-dimensional XXZ model

The Hamiltonian of the XXZ antiferromagnetic model reads

$$H_{xxz} = \sum_{\langle ij \rangle} \left(S_i^x S_j^x + S_i^y S_j^y + \eta S_i^z S_j^z \right), \tag{21}$$

where S_i^α $(\alpha = x, y, z)$ are the spin-1/2 operators at site i and $\eta = J_z/J_x\,(J_x = J_y)$ is a dimensionless parameter characterizing the anisotropy of the model. The summation runs over all the nearest neighbours on a square lattice. We begin with the two-sublattice model and Holstein–Primakoff transformation as follows:[33]

$$\begin{aligned} S_a^+ &= \sqrt{2S} a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2}, \quad S_a^- = \left(S_a^+\right)^+, \\ S_b^+ &= \sqrt{2S} b^\dagger \left(1 - \frac{b^\dagger b}{2S}\right)^{1/2}, \quad S_b^- = \left(S_b^+\right)^+, \\ S_a^z &= -S + a^\dagger a, \qquad\qquad S_b^z = S - b^\dagger b, \end{aligned}$$

where $a^\dagger$ and a (or $b^\dagger$ and b) are the boson creation operator and the boson annihilation operator on sublattice A (or sublattice B), respectively. The particle numbers $n_a = a^\dagger a$ and $n_b = b^\dagger b$ may not exceed $2S$. Transforming the operators into those in momentum space, we rewrite the Hamiltonian as

$$H_{xxz} = -2\eta z N S^2 + 2zS \sum_k H_k, \tag{22}$$

where z is the coordination number of the lattice and H_k is in the form of

$$H_k = \eta \left(a_k^\dagger a_k + b_k^\dagger b_k \right) + \gamma_k (a_k b_k + a_k^\dagger b_k^\dagger).$$

Here $\gamma_k = z^{-1} \sum_{\boldsymbol{R}} \mathrm{e}^{\mathrm{i}\,\boldsymbol{K}\cdot\boldsymbol{R}}$ with $\boldsymbol{R}$ being a vector relating an atom with its nearest neighbour. We can rewrite H_k in the $SU(1,1)$ form as

$$H_k = 2\eta A_z^k + \gamma_k (A_+^k + A_-^k), \tag{23}$$

by means of the generators of the algebra $SU(1,1)$: $A_+^k = a_k^\dagger b_k^\dagger$, $A_-^k = a_k b_k$ and $A_z^k = \frac{1}{2}\left(n_k^a + n_k^b + 1\right)$. Taking a similar transformation of H_k, we obtain the unitary operator $U = \exp(\xi_k A_+^k - \xi_k A_-^k)$ with $\tan\xi_k = -\gamma_k/\eta$. Within the spin-wave theory framework, we obtain the unitary operator and then calculate the fidelity susceptibility of the model in two dimensions. Substituting $\tanh\theta = \gamma_k/\eta$ and $\mathrm{e}^{-\mathrm{i}\,\Phi} = \gamma_k/\gamma_k = 1$ into Eq. (18), one can obtain

$$\chi = \int \frac{\left(\eta \partial_\lambda \gamma_\kappa - \gamma_\kappa \partial_\lambda \eta\right)^2}{32\left(\eta^2 - \gamma_\kappa^2\right)^2} \frac{\mathrm{d}\boldsymbol{K}}{(2\pi)^3}. \tag{24}$$

5.2. Fidelity susceptibility for BEC

The standard description of the Bose–Einstein condensate is given via an order parameter field $\Psi(x)$. The Hamiltonian takes the standard form

$$\begin{aligned} \mathscr{H}[\Psi] = & \int \mathrm{d}^3 x \left[\frac{\hbar^2}{2m} |\nabla \Psi(x)|^2 + U(x) |\Psi(x)|^2 \right] \\ & + \frac{1}{2} \int \mathrm{d}^3 x \int \mathrm{d}^3 y \, \Psi^*(y)\, \Psi^*(x) \\ & \times V(x,y)\, \Psi(y)\, \Psi(x). \end{aligned} \tag{25}$$

As in Ref. [35], equation (25) can be rewritten in the form of $SU(1,1)$ algebra

$$\begin{aligned} \mathscr{H} = & 2 \left[\sigma_0 A_3^0 + \frac{1}{2}\left(u_0 A_+^0 + u_0^* A_-^0\right) \right] \\ & + \sum_{k \neq 0} \left[\sigma_k A_3^k + \frac{1}{2}\left(u_k A_+^k + u_k^* A_-^k\right) \right] - E_*, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \sigma_0 &\equiv \epsilon_0 + \frac{1}{2} \sum_{k \neq 0} (V_0 + V_k)(\langle n_k \rangle + \langle n_{-k} \rangle), \\ u_0 &\equiv V_0 \langle a_0^2 \rangle + \sum_{k \neq 0} V_k \langle a_k a_{-k} \rangle, \\ u_k &\equiv V_k \langle a_0^2 \rangle, \quad \sigma_k \equiv \epsilon_k + \langle n_0 \rangle (V_0 + V_k), \end{aligned}$$

and

$$\begin{aligned} E_* = & \frac{1}{2}[V_0 |\langle a_0^2 \rangle|^2 + \sigma_0] \\ & + \frac{1}{2} \sum_{k \neq 0} [(\sigma_k - \epsilon_k)(\langle n_k + n_{-k} \rangle + \sigma_k] \\ & + \frac{1}{2} \sum_{k \neq 0} (u_k \left\langle a_k^\dagger a_{-k}^\dagger \right\rangle + u_k^* \langle a_k a_{-k} \rangle). \end{aligned}$$

Introducing the generators of the algebra $SU(1,1)$ $A_3^0 = \frac{1}{2}(n_0 + \frac{1}{2})$, $A_+^0 = a_0^{\dagger 2}/2$, $A_-^0 = a_0^2/2$, and $A_3^k = \frac{1}{2}(n_k + n_{-k} + 1)$, $A_+^k = a_k^\dagger a_{-k}^\dagger$, $A_-^k = a_k a_{-k}$, we can calculate the fidelity susceptibility of $\mathscr{H}_0 = 2[\sigma_0 A_3^0 + \frac{1}{2}(u_0 A_+^0 + u_0^* A_-^0)]$. From this equation, we can obtain

$$U = \exp(\xi_0 A_+^0 - \xi_0^* A_-^0), \quad \text{and} \quad \xi_0 = r \exp(\mathrm{i}\,\phi).$$

According to the same procedures as those in the two-dimensional XXZ model, we can obtain the fidelity susceptibility of the $\mathscr{H}_0$ as follows:

$$\chi_0 = \frac{[\sigma_0(u_0\partial_\lambda u_0^* + u_0^*\partial_\lambda\sigma) - 2|u_0|^2\partial_\lambda\sigma]^2}{32|u_0|^2(\sigma_0^2 - |u_0|^2)^2} - \frac{(\sigma_0^2 - |u_0|^2)(u_0^*\partial_\lambda u_0 - u_0\partial_\lambda u_0^*)^2}{32|u_0|^2(\sigma_0^2 - |u_0|^2)^2}.$$

Similarly the fidelity susceptibility of the $\mathscr{H}_k$ can be obtained as

$$\chi_k = \frac{[\sigma_k(u_k\partial_\lambda u_k^* + u_k^*\partial_\lambda\sigma) - 2|u_k|^2\partial_\lambda\sigma_k]^2}{32|u_k|^2(\sigma_k^2 - |u_k|^2)^2} - \frac{(\sigma_k^2 - |u_k|^2)(u_k^*\partial_\lambda u_k - u_k\partial_\lambda u_k^*)^2}{32|u_k|^2(\sigma_k^2 - |u_k|^2)^2}.$$

6. Conclusions

We have established the differential form of the fidelity susceptibility in terms of the general Lie algebras. Meanwhile we investigated the geometric phase which can also show the phase transition point. Therefore, we constructed the relation between fidelity susceptibility and geometric phase. We also applied the differential form of fidelity susceptibility to other physics models. In particular, we focus on the $SU(2)$ and $SU(1,1)$ algebras. The forms of the fidelity susceptibilities of $SU(2)$ and $SU(1,1)$ algebra can be used not only to evaluate the fidelity susceptibility easily, but also to build a relationship between quantum information theory and the Lie algebra in quantum many-body physics.

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