Chaotic solutions of (2+1)-dimensional Broek–Kaup equation with variable coefficients*

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In this paper, an improved projective approach is used to obtain the variable separation solutions with two arbitrary functions of the (2+1)-dimensional Broek–Kaup equation with variable coefficients (VCBK). Based on the derived solitary wave solution and using a known chaotic system, some novel chaotic solutions are investigated.

Keywords: VCBK equation, solitary wave solution, a known chaotic system, chaotic solutions

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1. Introduction

Over the past 30 years the concept of solitons has entered into various branches of many natural sciences such as chemistry, biology, mathematics, communication and, in particular, in almost all branches of physics such as fluid dynamics, plasma physics, field theory, optics and condensed matter physics. Searching for an analytical exact solution to a nonlinear system has long been an important and interesting topic in nonlinear science both for physicists and mathematicians. Various methods for obtaining exact solutions of nonlinear system have been proposed. For instance, the bilinear method, the standard Painlevé truncated expansion, the method of "coalescence of eigenvalue" or "wavenumbers", the homogenous balance method, the homotopy-perturbation method, the hyperbolic function method, the Jacobian elliptic method, the variable separation method, the (G'/G)-expansion method [1−14] and the projective equation approach,[15−20] etc. In this paper, with an improved projective approach and a linear variable separation approach, we obtain a new family of exact solutions to (2+1)-dimensional Broek–Kaup system with variable coefficients (VCBK)

$$\begin{aligned} &u_{ty} - a(t)[u_{xxy} - 2(uu_x)_y - 2v_{xx}] = 0, \\ &v_t + a(t)[v_{xx} + 2(uv)_x] = 0. \end{aligned} \tag{1}$$

Based on the derived solitary wave solution and using a known chaotic system, we obtain some special chaotic patterns. Here $a(t)$ is an arbitrary function of time t. It is evident that when $a(t) = 1$, the VCBK system will be degenerated to the well-known (2+1)-dimensional Broer–Kaup–Kupershmidt (BKK) system, which may be derived from the inner parameter-dependent symmetry constant of the Kadomtsev–Petviashvili model. When $y = x$, the (2+1)-dimensional Broer–Kaup–Kupershmidt system is reduced to the usual (1+1)-dimensional BKK system, which is often used to describe the propagation of long waves in shallow water.[21] Using some suitable dependent and independent variable transformations, the (2+1)-dimensional BKK system can be further transformed to the (2+1)-dimensional dispersive long-water wave equation and (2+1)-dimensional Ablowitz–Naup–Newell–Segur system.[22] Actually the (2+1)-dimensional BKK system has been widely investigated in detail by many researchers.

2. Solitary wave solutions to the VCBK system

The basic idea of the projective equation approach is as follows: Consider a given nonlinear partial differential equation (NPDE) with independent variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$, and one dependent

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variable u, in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{2}$$

where P is in general a polynomial function of its arguments and the subscripts denote the partial derivatives. One assumes its solution in the form,

$$u = A(x) + \sum_{i=1}^{n} B_i(x)\phi^i(q(x)) \tag{3}$$

with

$$\phi' = \sigma\phi + \phi^2, \tag{4}$$

where σ is a constant and the prime denotes differentiation with respect to q. To determine u explicitly, one may substitute Eqs. (3) and (4) into the given NPDE, collect the coefficients of the polynomials in ϕ, then eliminate each coefficient to derive a set of partial differential equations of A, B_i, and q, solves this system of partial differential equations to obtain A, B_i, and q. Finally, as equation (4) is known to possess the general solutions

$$\phi = \frac{-\sigma \mathrm{e}^{\sigma \mathrm{q} + \mathrm{c}\sigma}}{\mathrm{e}^{\sigma \mathrm{q} + \mathrm{c}\sigma} - 1}, \quad \sigma \neq 0, \tag{5}$$

by some appropriate transformation for Eq. (5), we can obtain the following solitary wave solutions of Eq. (4)

$$\phi = \begin{cases} -\dfrac{1}{2}\sigma\left[1 + \tanh\left(\dfrac{1}{2}\sigma q\right)\right], & \sigma = 0, \\ -\dfrac{1}{2}\sigma\left[1 + \coth\left(\dfrac{1}{2}\sigma q\right)\right], & \sigma \neq 0. \end{cases} \tag{6}$$

Substituting A_i, B_i, q and Eq. (6) into Eq. (3), one obtains the exact solutions to the given NPDE.

In order to find some new solutions of the VCBK system, we try to improve the above method. We rewrite the ansatz (3) as

$$\begin{aligned} u = {} & A(x) + \sum_{i=1}^{n} B_i(x)\phi^i(q(x)) \\ & + C_i(x)\phi^{i-1}(q(x))\sqrt{\sigma\phi + \phi^2(q(x))}). \end{aligned} \tag{7}$$

First, we make a transformation of Eq. (1): $v = u_y$. Substituting this transformation into Eq. (1) yields

$$u_{yt} + a[2u_x u_y + 2uu_{xy} + u_{xxy}] = 0. \tag{8}$$

Now we apply the improved projective equation approach to Eq. (8). By the balancing procedure, ansatz (7) becomes

$$\begin{aligned} u = {} & f(x, y, t) + g(x, y, t)\phi(q(x, y, t)) \\ & + h(x, y, t)\sqrt{\sigma\phi(q(x, y, t)) + \phi^2(q(x, y, t))}\, q, \end{aligned} \tag{9}$$

where f, g, h and q are functions of (x, y, t) to be determined. Substituting Eqs. (9) and (4) into Eq. (8) and collecting the coefficients of the polynomials of ϕ, then setting each coefficient to zero, we have

$$\begin{aligned} & f = -\frac{1}{4a}\frac{2q_t + 2aq_{xx} + a\sigma q_x^2}{q_x}, \\ & g = -\frac{1}{2}q_x, \quad h = \pm\frac{1}{2}q_x \end{aligned} \tag{10}$$

with

$$q = \chi(x, t) + \varphi(y), \tag{11}$$

where $\chi \equiv \chi(x, t)$, $\varphi \equiv \varphi(y)$ are two arbitrary variable separation functions of (x, t) and of y, respectively. Based on the solutions of Eq. (4), we can derive the following solitary wave solutions of Eq. (1):

$$u_1 = \frac{1}{2a}\frac{-a\chi_{xx} - \chi_t}{\chi_x} + \frac{1}{4a}\chi_x a\sigma\left\{\tanh\left(\frac{1}{2}\sigma(\chi + \varphi)\right) \pm \sqrt{-1 + \tanh\left(\frac{1}{2}\sigma(\chi + \varphi)\right)^2}\right\}, \tag{12}$$

$$v_1 = -\frac{1}{8}\sigma^2\chi_x\varphi_y\sqrt{-1 + \tanh\left(\frac{1}{2}\sigma(\chi + \varphi)\right)^2}\left\{\pm\tanh\left(\frac{1}{2}\sigma(\chi + \varphi)\right) + \sqrt{-1 + \tanh\left(\frac{1}{2}\sigma(\chi + \varphi)\right)^2}\right\}, \tag{13}$$

$$u_2 = \frac{1}{2a}\frac{-a\chi_{xx} - \chi_t}{\chi_x} + \frac{1}{4a}\chi_x a\sigma\left\{\coth\left(\frac{1}{2}\sigma(\chi + \varphi)\right) \pm \sqrt{-1 + \coth\left(\frac{1}{2}\sigma(\chi + \varphi)\right)^2}\right\}, \tag{14}$$

$$v_2 = -\frac{1}{8}\sigma^2\chi_x\varphi_y\sqrt{-1 + \coth\left(\frac{1}{2}\sigma(\chi + \varphi)\right)^2}\left\{\pm\coth\left(\frac{1}{2}\sigma(\chi + \varphi)\right) + \sqrt{-1 + \coth\left(\frac{1}{2}\sigma(\chi + \varphi))^2\right.}\right\}, \tag{15}$$

with two arbitrary functions being $\chi(x, t)$ and $\varphi(y)$.

3. Dromion excitation of the VCBK system

Due to the arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ included in the above cases, the physical quantities u may possess rich structures. For example, when $\chi = ax + ct$ and $\varphi = ky$, all the solutions of the above cases become simple travelling wave excitations. Moreover, based on the derived solutions, we may obtain rich localized structures such as dromion and chaotic patterns. In the following discussion, we merely analyse some special localized excitations of the solution v_2 [Eq. (15)], namely

$$V = v_2 = -\frac{1}{8}\sigma^2\chi_x\varphi_y\sqrt{-1+\coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2}\left\{\coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)+\sqrt{-1+\coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2}\right\}. \quad (16)$$

In (2+1) dimensions, one of the most important nonlinear solutions is the dromion excitation, which is localized in all directions exponentially. According to the solution V [Eq. (16)], we first discuss the dromion excitation. For instance, if we choose χ and φ as

$$\chi = 1 + 0.1\exp(x+t), \quad \varphi = 1 + 0.1\exp(y), \quad (17)$$

we can obtain a dromion excitation for the physical quantity V under the condition (17) as shown in Fig. 1 with fixed parameters $\sigma = -1$ and $t = 3$.

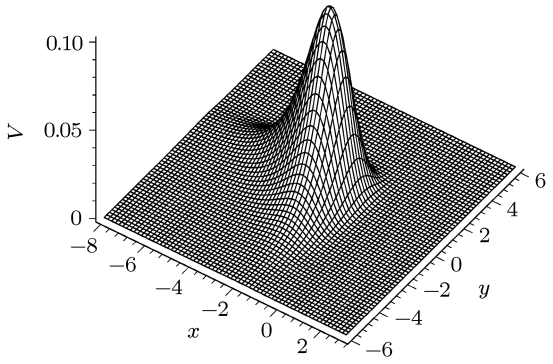


Fig. 1. A plot of the dromion structure for the physical quantity V given by the solution (16) with the choice (17) and $\sigma = -1, t = 3$.

4. Chaotic behaviours of the VCBK system

Just as solitons, chaos is another important part of nonlinear science. It has been widely applied in many natural sciences. In this section, we mainly discuss some localized coherent excitations with chaotic behaviour in the (2+1)-dimensional VCBK system.

4.1. Novel chaotic attractor

Recently, Wu and Guo[23] have introduced a new chaotic system (WG) of three-dimensional (3D) quadratic autonomous ordinary differential equations

$$m_\xi = 0.2m - 0.2n, \quad n_\xi = m - 0.15n + l,$$
$$l_\xi = m + n^2 - 0.6l, \quad (18)$$

where m, n, and l are functions of ξ ($\xi = (x + ct)$ or $\xi = y$). A novel chaotic attractor for the WG system (18) is depicted in Fig. 2 when

$$m(0) = 0.02, \quad n(0) = 0.13, \quad l(0) = 0.06. \quad (19)$$

4.2. Chaotic patterns

If the functions χ and/or φ are assumed to be solutions of a chaotic dynamical system, we can derive some localized excitations with chaotic behaviour. For example, χ is defined to be a solution of the WG system (18), and takes

$$\chi = 1 + 0.1m(x+ct), \quad \varphi = 1 + 0.1\exp(y), \quad (20)$$

where $m(x + ct)$ is a solution of the WG system (18) with the initial conditions (19). By this choice, the dromion localized in all directions is changed into a chaotic line soliton, which presents chaotic behaviour in the x direction though still localized in y direction. Figure 3(a) shows the corresponding plot of the chaotic line soliton for the field V of Eq. (16) with parameter $c = 1$ at time $t = 0$.

Furthermore, if χ and φ are both selected as chaotic solutions of the WG system, the field V of Eq. (16) will behave chaotically in all directions and will yield a chaotic pattern. For example, χ and φ may be chosen as

$$\chi(x) = 1 + 0.05m(x+ct),$$
$$\varphi(y) = 1 + 0.05m(y), \quad (21)$$

where $m(x+ct)$, and $m(y)$ are the solutions of the WG system (18) under the initial conditions (19). Figure 3(b) shows a plot of the special chaotic pattern for the field V expressed by Eq. (16) under the condition (21) at time $t = 0$. In order to show that the chaotic behaviour is due to the peak value of solitons, we enlarge small regions ($x \in [60, 76]$, $y \in [60, 76]$) of Fig. 3(b). The result is shown in Fig. 3(c), which presents a kind of dromion with a chaotic structure clearly.

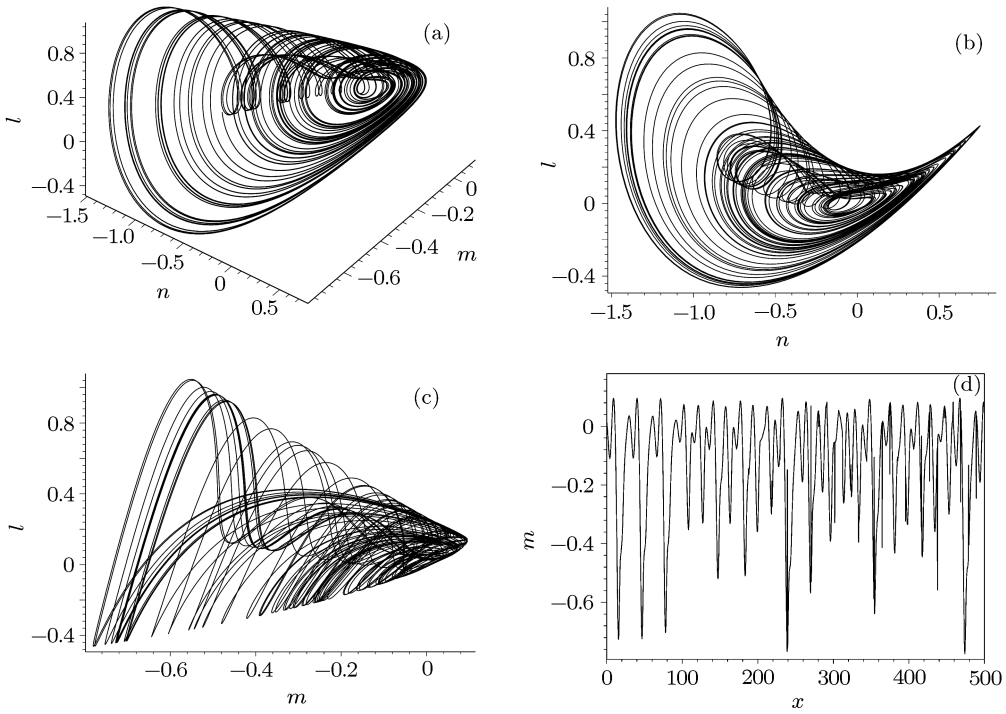


Fig. 2. (a) A novel attractor plot of the chaotic WG system (18) with the initial condition (19). (b) Two-dimensional phase diagram of n, l. (c) Two-dimensional phase diagram of m, l. (d) A typical plot of the chaotic solution m of WG system (18) related to (a).

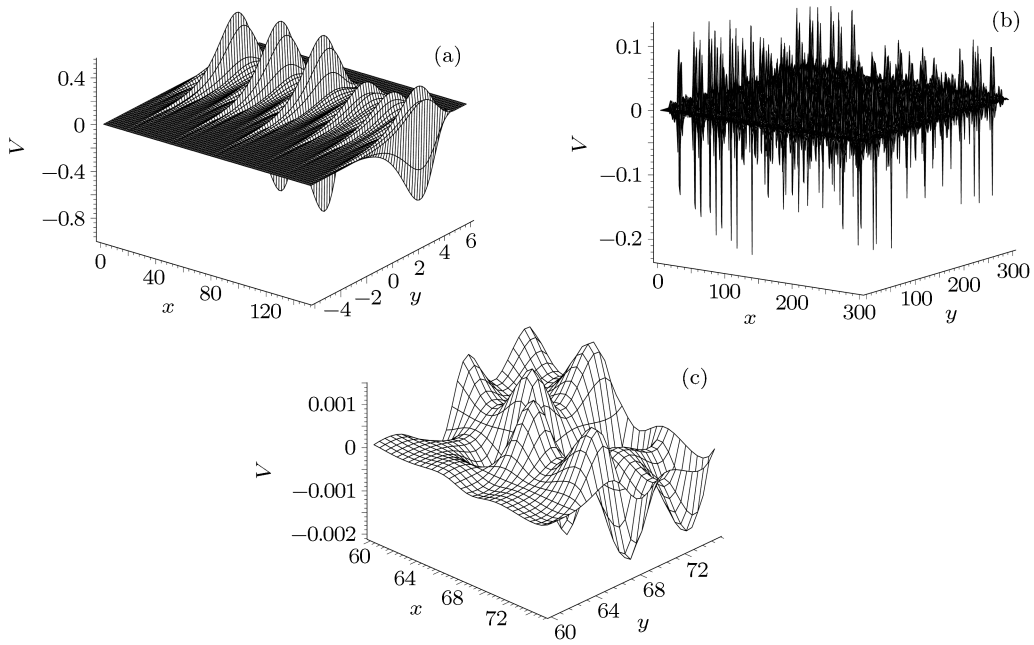


Fig. 3. (a) Plot of the chaotic line soliton for the field V determined by Eq. (16) under condition (20) at $c = 1$, $t = 0$. (b) Plot of the chaotic patten for the field V expressed by Eq. (16) under condition (21) at $c = 1$, $t = 0$. (c) An enlargement of the centre area ($x \in [60, 76]$, $y \in [60, 76]$) of (b).

5. Summary and discussion

In summary, via an improved projective approach and a special variable separation form $q = \chi(x,t) + \varphi(y)$, we find some new exact solutions of the (2+1)-dimensional Broek–Kaup system with variable coefficients. Based on the derived solitary wave solution

and using a known chaotic system, we obtain some chaotic solitons of the variable coefficient Broek–Kaup equation system, which are different from the ones of the previous work. Since the wide applications of the soliton theory, to learn more about the localized excitations and their applications in practice is worthy of further study.

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