Solitons for a generalized variable-coefficient nonlinear Schrödinger equation*

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In this paper, we investigate some exact soliton solutions for a generalized variable-coefficients nonlinear Schrödinger equation (NLS) with an arbitrary time-dependent linear potential which describes the dynamics of soliton solutions in quasi-one-dimensional Bose–Einstein condensations. Under some reasonable assumptions, one-soliton and two-soliton solutions are constructed analytically by the Hirota method. From our results, some previous one- and two-soliton solutions for some NLS-type equations can be recovered by some appropriate selection of the various parameters. Some figures are given to demonstrate some properties of the one- and the two-soliton and the discussion about the integrability property and the Hirota method is given finally.

Keywords: generalized NLS equation, Hirota method, solitons

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1. Introduction

The construction of exact solutions for a given nonlinear evolution equation is an important topic in the study of nonlinear physical phenomena. In recent years, with the availability of symbolic computation systems like Mathematica or Maple, there has been much work on it and many sophisticated methods have been proposed. We can cite the Darboux transformation,[1] inverse scattering transformation,[2] Lie group method,[3−7] CK direct method and modified CK method,[8−10] symmetry group direct method,[11−16] Hirota method,[17] various tanh methods,[18−23] the formal variable separation approach,[24−26] various sub-equation expansion methods,[27−32] and so on. Among them, the Hirota method is one of the powerful methods in searching for the exact solution of nonlinear evolution equations.[33−35] In the present paper, we explore some exact solutions for a generalized variable-coefficients nonlinear Schrödinger (NLS) equation by the Hirota method.

It is well known that the Gross–Pitaevskii (GP) equation has either dark or bright solitons depending on the repulsive or attractive nature of the interatomic interactions, respectively. A dark soliton in Bose–Einstein condensate (BEC) is a macroscopic excitation of the condensate which is characterized by a local density minimum and a phase gradient of the wave function at the position of the minimum.[36] A bright soliton in BEC is expected to have the balance between the dispersion and the attractive mean-field energy.[37] Several methods have been used to obtain the soliton solutions of GP equation with different potentials,[38−43] and the dynamics of the excitation of the condensate was discussed. When the longitudinal dimension of the BEC is much longer than its transverse dimension which is the order of its healing length, the GP equation can be reduced to the quasi-one-dimensional (quasi-1D) regime. This trapped quasi-low-dimensional[44] condensate has offered a useful tool for investigating the nonlinear excitations such as solitons and vortices, which is more stable than that in the three-dimensional (3D) case, where the solitons suffer the transverse instability and the vortices can bend. So it is very important to study the soliton excitations in quasi-low-dimensional BECs both in theory and in experiment.

In the present paper, we consider a generalized variable-coefficients nonlinear Schördinger equation with an arbitrary linear time-dependent potential as follows:

$$\mathrm{i}\frac{\partial u}{\partial t} + b(t)\frac{\partial^2 u}{\partial x^2} + h(t)|u|^2 u + v(x,t)u = 0, \quad (1)$$

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where $v(x,t) = xv_1(t) + v_2(t)$. In Bose–Einstein condensations (BECs), $b(t)$ is a parameter related to kinetic energy, $h(t)$ is a scattering length of attractive interaction $[h(t) > 0]$ or repulsive interaction $[h(t) < 0]$ between atoms, $v_1(t)$ is a parameter related to nonlinear potential and $v_2(t)$ is a small parameter related to the feeding of condensate from the thermal cloud.[45] While in the context of nonlinear optics, $b(t)$ is group velocity dispersion parameter, $h(t)$ is related to the Kerr nonlinearity, and $v_1(t)$ and $v_2(t)$ are inhomogeneous parameters related to phase modulation and phase loss/gain respectively.[29] Under some special parameters of $\{b(t), v_1(t), v_2(t), h(t)\}$, the study of nonlinear wave propagation for Eq. (1) is of great interest and has a wide range of applications.[44−48] For example, Li *et al.*[47] constructed one- and two-solitons for the following equation:

$$\mathrm{i}\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + xv_1(t)u + \mu|u|^2 u = 0 \tag{2}$$

by Hirota method; in Ref. [48], the authors investigated one- and two-solitons of Eq. (1) with $v_1(t) = 0$ and $\{b(t), v_2(t), h(t)\}$ being all constants by a Darboux transformation; in Ref. [49], Khawaja derived some classes of exact solitonic solutions of Eq. (1) with $b(t) = 1, v_1(t) = -1, v_2(t) = 0, h(t) = \pm 1$ by a Darboux transformation.

The rest of the present paper is organized as follows. In Section 2, first, we transform Eq. (1) into a bilinear form; then on the basis of the bilinear form, the general one- and two-soliton solutions are constructed successfully under some reasonable assumptions. With some special parameters, the soliton solutions obtained in Ref. [47] can be recovered by our results. Furthermore, some figures are given to demonstrate the evolutions and the stabilities of one- and two-solitons and the discussion about the integrability property and the Hirota method is also performed. In Section 3, a short summary and discussion are presented.

2. Soliton solutions

2.1. One-soliton solution

According to the Hirota method, the solution of Eq. (1) is assumed to be

$$u = \frac{g}{f}, \tag{3}$$

where $g \equiv g(x,t)$ is a complex function and $f \equiv f(x,t)$ is a real function. With this transformation, equation (1) becomes

$$f(\mathrm{i}D_t + bD_x^2 + v)g \cdot f + g(hg\bar{g} - bD_x^2 f \cdot f) = 0, \tag{4}$$

where the overbar denotes complex conjugate, and the bilinear derivative operator is defined as

$$D_x^m D_t^n g \cdot f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n g(x,t) \cdot f(x',t')|_{x=x',t=t'}. \tag{5}$$

Many attempts show that equation (4) can be decoupled into

$$\begin{aligned} &(\mathrm{i}D_t + bD_x^2 + v)g \cdot f = 0, \\ &bD_x^2(f \cdot f) = hg\bar{g}. \end{aligned} \tag{6}$$

We now assume that

$$g = \chi g_1, \quad f = 1 + \chi^2 f_1, \tag{7}$$

where χ is a parameter. Substituting Eq. (7) into Eq. (6), then collecting the coefficients of the different powers of χ, we have

$$\chi: \quad (\mathrm{i}D_t + bD_x^2 + v)g_1 \cdot 1 = 0, \tag{8}$$

$$\chi^2: \quad 2bD_x^2 f_1 \cdot 1 = hg_1\bar{g_1}, \tag{9}$$

$$\chi^3: \quad (\mathrm{i}D_t + bD_x^2 + v)g_1 \cdot f_1 = 0, \tag{10}$$

$$\chi^4: \quad D_x^2 f_1 \cdot f_1 = 0. \tag{11}$$

Employing definition (5), the above equations can be expressed in detail. In order to satisfy Eq. (8) we can assume that g_1 has the form

$$g_1 = \exp \eta_1. \tag{12}$$

Substituting Eq. (12) into Eq. (8), we have

$$\mathrm{i}\eta_{1t} + b\eta_{1xx} + b\eta_{1x}^2 + v = 0. \tag{13}$$

It is necessary to point out that there are some difficulties in acquiring general solutions of the above equation. For convenience, we assume

$$\eta_1 = p_1(t)x + k_1(t), \tag{14}$$

with the time-dependent functions $p_1(t)$ and $k_1(t)$ to be determined. With the restriction of Eq. (9) and Eq. (11), we have

$$f_1 = \exp(\eta_1 + \bar{\eta_1} + A_{11}), \tag{15}$$

where

$$A_{11} = \ln \frac{\lambda}{(p_1 + \bar{p_1})^2}, \quad \lambda = \frac{h(t)}{2b(t)}.$$

Substituting solutions Eq. (12) and Eq. (15) into Eq. (10), we obtain the equations of $p_1(t)$ and $k_1(t)$ as

$$(\mathrm{i}p_{1t} + v_1)x + \mathrm{i}k_{1t} + bp_1^2 + v_2 = 0. \tag{16}$$

We can obtain the following conditions from Eq. (16):

$$\begin{aligned} &\mathrm{i}p_{1t} + v_1 = 0, \\ &\mathrm{i}k_{1t} + bp_1^2 + v_2 = 0. \end{aligned} \tag{17}$$

From the above results, we can also have

$$\begin{aligned} &\mathrm{i}\bar{p_{1t}} - v_1 = 0, \\ &\mathrm{i}\bar{k_{1t}} - b\bar{p_1^2} - v_2 = 0. \end{aligned} \tag{18}$$

Then we solve the above two equations, we obtain the expressions to be

$$\begin{aligned} p_1 &= \mathrm{i}\int_0^t v_1(\tau)\mathrm{d}\tau + \xi_1, \\ k_1 &= \mathrm{i}\int_0^t \left[b(\tau)p_1^2(\tau) + v_2(\tau)\right]\mathrm{d}\tau + \xi_2, \end{aligned} \tag{19}$$

where ξ_1 and ξ_2 are complex parameters. With Eqs. (12), (15) and (3), after absorbing $\chi = 1$, the bright one-soliton solution of Eq. (1) can be derived as

$$u = \frac{\exp\eta_1}{1+\exp(\eta_1+\bar{\eta_1}+A_{11})}. \tag{20}$$

Remark 1 From the one-soliton solution (20) of Eq. (1), if setting $b(t) = 1/2, h(t) = \mu, v_2(t) = 0$, we can immediately derive a one-soliton solution for Eq. (2), which is the same as solution (19) in Ref. [47]. However, to our knowledge, other one-soliton solutions from Eq. (20) are not reported earlier.

In the following, the evolution of one-soliton solution (20) with some different parameters are given in Figs. 1(a) and 1(b), where $U = |u|^2$.

In order to consider the stability of one-soliton (20), first we can derive the variation equation of Eq. (1) as follows:

$$\mathrm{i}w_t + b(t)w_{xx} + v(x,t)w + h(t)u^2w^* + 2|u|^2w = 0, \tag{21}$$

where $w = w(x,t)$ and u is a solution from Eq. (20). Usually, it is easy to investigate the stability of a static solution. However, the solutions u from Eq. (21) are time-dependant, so it is difficult to give a thorough analysis of the stability of u. In the present paper, we directly solve the initial-boundary problem of Eq. (21) numerically. If we can verify the stability for many enough initial values, then we can prove the stability of u. For real problems, we generally select some initial values to illustrate the stability of solutions.

As shown from Figs. 2 and 3, all of the quantities are oscillatory surrounding some fixed values, which suggests that the considered solution is stable. At the same time, it is necessary to point out that in order to guarantee the accuracy of numerical solutions of w, we solve Eq. (21) within a short time compared with time in Fig. 1.

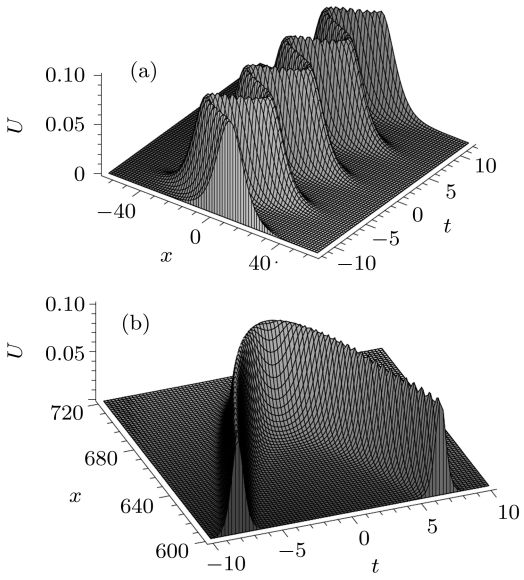
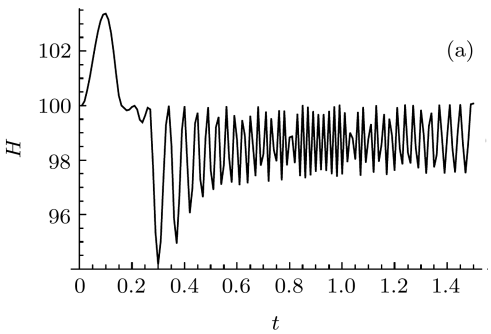
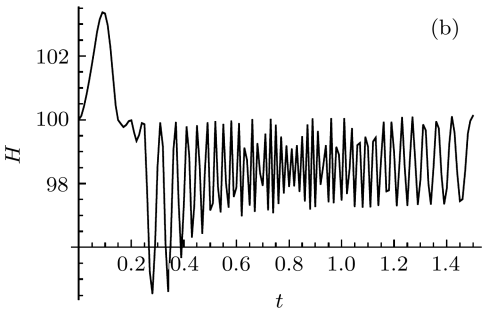


Fig. 1. Evolutions of one-soliton solution (20) with parameters being (a) $v_1(t) = 3\sin(0.5t)$, $b(t) = 2\sin(0.5t)$, $v_2(t) = \mathrm{sech(t)}/2$, $h(t) = 0.2b(t)$, $\xi_1 = 0.1 + 0.1\mathrm{i}$, $\xi_2 = 0.1 + 0.1\mathrm{i}$; (b) $v_1(t) = 0.1\sin(0.01t)$, $b(t) = 0.2t/(1 + 0.01t^2) + 0.1$, $v_2(t) = \sin(t)$, $h(t) = 0.2b(t)$, $\xi_1 = 0.1 + 0.1\mathrm{i}$, $\xi_2 = 0.1 + 0.1\mathrm{i}$.





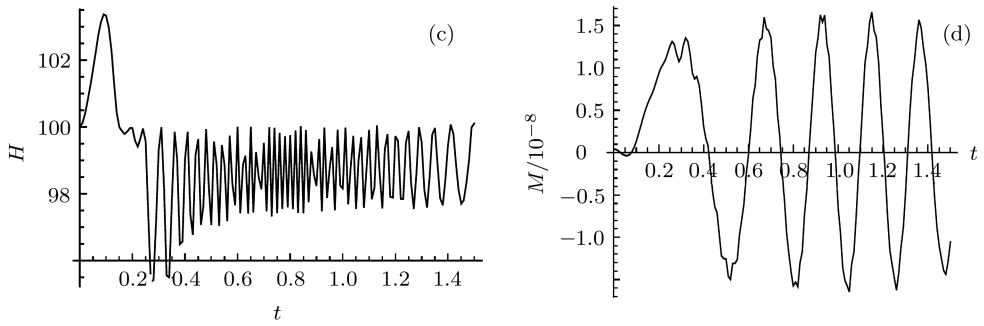


Fig. 2. Evolutions of perturbation and energy of interaction of u and w, where $H = \int_{-L}^{L} |w|^2 \mathrm{d}x, M = \int_{-L}^{L} (uw^* + u^*w)/2\mathrm{d}x$, u is the same as the solution in Fig. 1(a). Input initial values are (a) $w(x,0) = \sin(\pi x/L)$; (b) $w(x,0) = \sin(5\pi x/L)$; (c) $w(x,0) = \sin(10\pi x/L)$; (d) $w(x,0) = \sin(\pi x/L)$; boundary conditions are $w(-L,t) = w(L,t) = 0$ and $L = 100$.

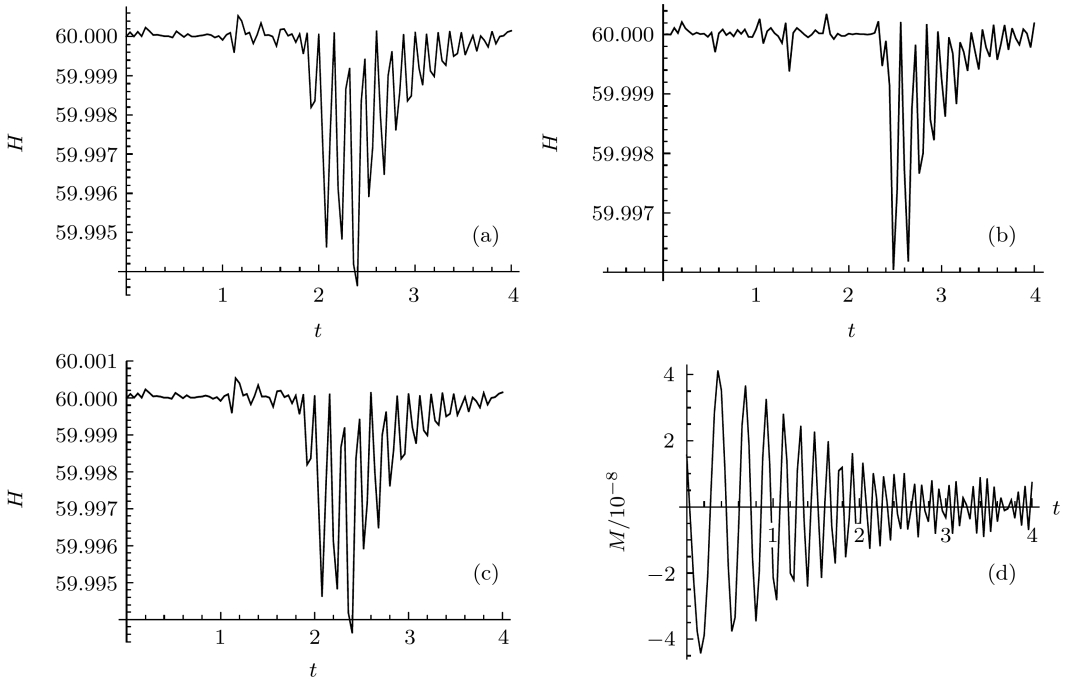


Fig. 3. Evolutions of perturbation and energy of interaction of u and w, where $H = \int_{L_1}^{L_2} |w|^2 \mathrm{d}x, M = \int_{L_1}^{L_2} [(uw^* + u^*w)/2]\mathrm{d}x$, u is the same as the solution in Fig. 1(b). Input initial values are (a) $w(x,0) = \sin(\pi x/(L_2 - L_1))$; (b) $w(x,0) = \sin(5\pi x/(L_2 - L_1))$; (c) $w(x,0) = \sin(10\pi x/(L_2 - L_1))$; (d) $w(x,0) = \sin(\pi x/(L_2 - L_1))$; boundary conditions are $w(L_1,t) = w(L_2,t) = 0, L_1 = 600$ and $L_2 = 720$.

2.2. Two-soliton solution

In this subsection, we will analyse the expressions of two-soliton solutions of Eq. (1). So we now assume that the forms of g and f in Eq. (6) are as follows:

$$g = \chi g_1 + \chi^3 g_2, \quad f = 1 + \chi^2 f_1 + \chi^4 f_2. \tag{22}$$

By employing the same procedure as the previous one, we obtain the following set of equations from Eq. (6), corresponding to the different powers of χ

$$\chi: \ (\mathrm{i}D_t + bD_x^2 + v) g_1 \cdot 1 = 0, \tag{23}$$

$$\chi^2: \ D_x^2 f_1 \cdot 1 = \lambda g_1 \bar{g_1}, \tag{24}$$

$$\chi^3: \ (\mathrm{i}D_t + bD_x^2 + v)(g_1 \cdot f_1 + g_2 \cdot 1) = 0, \tag{25}$$

$$\chi^4: \ bD_x^2 f_1 \cdot f_1 + 2bD_x^2 f_2 \cdot 1 = h(g_1 \bar{g_2} + g_2 \bar{g_1}), \tag{26}$$

$$\chi^5: \ (\mathrm{i}D_t + bD_x^2 + v)(g_1 \cdot f_2 + g_2 \cdot f_1) = 0, \tag{27}$$

$$\chi^6: \ D_x^2 f_1 \cdot f_2 = \lambda g_2 \bar{g_2}, \tag{28}$$

$$\chi^7: \ (\mathrm{i}D_t + bD_x^2 + v) g_2 \cdot f_2 = 0, \tag{29}$$

$$\chi^8: \ D_x^2 f_2 \cdot f_2 = 0. \tag{30}$$

In order to construct the two-soliton solution of Eq. (1), as discussed in the one-soliton solution, we can solve the equations from Eq. (23) to Eq. (30) in turn for acquiring the expressions of g and f. We

assume that g_1 has the form

$$g_1 = \exp\eta_1 + \exp\eta_2,$$
$$\eta_j = p_j(t)x + k_j(t), \tag{31}$$

where the time-dependent functions $p_j(t)$ and $k_j(t)$, $j = 1, 2$ are to be determined. Substituting Eq. (31) into Eq. (23), we have

$$[(\mathrm{i}p_{1t} + v_1)x + \mathrm{i}k_{1t} + bp_1^2 + v_2]\exp\eta_1 + [(\mathrm{i}p_{2t} + v_1)x + \mathrm{i}k_{2t} + bp_2^2 + v_2]\exp\eta_2 = 0, \tag{32}$$

which implies that

$$\mathrm{i}p_{jt} + v_1 = 0,$$
$$\mathrm{i}k_{jt} + bp_j^2 + v_2 = 0, \tag{33}$$

where $j = 1, 2$. So we also obtain

$$\mathrm{i}\bar{p}_{jt} - v_1 = 0,$$
$$\mathrm{i}\bar{k}_{jt} - b\bar{p_j}^2 - v_2 = 0. \tag{34}$$

From the above equations, one can find the solutions

$$p_j = \mathrm{i}\int_0^t v_1(\tau)\,\mathrm{d}\tau + \xi_{j1},$$
$$k_j = \mathrm{i}\int_0^t [b(\tau)p_j^2(\tau) + v_2(\tau)]\,\mathrm{d}\tau + \xi_{j2}, \tag{35}$$

where ξ_{j1} and ξ_{j2} $(j = 1, 2)$ are complex parameters in general. Combining Eq. (31) with Eq. (24) we obtain the expression of f_1 as

$$f_1 = \exp(\pi_{11} + A_{11}) + \exp(\pi_{21} + A_{21}) + \exp(\pi_{12} + A_{12}) + \exp(\pi_{22} + A_{22}), \tag{36}$$

where

$$A_{mn} = \ln\frac{\lambda}{(\gamma_{mn})^2}, \quad \gamma_{mn} = p_m + \bar{p_n},$$
$$\pi_{mn} = \eta_m + \bar{\eta_n}, \quad (m, n = 1, 2).$$

With the help of Eqs. (31) and (36), we can simplify Eq. (25) into

$$\begin{aligned}
&\mathrm{i}g_{2t} + bg_{2xx} + vg_2 \\
&= \left(\frac{2b\lambda\beta^2}{\gamma_{11}\gamma_{21}} + \mathrm{i}\frac{\lambda_t(\gamma_{11}^2 + \gamma_{21}^2)}{\gamma_{11}^2\gamma_{21}^2}\right)\exp(\pi_{11} + \eta_2) \\
&\quad + \left(\frac{2b\lambda\beta^2}{\gamma_{12}\gamma_{22}} + \mathrm{i}\frac{\lambda_t(\gamma_{12}^2 + \gamma_{22}^2)}{\gamma_{12}^2\gamma_{22}^2}\right)\exp(\eta_1 + \pi_{22}) \\
&\quad + \mathrm{i}\frac{\lambda_t}{\gamma_{11}^2}\exp(\eta_1 + \pi_{11}) + \mathrm{i}\frac{\lambda_t}{\gamma_{12}^2}\exp(\eta_1 + \pi_{12}) \\
&\quad + \mathrm{i}\frac{\lambda_t}{\gamma_{21}^2}\exp(\eta_2 + \pi_{21}) + \mathrm{i}\frac{\lambda_t}{\gamma_{22}^2}\exp(\eta_2 + \pi_{22}),
\end{aligned} \tag{37}$$

where $\beta = p_2 - p_1$, which shows that the expression of g_2 has the form

$$\begin{aligned}
g_2 &= \delta_1\exp(\pi_{11} + \eta_2) + \delta_2\exp(\pi_{22} + \eta_1) \\
&\quad + \delta_3\exp(\eta_1 + \pi_{11}) + \delta_4\exp(\eta_1 + \pi_{12}) \\
&\quad + \delta_5\exp(\eta_2 + \pi_{21}) + \delta_6\exp(\eta_2 + \pi_{22}),
\end{aligned} \tag{38}$$

where the parameters δ_j $(j = 1, 2, \ldots, 6)$ are given as

$$\begin{aligned}
&\delta_{1t} + 2b\delta_1\gamma_{11}\gamma_{21} = \mathrm{i}\frac{\lambda_t(\gamma_{11}^2 + \gamma_{21}^2)}{\gamma_{11}^2\gamma_{21}^2} + \frac{2b\lambda\beta^2}{\gamma_{11}\gamma_{21}}, \\
&\delta_{2t} + 2b\delta_2\gamma_{12}\gamma_{22} = \mathrm{i}\frac{\lambda_t[\gamma_{12}^2 + \gamma_{22}^2]}{\gamma_{12}^2\gamma_{22}^2} + \frac{2b\lambda\beta^2}{\gamma_{12}\gamma_{22}}, \\
&\delta_{3t} + 2b\delta_3\gamma_{11} = \mathrm{i}\frac{\lambda_t}{\gamma_{11}^2},\ \delta_{4t} + 2b\delta_4\gamma_{12} = \mathrm{i}\frac{\lambda_t}{\gamma_{12}^2}, \\
&\delta_{5t} + 2b\delta_5\gamma_{21} = \mathrm{i}\frac{\lambda_t}{\gamma_{21}^2},\ \delta_{6t} + 2b\delta_6\gamma_{22} = \mathrm{i}\frac{\lambda_t}{\gamma_{22}^2}.
\end{aligned} \tag{39}$$

Now we have obtained the expressions of g_1, g_2 and f_1 in Eq. (22). Substituting Eqs. (31), (36) and (38) into Eq. (26) we obtain

$$\begin{aligned}
f_2 &= 2\,\mathrm{Re}\left[\left(\frac{\lambda(\delta_1 + \bar{\delta_3} + \bar{\delta_4})}{(\gamma_{11} + \gamma_{21})^2} - \frac{\lambda^2\beta^2}{\gamma_{21}^2\gamma_{11}^2(\gamma_{11} + \gamma_{21})^2}\right) \times \exp(\pi_{11} + \pi_{21})\right] + \frac{\lambda\,\mathrm{Re}\delta_3}{2\gamma_{11}^2}\exp(2\pi_{11}) + \frac{\lambda\,\mathrm{Re}\delta_6}{2\gamma_{22}^2}\exp(2\pi_{22}) \\
&\quad + \left[\frac{\lambda(2\,\mathrm{Re}\delta_1 + 2\,\mathrm{Re}\delta_2)}{(\gamma_{11} + \gamma_{22})^2} - \frac{\lambda^2(\beta + \bar{\beta})^2}{\gamma_{22}^2\gamma_{11}^2(\gamma_{11} + \gamma_{22})^2} - \frac{\lambda^2(\beta - \bar{\beta})^2}{\gamma_{21}^2\gamma_{12}^2(\gamma_{11} + \gamma_{22})^2}\right] \times \exp(\pi_{11} + \pi_{22}) \\
&\quad + 2\mathrm{Re}\left[\frac{\lambda(\delta_4 + \bar{\delta_5})}{4\gamma_{12}^2}\exp(2\pi_{12})\right] + 2\,\mathrm{Re}\left[\left(\frac{\lambda(\bar{\delta_5} + \bar{\delta_6} + \delta_2)}{(\gamma_{12} + \gamma_{22})^2} - \frac{\lambda^2\beta^2}{\gamma_{22}^2\gamma_{12}^2(\gamma_{12} + \gamma_{22})^2}\right) \times \exp(\pi_{12} + \pi_{22})\right].
\end{aligned} \tag{40}$$

So the two-soliton solution has the form

$$u = \frac{g}{f}, \tag{41}$$

where

$$\begin{aligned}
g &= \exp\eta_1 + \exp\eta_2 + \delta_1\exp(\pi_{11} + \eta_2) + \delta_2\exp(\pi_{22} + \eta_1) + \delta_3\exp(\eta_1 + \pi_{11}) \\
&\quad + \delta_4\exp(\eta_1 + \pi_{12}) + \delta_5\exp(\eta_2 + \pi_{21}) + \delta_6\exp(\eta_2 + \pi_{22}),
\end{aligned}$$

$$
\begin{aligned}
f = {} & 1+\exp(\pi_{11}+A_{11})+\exp(\pi_{21}+A_{21})+\exp(\pi_{12}+A_{12})+\exp(\pi_{22}+A_{22})+\frac{\lambda\ \mathrm{Re}\delta_3}{2\gamma_{11}^2}\exp(2\pi_{11})\\
& +2\mathrm{Re}\left[\left(\frac{\lambda(\delta_1+\bar{\delta_3}+\bar{\delta_4})}{(\gamma_{11}+\gamma_{21})^2}-\frac{\lambda^2\beta^2}{\gamma_{21}^2\gamma_{11}^2(\gamma_{11}+\gamma_{21})^2}\right)\times\exp(\pi_{11}+\pi_{21})\right]+\frac{\lambda\ \mathrm{Re}\delta_6}{2\gamma_{22}^2}\exp(2\pi_{22})\\
& +\left[\frac{\lambda(2\ \mathrm{Re}\delta_1+2\ \mathrm{Re}\delta_2)}{(\gamma_{11}+\gamma_{22})^2}-\frac{\lambda^2(\beta+\bar{\beta})^2}{\gamma_{22}^2\gamma_{11}^2(\gamma_{11}+\gamma_{22})^2}-\frac{\lambda^2(\beta-\bar{\beta})^2}{\gamma_{21}^2\gamma_{12}^2(\gamma_{11}+\gamma_{22})^2}\right]\exp(\pi_{11}+\pi_{22})\\
& +2\mathrm{Re}\left[\frac{\lambda(\delta_4+\bar{\delta_5})}{4\gamma_{12}^2}\exp(2\pi_{12})\right]+2\mathrm{Re}\left[\left(\frac{\lambda(\bar{\delta_5}+\bar{\delta_6}+\delta_2)}{(\gamma_{12}+\gamma_{22})^2}-\frac{\lambda^2\beta^2}{\gamma_{22}^2\gamma_{12}^2(\gamma_{12}+\gamma_{22})^2}\right)\exp(\pi_{12}+\pi_{22})\right].
\end{aligned}
$$

Then from Eqs. (39), we can derive

$$
\begin{aligned}
\delta_1 &= \left(\int_0^t M_1\exp\left(2\gamma_{11}\gamma_{21}\int_0^t b\mathrm{d}\tau\right)\mathrm{d}\tau+c_1\right)\exp\left(\int_0^t -2b\gamma_{11}\gamma_{21}\mathrm{d}\tau\right),\\
\delta_2 &= \left(\int_0^t M_2\exp\left(2\gamma_{12}\gamma_{22}\int_0^t b\mathrm{d}\tau\right)\mathrm{d}\tau+c_2\right)\exp\left(\int_0^t -2b\gamma_{12}\gamma_{22}\mathrm{d}\tau\right),\\
\delta_3 &= \left(\int_0^t M_3\exp\left(2\gamma_{11}\int_0^t b\mathrm{d}\tau\right)\mathrm{d}\tau+c_3\right)\exp\left(\int_0^t -2b\gamma_{11}\mathrm{d}\tau\right),\\
\delta_4 &= \left(\int_0^t M_4\exp\left(2\gamma_{12}\int_0^t b\mathrm{d}\tau\right)\mathrm{d}\tau+c_4\right)\exp\left(\int_0^t -2b\gamma_{12}\mathrm{d}\tau\right),\\
\delta_5 &= \left(\int_0^t M_5\exp\left(2\gamma_{21}\int_0^t b\mathrm{d}\tau\right)\mathrm{d}\tau+c_5\right)\exp\left(\int_0^t -2b\gamma_{21}\mathrm{d}\tau\right),\\
\delta_6 &= \left(\int_0^t M_6\exp\left(2\gamma_{22}\int_0^t b\mathrm{d}\tau\right)\mathrm{d}\tau+c_6\right)\exp\left(\int_0^t -2b\gamma_{22}\mathrm{d}\tau\right),
\end{aligned}
\tag{42}
$$

where

$$
M_1=\mathrm{i}\frac{\lambda_t(\gamma_{11}^2+\gamma_{21}^2)}{\gamma_{11}^2\gamma_{21}^2}+\frac{2b\lambda\beta^2}{\gamma_{11}\gamma_{21}},
$$

$$
M_2=\mathrm{i}\frac{\lambda_t(\gamma_{12}^2+\gamma_{22}^2)}{\gamma_{12}^2\gamma_{22}^2}+\frac{2b\lambda\beta^2}{\gamma_{12}\gamma_{22}},
$$

$M_3=\mathrm{i}\lambda_t/\gamma_{11}^2$, $M_4=\mathrm{i}\lambda_t/\gamma_{12}^2$, $M_5=\mathrm{i}\lambda_t/\gamma_{21}^2$, $M_6=\mathrm{i}\lambda_t/\gamma_{11}^2$, c_j $(j=1,2,\ldots,6)$ are arbitrary constants.

The evolutions of two-soliton solution (41) with Eq. (42) under some different parameters and $c_i=0$ $(i=1,2,\ldots,6)$ are given in Figs. 4(a) and 4(b), where $U=|u|^2$.

Remark 2

(i) If setting $c_j=0$ $(j=1,2,\ldots,6)$ in Eq. (42) and setting $b(t)=1/2, h(t)=\mu=$ constant, from two-soliton solution (41), we can obtain a two-soliton solution for Eq. (2) which is the same as the results in Ref. [47]. To our knowledge, other two-soliton solutions (41) with Eq. (42) of Eq. (1) have not been reported earlier.

(ii) It is known that the integrability for NEEs is an important topic in nonlinear science research. In Refs. [49] and [50], the authors investigated the integrability for two NLS-type equations: Khawaja gave a Lax pairs for Eq. (1) with $b(t)=1, v_1(t)=-1, v_2(t)=0, h(t)=\pm1$ and Serkin *et al.* derived a Lax pair for a generalized NLS equation with a linear and harmonic oscillator potential under an integrable condition.

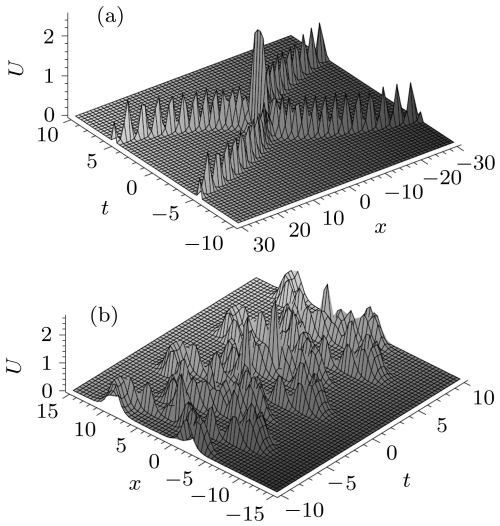


Fig. 4. Evolutions of two-soliton solution (41) with Eq. (42) with parameters being $\lambda=1, \xi_{11}=1+\mathrm{i}, \xi_{21}=1-\mathrm{i}, \xi_{12}=\mathrm{i}, \xi_{22}=-\mathrm{i}$ and (a) $v_1(t)=\exp(0.01t), b(t)=t, v_2(t)=t$; (b) $v_1(t)=2\sin(2t), b(t)=3\sin(t), v_2(t)=\exp(-t)$.

Now we investigate the integrability of Eq. (1). After some analysis and computation, we can obtain the following linear spectral problem:

$$
\psi_x=U\psi(x,t),\quad \psi_t=V\psi(x,t),
\tag{43}
$$

where

$$V=\begin{pmatrix} \dfrac{\mathrm{i}\xi^2 h(t)}{\rho^2}-\mathrm{i}\xi_t x+\dfrac{\mathrm{i}v_2(t)}{2}+\dfrac{\mathrm{i}h(t)}{2}|Q|^2 & \left(2\rho_t x-\dfrac{\xi h(t)}{\rho}\right)Q-\dfrac{\mathrm{i}h(t)}{2\rho}Q_x \\ \left(2\rho_t x-\dfrac{\xi h(t)}{\rho}\right)\bar{Q}+\dfrac{\mathrm{i}h(t)}{2\rho}\bar{Q}_x & -\dfrac{\mathrm{i}\xi^2 h(t)}{\rho^2}+\mathrm{i}\xi_t x-\dfrac{\mathrm{i}v_2(t)}{2}-\dfrac{\mathrm{i}h(t)}{2}|Q|^2 \end{pmatrix}, \tag{44}$$

$$U=\begin{pmatrix} -\mathrm{i}\xi & \rho Q \\ \rho\bar{Q} & -\mathrm{i}\xi \end{pmatrix}, \tag{45}$$

with the overbar denoting the complex conjugate, $\xi \equiv \xi(t), \rho \equiv \rho(t)$, $Q = q\exp((\mathrm{i}/2)k(t)x^2)$, $\rho^2 = -h(t)/(2b(t))$, $\xi = \rho^2(\int -(v_1(t)/2\rho^2)\mathrm{d}t + \xi_0)$, and $k(t) = 2\rho'\rho/h(t)$. From the zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{46}$$

one can derive Eq. (1) only under the following compatibility condition

$$h(t)\left[\left(\frac{2h'(t)}{h(t)}\right)' - \left(\frac{b'(t)}{b(t)}\right)'\right] - b(t)\left(\frac{h'(t)}{b(t)}\right)' = 0, \tag{47}$$

where prime denotes the derivative with respect to t. Thus equation (1) is integrable under the constraint of Eq. (47).

In the process of computation by the Hirota method, we have not imposed any constraint on two-soliton solution (41) with Eq. (42) obtained in the paper, i.e., solution (41) need not satisfy integrable condition (47). Thus for some non-integrable nonlinear partial differential equations (PDEs), some exact solutions can be expected to be obtained by the Hirota method. Of course, for some non-integrable nonlinear PDEs, one can use the technique of perturbation theory based on the inverse scattering transformation.[51]

(iii) In order to discuss the real application in BECs under integrable condition Eq. (47), we set $b(t) = 1/2$, then from Eq. (47) we can obtain $h(t) = 1/(c_1 t + c_0)$, which can modulate the s-wave scattering length by using the Feshbach resonance with a tunable time-dependent magnetic field.

3. Summary and discussion

One- and two-soliton solutions are presented explicitly for a generalized variable-coefficient nonlinear Schrödinger equation (NLS) with an arbitrary time-dependent linear potential by the Hirota method. From our results, some previous one- and two-soliton solutions for some simple NLS-type equations can be recovered by some appropriate selections of the various parameters. Some figures are given to demonstrate the evolution and stability properties of these solutions and discussion about the integrability property and the Hirota method is also given. The results in the paper can describe the dynamics of quasi-one-dimensional Bose–Einstein condensation and optical soliton communication and will have useful applications in these fields.

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