The periodic solutions for coupled integrable dispersionless equations*

Liu Shi-Kuo(刘式适), Zhao Qiang(赵 强)†, and Liu Shi-Da(刘式达)

School of Physics, Peking University, Beijing 100871, *China*

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By using the Jacobi elliptic-function method, this paper obtains the periodic solutions for coupled integrable dispersionless equations. The periodic solutions include some kink and anti-kink solitons.

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1. Introduction

For the following coupled integrable dispersionless equations,[1,2]

$$u_{xt} + (vw)_x = 0, \tag{1a}$$

$$v_{xt} - 2vu_x = 0, \tag{1b}$$

$$w_{xt} - 2wu_x = 0, \tag{1c}$$

Konno and co-workers[1,2] have shown that the coupled system is solvable by using the inverse scattering method. This special case has also been discussed by different authors in different contexts.[3,4] Its generalized form has been studied through the inverse scattering method and some remarkable soliton properties have been found.[2,5,6] Recently, Naranmandula *et al.*[7,8] have obtained some exact solutions and constructed some new coherent structures by using the improved homogeneous balance method. However, these solutions are all soliton-type ones. In this paper, using the Jacobi elliptic function expansion method,[9,10] we obtain the periodic solutions which include some soliton-type ones for Eqs. (1a)–(1c).

2. Transformed equations

We seek the travelling wave solutions of Eqs. (1a)–(1c) in the form

$$u = u(\xi), \quad v = v(\xi), \quad w = w(\xi), \quad \xi = x - ct, \tag{2}$$

where c is the wave speed and $c \neq 0$. Substituting Eq. (2) into Eqs. (1a)–(1c) yields

$$cu_{\xi\xi} - (vw)_\xi = 0, \tag{3a}$$

$$cv_{\xi\xi} + 2vu_\xi = 0, \tag{3b}$$

$$cw_{\xi\xi} + 2wu_\xi = 0. \tag{3c}$$

Integrating Eq. (3a), we obtain

$$u_\xi = \frac{1}{c}vw + A, \tag{4}$$

where A is the integration constant. Substituting Eq. (4) into Eqs. (3b) and (3c) yields

$$v_{\xi\xi} + \frac{2}{c^2}v^2 w + Bv = 0, \tag{5a}$$

$$w_{\xi\xi} + \frac{2}{c^2}vw^2 + Bw = 0, \tag{5b}$$

where $B = 2A/c$. Equations (5a) and (5b) are symmetrical about v and w. Multiplying Eqs. (5b) and (5a) by v and w respectively and subtracting the latter from the former we have

$$vw_{\xi\xi} - wv_{\xi\xi} = 0. \tag{6}$$

Equation (6) can also be written as

$$(vw_\xi - wv_\xi)_\xi = 0, \tag{7}$$

or

$$\left(\frac{w}{v}\right)_\xi = 0, \tag{8}$$

so that

$$w = Dv, \tag{9}$$

where D is the integration constant. It is readily seen from Eq. (9) that w is proportional to v. Evidently equation (9) enlarges the results in Refs. [7] and [8]. Substituting Eq. (9) into Eq. (5a) or (5b), we

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†Corresponding author. E-mail: zhqing@pku.edu.cn

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obtain that a second-order nonlinear ordinary differential equation in v is given by

$$v_{\xi\xi} + Bv + \frac{2D}{c^2}v^3 = 0. \tag{10}$$

3. The periodic solutions for Eqs. (1a)–(1c)

According to Eqs. (10), (9) and (4), we will find the periodic solutions for Eqs. (1a)–(1c). Solutions of this kind are very rich.[11] We give only the following two types of periodic solutions.

3.1. Type 1: Jacobi elliptic sine function

Based on the Jacobi elliptic function expansion method,[9] we obtain that the formal solution to Eq. (10) is given by

$$v = a\,\mathrm{sh}(k\xi, m), \tag{11}$$

where sh is the Jacobi elliptic sine function, a is the amplitude, k is the wavenumber, $0 \leq m \leq 1$ is the modulus, and

$$B = k^2(1 + m^2), \quad D = -\frac{m^2k^2c^2}{a^2}. \tag{12}$$

It implies that $B > 0$, $D < 0$ and the wave amplitude is proportional to wave speed. Substituting Eq. (11) into Eq. (9), we can determine w as

$$w = Da\,\mathrm{sh}(k\xi, m). \tag{13}$$

Since $D < 0$, thus v and w have opposite sign. Substituting Eqs. (11) and (13) into Eq. (4), we obtain

$$u_\xi = \frac{Da^2}{c}\,\mathrm{sh}^2(k\xi, m) + A. \tag{14}$$

Equation (14) can be integrated to give

$$u = -\frac{Da^2}{kcm^2}E(k\xi) + F\xi + G, \tag{15}$$

where

$$E(u) = \int_0^u \mathrm{dn}^2\,x\,\mathrm{d}x \tag{16}$$

is called the Legendre elliptic integral of the second kind. The dn is the Jacobi elliptic function of the third kind, $F = A + Da^2/cm^2$ and G is the integration constant. When $m = 1$, equations (15), (11) and (13) degenerate to the following kink-type soliton, respectively

$$u = -\frac{Da^2}{kc}\tanh(k\xi) + F\xi + G, \tag{17}$$

$$v = a\tanh(k\xi), \tag{18}$$

$$u = Da\tanh(k\xi), \tag{19}$$

where

$$B = 2k^2, \quad D = -\frac{k^2c^2}{a^2},$$
$$F = A + \frac{Da^2}{c}, \quad \left(B = \frac{2A}{c}, \quad A = k^2c\right). \tag{20}$$

3.2. Type 2: Jacobi elliptic cosine function

Similar to Eq. (11), we can obtain

$$v = a\,\mathrm{cn}(k\xi, m), \tag{21}$$

where cn is the Jacobi elliptic cosine function and the amplitude a, wavenumber k and modulus m satisfy

$$B = k^2(1 - m^2), \quad D = \frac{m^2k^2c^2}{a^2}. \tag{22}$$

It shows that the wave amplitude is also proportional to wave speed and $D > 0$. Substituting Eq. (21) into Eq. (9) we have

$$w = Da\,\mathrm{cn}(k\xi, m). \tag{23}$$

Comparing Eq. (23) with Eq. (21) we conclude that v and w have the same sign. Substituting Eqs. (21) and (23) into Eq. (4) yields

$$u_\xi = \frac{Da^2}{c}\,\mathrm{cn}^2(k\xi, m) + A. \tag{24}$$

On integration of Eq. (24) we obtain

$$u = \frac{Da^2}{kcm^2}E(k\xi) + H\xi + I, \tag{25}$$

where E is defined by Eq. (16),

$$H = A - \frac{Da^2(1 - m^2)}{cm^2},$$

I is the integration constant. When $m = 1$, equations (25), (21) and (23) reduce to, respectively

$$u = -\frac{Da^2}{kc}\tanh(k\xi) + G\xi + I, \tag{26}$$

$$v = a\,\mathrm{sech}(k\xi), \tag{27}$$

$$u = Da\,\mathrm{sech}(k\xi), \tag{28}$$

where

$$B = -k^2, \quad D = -\frac{k^2c^2}{a^2},$$
$$H = A, \quad \left(B = \frac{2A}{c}, \quad A = -\frac{k^2c}{2}\right). \tag{29}$$

Equations (26), (27) and (28) correspond to the case for kink-type soliton and antikink-type soliton collision or interaction.

4. Conclusions

In this paper, by applying the Jacobi elliptic function expansion method, the periodic solutions for the coupled integrable dispersionless equations are obtained. We believe that our results play an important role for the analysis of nonlinear coherent structures.

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