A new type of conserved quantity of Mei symmetry for the motion of mechanico–electrical coupling dynamical systems*

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We obtain a new type of conserved quantity of Mei symmetry for the motion of mechanico–electrical coupling dynamical systems under the infinitesimal transformations. A criterion of Mei symmetry for the mechanico–electrical coupling dynamical systems is given. Simultaneously, the condition of existence of the new conserved quantity of Mei symmetry for mechanico–electrical coupling dynamical systems is obtained. Finally, an example is given to illustrate the application of the results.

Keywords: new conserved quantity, Mei symmetry, mechanico–electrical coupling systems

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1. Introduction

Symmetries play a very important role in the fields of mathematics, physics and mechanics. It is well known that the methods of finding conserved quantity by using symmetry are mainly: Lie symmetry, Noether symmetry and Mei symmetry. The Lie symmetry is an invariance of a differential equation and Noether symmetry is an invariance of Hamilton action.[1−15] In recent years, with continual attention on the Lie group and Noether theory, many important results have been obtained and the properties of mechanico–electrical dynamical systems are studied by using the Lie group analysis method.[10,16−20]

A new invariance was studied by Mei in 2000. Mei symmetry is an invariance under which transformed physical quantities such as Lagrangians, Hamiltionians, non-potential generalized forces and generalized constrained forces, etc. still satisfy original dynamical equations. Mei symmetry has been studied many useful results have been obtained as given in Refs. [21]–[25], also in the mechanico–electrical systems.[26,27] However, all the symmetries are nearly induced by Lie symmetry and Mei symmetry. Fang gives new conserved quantities of Mei symmetry of the holonomic mechanical system.[28]

Recently, we have investigated a new type of conserved quantity for mechanico–eleclectrical coupling dynamical systems, which is induced directly from Mei symmetry. The criterion of Mei symmetry for mechanico–electrical coupling dynamical systems is given. The new conserved quantity of the system is obtained. The results will have more general significances. Finally, there will be an example to illustrate the results.

2. Equations of motion of the mechanico–electrical coupling dynamical systems

A mechanico–electrical dynamical system couples a mechanical process to an electromagnetic process. The mechanical part consisting of N particles is described by n generalized coordinates q_s ($s = 1, 2, \ldots, n$); the electrodynamic part corresponds to m electrical circuits composed of linear conductors and capacitors. For the circuit k, we denote the current by i_k, the electromotive potential by u_k, the capacitor charge by e_k (with $\dot{e}_k = i_k$), the resistance by R_k and the capacitance by C_k. Then the Lagrangian of the mechanico–electrical system is

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q}) + W_{\rm e}(\boldsymbol{q}, \dot{\boldsymbol{e}}) - W_{\rm m}(\boldsymbol{q}, \boldsymbol{e}), \quad (1)$$

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where

$$W_{\rm e}=\frac{1}{2}\sum_{k=1}^{m}\frac{e_k^2}{C_k}\,,\quad W_{\rm m}=\frac{1}{2}\sum_{k=1}^{m}\sum_{r=1}^{m}L_{kr}i_ki_r \tag{2}$$

are, respectively, electric field energy and the magnetic field energy of the m-th circuit; T and V are, respectively, the kinetic energy and potential energy. In Eq. (2) L_{kr} (with $k\neq r$) is the mutual inductance between the k-th and the r-th circuits, and L_{kk} is the self-inductance of the k-th return circuit.

The Lagrange–Maxwell equations are

$$\begin{cases}\dfrac{\rm d}{{\rm d}t}\dfrac{\partial L}{\partial\dot{q}_s}-\dfrac{\partial L}{\partial q_s}=Q_s-\dfrac{\partial F}{\partial\dot{q}_s} & (s=1,2,\ldots,n),\\ \dfrac{\rm d}{{\rm d}t}\dfrac{\partial L}{\partial\dot{e}_k}-\dfrac{\partial L}{\partial e_k}=u_k-\dfrac{\partial F}{\partial\dot{e}_k} & (k=1,2,\ldots,m),\end{cases} \tag{3}$$

where Q_s $(s=1,2,\ldots,n)$ are general non-potential forces. They express a system of $n+m$ ordinary differential equations. The dissipative function F of the system is given by

$$F=F_e(\dot{\boldsymbol{e}})+F_m(\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{4}$$

where the electric dissipative function is

$$F_e=\frac{1}{2}\sum_{k=1}^{m}R_ki_k^2=\frac{1}{2}\sum_{k=1}^{m}R_k\dot{e}_k^2, \tag{5}$$

and F_m is the dissipative function of the viscous frictional damping force. Both $\partial F/\partial\dot{q}_s$ and $\partial F/\partial\dot{e}_k$ are dissipative forces.

3. The Mei symmetry of the system

We want to gain a new type of conserved quantity of the system under infinitesimal transformations. So, first we introduce the infinitesimal transformations as

$$\begin{aligned}&t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}}),\\ &q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}}),\\ &e_k^*(t^*)=e_k(t)+\varepsilon\eta_k(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}}),\end{aligned} \tag{6}$$

where ε is an infinitesimal parameter, ξ_0, ξ_s, and η_k are infinitesimal generators.

Then introducing the Euler operator

$$E_s=\frac{\rm d}{{\rm d}t}\frac{\partial}{\partial\dot{q}_s}-\frac{\partial}{\partial q_s}, \tag{7}$$

equation (3) can be written as

$$E_s(L)=Q_s-\frac{\partial F}{\partial\dot{q}_s},\quad E_k(L)=u_k-\frac{\partial F}{\partial\dot{e}_k}. \tag{8}$$

The criterion equation of Mei symmetry for the mechanico–electrical systems is

$$\begin{aligned}E_s[X^{(1)}(L)]&=X^{(1)}\left(Q_s-\frac{\partial F}{\partial\dot{q}_s}\right),\\ E_k[X^{(1)}(L)]&=X^{(1)}\left(u_k-\frac{\partial F}{\partial\dot{e}_k}\right),\end{aligned} \tag{9}$$

where

$$\begin{aligned}X^{(1)}=\;&\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}+\eta_k\frac{\partial}{\partial e_k}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial\dot{q}_s}\\ &+(\dot{\eta}_k-\dot{e}_k\dot{\xi}_0)\frac{\partial}{\partial\dot{e}_k}.\end{aligned} \tag{10}$$

If the infinitesimal generators ξ_0, ξ_s, and η_k of Mei symmetry and a gauge function $G_{\rm M}=G_{\rm M}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy the following equation

$$\begin{aligned}&X^{(1)}(L)\dot{\xi}_0+X^{(1)}[X^{(1)}(L)]\\ &\quad+X^{(1)}\left(Q_s-\frac{\partial F}{\partial\dot{q}_s}\right)(\xi_s-\dot{q}_s\xi_0)\\ &\quad+X^{(1)}\left(u_k-\frac{\partial F}{\partial\dot{e}_k}\right)(\eta_k-\dot{e}_k\xi_0)+\dot{G}_{\rm M}=0,\end{aligned} \tag{11}$$

the Mei symmetry leads to the Mei conserved quantity

$$\begin{aligned}I_{\rm M}=\;&X^{(1)}(L)\xi_0+\frac{\partial X^{(1)}(L)}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)\\ &+\frac{\partial X^{(1)}(L)}{\partial\dot{e}_k}(\eta_k-\dot{e}_k\xi_0)+G_{\rm M}\\ =\;&{\rm const.}\end{aligned} \tag{12}$$

Now there is a new expression of the conserved quantity of Mei symmetry. For the mechanico–electrical coupling dynamical systems Eq. (3), if the infinitesimal generators ξ_0,ξ_s, and η_k of Mei symmetry and a gauge function $G=G(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy the following condition

$$\begin{aligned}&\frac{\partial X^{(1)}(L)}{\partial t}+\frac{\partial L}{\partial\dot{q}_s}\dot{\xi}_s+\frac{\partial L}{\partial\dot{e}_k}\dot{\eta}_k+\frac{\partial L}{\partial q_s}\xi_s+\frac{\partial L}{\partial e_k}\eta_k-X^{(1)}\left(Q_s-\frac{\partial F}{\partial\dot{q}_s}\right)\dot{q}_s-X^{(1)}\left(u_k-\frac{\partial F}{\partial\dot{e}_k}\right)\dot{e}_k\\ &\quad+\left(Q_s-\frac{\partial F}{\partial\dot{q}_s}\right)\xi_s+\left(u_k-\frac{\partial F}{\partial\dot{e}_k}\right)\eta_k+\dot{G}=0,\end{aligned} \tag{13}$$

there is a new conserved quantity of Mei symmetry for the system

$$I = X^{(1)}(L) - \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\dot{q}_s - \frac{\partial X^{(1)}(L)}{\partial \dot{e}_k}\dot{e}_k + \frac{\partial L}{\partial \dot{q}_s}\xi_s + \frac{\partial L}{\partial \dot{e}_k}\eta_k + G = \text{const.} \tag{14}$$

Proof

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\partial X^{(1)}(L)}{\partial t} + \frac{\partial X^{(1)}(L)}{\partial q_s}\dot{q}_s + \frac{\partial X^{(1)}(L)}{\partial e_k}\dot{e}_k + \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\ddot{q}_s + \frac{\partial X^{(1)}(L)}{\partial \dot{e}_k}\ddot{e}_k \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right]\dot{q}_s - \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\ddot{q}_s - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial X^{(1)}(L)}{\partial \dot{e}_k}\right]\dot{e}_k - \frac{\partial X^{(1)}(L)}{\partial \dot{e}_k}\ddot{e}_k \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\xi_s + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{e}_k}\right)\eta_k + \frac{\partial L}{\partial \dot{e}_k}\dot{\eta}_k + \dot{G} \\
&= \frac{\partial X^{(1)}(L)}{\partial t} + \frac{\partial X^{(1)}(L)}{\partial q_s}\dot{q}_s + \frac{\partial X^{(1)}(L)}{\partial e_k}\dot{e}_k - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right]\dot{q}_s - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial X^{(1)}(L)}{\partial \dot{e}_k}\right]\dot{e}_k \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\xi_s + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{e}_k}\right)\eta_k + \frac{\partial L}{\partial \dot{e}_k}\dot{\eta}_k + \dot{G} \\
&= \frac{\partial X^{(1)}(L)}{\partial t} + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \frac{\partial L}{\partial \dot{e}_k}\dot{\eta}_k + E_s(L)\xi_s + \frac{\partial L}{\partial q_s}\xi_s + E_k(L)\eta_k + \frac{\partial L}{\partial e_k}\eta_k \\
&\quad - E_s[X^{(1)}(L)]\dot{q}_s - E_k[X^{(1)}(L)]\dot{e}_k + \dot{G} \\
&= \frac{\partial X^{(1)}(L)}{\partial t} + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \frac{\partial L}{\partial \dot{e}_k}\dot{\eta}_k + \frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial e_k}\eta_k + \left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\xi_s + \left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\eta_k \\
&\quad - X^{(1)}\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\dot{q}_s - X^{(1)}\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\dot{e}_k + \dot{G} \\
&= 0.
\end{aligned}$$

When $Q_s - \partial F/\partial \dot{q}_s = 0,\ u_k - \partial F/\partial \dot{e}_k = 0$, according to the above theorem, if the infinitesimal generators $\xi_0,\ \xi_s$, and η_k of Mei symmetry and a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy Eq. (13), i.e.

$$\begin{aligned}
&\frac{\partial X^{(1)}(L)}{\partial t} + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \frac{\partial L}{\partial \dot{e}_k}\dot{\eta}_k + \frac{\partial L}{\partial q_s}\xi_s \\
&\qquad + \frac{\partial L}{\partial e_k}\eta_k + \dot{G} = 0,
\end{aligned} \tag{15}$$

it can lead to the new conserved quantity Eq. (14) of the Mei symmetry of mechanico–electrical coupling dynamical systems. Now, an example is given to illustrate the result.

4. An example

There is an electrocircuit of capacitor and the capacitor plate is connected to a spring with elastic coefficient k. Free vibration along vertical direction is considered between capacitor plates, under action of gravity, elastic and electric field forces. Let A denote the plate area. The power of the system is supplied by electromotive force U. Take coordinates q and electric charge e as generalized coordinates. The kinetic energy of the system is $T = \dot{q}^2/2$, the spring potential energy is $V = kq/2$, the magnetic energy is $W_{\mathrm{m}} = \dot{e}^2/2$ and the electric energy of the capacitor is $W_{\mathrm{e}} = Ae/2$.

Here we study the momentum dependence of capacitor plate movement symmetry.

Then the Lagrangian of the system is

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\dot{e}^2 - \frac{1}{2}kq - \frac{1}{2}Ae. \tag{16}$$

The dissipative function is

$$F = \frac{1}{2}k\dot{q}^2 + \frac{1}{2}R\dot{e}^2, \tag{17}$$

and the non-conservative generalized force is

$$u = R\dot{e}. \tag{18}$$

So the equation of motion of the system is

$$\ddot{q} + \frac{1}{2}k + k\dot{q} = 0, \quad \ddot{e} + \frac{1}{2}A = U. \tag{19}$$

According to the above theory, we choose the infinitesimal generators

$$\xi_0 = \xi = \eta = 1. \tag{20}$$

Substituting it into Eqs. (8), (9), and (10), we can obtain

$$X^{(1)}(L) = -\frac{1}{2}k - \frac{1}{2}A, \quad X^{(1)}[X^{(1)}(L)] = 0,$$

$$E_s[X^{(1)}(L)] = E_k[X^{(1)}(L)] = 0. \tag{21}$$

Clearly, the infinitesimal generators meet Eq. (9). Substituting Eq. (21) into Eq. (11), we have the gauge function

$$G_{\rm M} = c \tag{22}$$

with $c = \text{const}$.

According to Eq. (12), the conserved quantity is

$$I_{\rm M} = -\frac{1}{2}k - \frac{1}{2}A + c = \text{const}. \tag{23}$$

Now substituting Eq. (21) into Eq. (13), the gauge function is obtained

$$G = \left(\frac{1}{2}k + \frac{1}{2}A - U\right) + kq + Re. \tag{24}$$

The new conserved quantity of Mei symmetry for the mechanico–electrical system is

$$\begin{aligned} I = & -\frac{1}{2}k - \frac{1}{2}A + \dot{q} + \dot{e} \\ & + \left(\frac{1}{2}k + \frac{1}{2}A - U\right)t + kq + Re \\ = & \ \text{const}. \end{aligned} \tag{25}$$

Clearly, the conserved quantity equation (25) is different from the original conserved quantity Eq. (23).

So we have obtained the new form of a conserved quantity of Mei symmetry for the motion of mechanico–electrical systems and at the same time, the definition and criterion are also gained.

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