Control of random Boolean networks via average sensitivity of Boolean functions*

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In this paper, we discuss how to transform the disordered phase into an ordered phase in random Boolean networks. To increase the effectiveness, a control scheme is proposed, which periodically freezes a fraction of the network based on the average sensitivity of Boolean functions of the nodes. Theoretical analysis is carried out to estimate the expected critical value of the fraction, and shows that the critical value is reduced using this scheme compared to that of randomly freezing a fraction of the nodes. Finally, the simulation is given for illustrating the effectiveness of the proposed method.

Keywords: random Boolean network, control, average sensitivity

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1. Introduction

Complex systems and networks have become hot topics in recent years and various ideas have been proposed to deal with complexities resulting from a large number of elements interacting with each other.[1−4] The Boolean network is one of the important models of complex systems due to its broad applications in bio-systems (including gene networks, neuron networks and metabolic networks), engineering systems and social systems.[5−7]

It is known that the random Boolean network (RBN) was first proposed to handle genetic regulatory networks several decades ago by Kauffman,[5] and it was later found to provide a better understanding of living organisms or complex dynamics whose behaviors are determined by thresholds or logic. RBNs can exhibit various complex phenomena, which can be roughly classified as two different phases: ordered and disordered phases. In the ordered phase, the network dynamics tend to be an ordered pattern after a transient period, while in the disordered phase, the evolution procedure seems to be unpredictable.

Then, it is natural for people to consider how to design a control strategy and how to implement the control in RBNs in order to achieve desired behaviors of RBNs. In fact, many efforts have been made in the control of phase transition and chaotic behavior in RBNs.[8−11] For example, Refs. [10] and [11] developed methods by freezing an quenched fraction of the nodes for every fixed period of time in order to suppress the chaos, and a theoretical relation between the fraction and the period was also derived to ensure the Boolean network to behave periodically. In a follow-up work,[12] the control scheme was developed with freezing a varying fraction of the nodes in accordance to a known periodic external signal, inspired by the circadian rhythm. A rough approximation of the maximal fraction of nodes to be frozen was done and numerically verified. The maximal fraction of the nodes to be frozen is relatively large in the existing results, which may not be effective in practice.

The objective of this paper is to propose a control scheme to increase the effectiveness by reducing the number of frozen nodes. To decrease the fraction of the controlled nodes, the average sensitivity, a known complexity measure of Boolean functions, is employed since it characterizes each Boolean function by its ability of propagating damage. Our strategy is to freeze the high sensitive nodes (that is, the nodes whose Boolean functions have high average sensitivity value) prior to the low sensitive nodes. However, this ordered selection renders the expected average sensitivity (EAS) of the acting nodes to be time-varying, which makes the expectation of the one-period damage percolation hard to be obtained, and therefore, we have to find ways to simplify the calculation of EAS.

This paper is organized as follows. Basic concepts of RBN including EAS are given in Section 2. Then a new control strategy is given and the critical value of

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the fraction of nodes to be frozen (denoted by $\gamma_{\rm c}^{\max}$) is analysed for our new method in Section 3. Following that, simulations for a standard case of RBN are given to calculate some critical condition or values in order to verify our analysis. Finally, the concluding remarks are given.

Notations: $\mathbb{F}_2 = \{0, 1\}$ denotes the Galois field of two elements. The addition of this field is denoted by $\oplus$, defined modulo 2. $\mathbb{F}_2^N$ stands for the set of vectors of length N over $\mathbb{F}_2$. If $\boldsymbol{x} \in \mathbb{F}_2^N$, the ith component of the vector is denoted by x_i. Define $\boldsymbol{u}^{(i)} \in \mathbb{F}_2^N$ to be the unit vector with the ith component being 1 while all other components being 0. $\boldsymbol{x}^{\rm T}\boldsymbol{y}$ denotes the inner product of these two vectors with the same length. $\langle a \rangle$ denotes the expectation (or expected value) of random variable a.

2. Preliminaries

In this section, we introduce some basic knowledge related to random Boolean network (RBN). Consider a RBN consisting of N nodes, and each node just has two (logic) states (usually denoted as 0 and 1). Define a vector $\boldsymbol{x}(t) \in \mathbb{F}_2^N$ to indicate the state of the network at time t. Each node i receives the value of k randomly selected nodes $(x_{i1}, x_{i2}, \ldots, x_{ik})$ to update its state, that is,

$$x_i(t+1) = f(x_{i1}(t), x_{i2}(t), \ldots, x_{ik}(t)),$$

where f is a mapping $f : \mathbb{F}_2^K \to \mathbb{F}_2$ randomly selected for the ith node, called a Boolean function. Once selected, both f and the inputs for each node are fixed during the evolution of the network.

The Boolean function f can be generated as follows: for each combination of its inputs, the probability of the output being 1 is p and the probability of the output being 0 is $1 - p$. The probability p is called the bias of the Boolean network. Like the state of the Boolean network, f can also be represented by its truth table vector $\boldsymbol{t}_f$, which belongs to $\mathbb{F}_2^{2^K}$.

For all 2^{2^K} possible Boolean functions of connectivity K, we establish a one-to-one correspondence between the numbers from 1 to 2^{2^K} to vectors representing the Boolean functions. For $\boldsymbol{t}_f = (t_1, t_2, \ldots, t_{2^K})^{\rm T} \in \mathbb{F}_2^{2^K}$, we define

$$\psi_K(\boldsymbol{t}_f) = \sum_{n=1}^{2^{2^K}} t_n 2^{n-1}, \tag{1}$$

which is a binary expansion of $\boldsymbol{t}_f$. Another useful concept, Hamming weight of $\boldsymbol{x} \in \mathbb{F}_2^N$, can be defined as:

$$W_{\rm H}(\boldsymbol{x}) = \boldsymbol{x}^{\rm T}\boldsymbol{x}.$$

Next, we introduce an important concept, the average sensitivity of Boolean functions for the following control strategy design. It measures the sensitivity of Boolean functions subject to change of its inputs.

Definition 1[13] Let f denote a Boolean function $\mathbb{F}_2^K \to \mathbb{F}_2$ and $\boldsymbol{u}^{(i)}$ a unit vector.

1) The sensitivity $s_f(\boldsymbol{w})$ is defined as:

$$s_f(\boldsymbol{w}) = |\{i | f(\boldsymbol{w}) \neq f(\boldsymbol{w} \oplus \boldsymbol{u}^{(i)}, i = 1, \ldots, K\}|,$$

where $\boldsymbol{w} \in \mathbb{F}_2^K$.

2) The average sensitivity s_f is defined as the average of $s_f(\boldsymbol{w})$ over all $\boldsymbol{w} \in \mathbb{F}_2^K$:

$$s_f = 2^{-K} \sum_{\boldsymbol{w} \in \mathbb{F}_2^K} s_f(\boldsymbol{w}).$$

It is known that the critical condition for the transition between ordered and disordered phase can be interpreted by percolation, which is based on the expected value of average sensitivity.[12]

Define a random variable $F_{K,p} : \Omega \to \mathcal{F}_K$, where Ω is the sample space and $\mathcal{F}_K$ is the set of vector representations of all possible Boolean functions of connectivity K. The probability of each Boolean function is:

$$P(F_{K,p} = \boldsymbol{t}_f) = p^{W_{\rm H}(\boldsymbol{t}_f)}(1-p)^{2^K - W_{\rm H}(\boldsymbol{t}_f)}.$$

For this random variable, the expected value of average sensitivity is denoted by $\langle s_f \rangle$. It is expressed as:

$$\langle s_f \rangle = \sum_f P(F_{K,p} = \boldsymbol{t}_f) s_f.$$

3. Control using periodic functions

In this section, we propose the new control scheme and estimate the expected critical value of the fraction of the frozen nodes in the RBN (denoted by $\gamma_{\rm c}^{\max}$). An approximation approach is given with the help of a proposed method to calculate the expected average sensitivity of the Boolean functions.

We employ the expected average sensitivity (EAS) of RBN, denoted by ξ, as the order parameter to describe the phase of RBN. It is known[15] that, in the case without control, the EAS can be derived as

$$\xi := \langle s_f \rangle = 2Kp(1-p).$$

It was proved that $\langle s_f \rangle$ is the parameter ξ to determine the phase of RBN in Ref. [13]. Namely, if the order parameter ξ is larger than 1, the RBN is ordered; otherwise it is disordered.

In Ref. [12], a detailed physical explanation of ξ was proposed in light of the damage percolation, which provides a better way to understand the chaotic behaviour of RBN. Moreover, a method with an arbitrary periodic function to shift the disordered phase of the RBN to an ordered phase was proposed.[12] The basic idea was to periodically freeze a portion of the nodes in order to block the damage percolation. Denote $Pe_{\rm r}(\gamma^{\max})$ as the damage percolation during one period when nodes to be blocked are randomly chosen. Define $\xi_{\rm r}(\gamma)$ as the mean average sensitivity of the remaining nodes when a fraction γ of the nodes is frozen:

$$\xi_{\rm r}(\gamma) = \frac{1}{N-M} \sum_{i=M+1}^{N} s_{f_i},$$

where M is $N\gamma$, $\xi_{\rm r}(1) = 0$, and the Boolean functions $f_i, i = M+1, \ldots, N$ are the functions that are not frozen.

$Pe_{\rm r}(\gamma^{\max})$ can be expressed as:

$$Pe_{\rm r}(\gamma^{\max}) = \prod_{t=1}^{\tau} \xi_{\rm r}(\gamma^{\max} F_\tau(t))(1, -\gamma^{\max} F_\tau(t)), \quad (2)$$

where $\gamma^{\max}$ is the maximal fraction of the nodes to be frozen. $F_\tau(t)$ is a periodic function with period τ ranging from 0 to 1.

As the nodes to be frozen are randomly selected,

$$\langle \xi_{\rm r}(\gamma^{\max} F_\tau(t)) \rangle = \xi.$$

Then we have:

$$\langle Pe_{\rm r}(\gamma^{\max}) \rangle = \prod_{t=1}^{\tau} \xi(1 - \gamma^{\max} F_\tau(t)). \quad (3)$$

Therefore, in each step the probability of damage percolation is decreased in average just by the blocking effect formulated by the term $1 - \gamma^{\max} F_\tau(t)$. In a disordered phase, ξ is larger than 1, which amplifies the damage percolation by a fixed factor step by step.

An intuitive idea to better block the damage percolation is to block the more sensitive nodes in advance. This action can weaken the ability of the acting part of the RBN to propagate damage. A permutation can accomplish this task. For an arbitrary set of Boolean function $\mathcal{B} = \{\boldsymbol{t}_{f_i} | i = 1, \ldots, N\}$, denote σ as a permutation:

$$\sigma = \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_N \\ \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_N} \end{pmatrix}$$

satisfying that, for any $i \leqslant j$, $s_{f_{\sigma(i)}} \geq s_{f_{\sigma(j)}}$.

Hence, for a RBN whose Boolean function set is $\mathcal{B}$, the blocking order is $f_{\sigma(1)}, f_{\sigma(2)}, \ldots, f_{\sigma(M)}$, where M is the maximal number of nodes to be frozen. In this situation, $\xi_{\rm r}$ in equation (2) is decreased to $\xi_{\rm o}$, which can be expressed as:

$$\xi_{\rm o}(\gamma) = \frac{1}{N-M} \sum_{i=M+1}^{N} s_{f_{\sigma(i)}}, \quad (4)$$

where M is $N\gamma$, and $\xi_{\rm o}(1) = 0$.

Define $Pe_{\rm o}(\gamma)$ as the percolation during one period in ordered situation. Similar to equation (2), we get

$$Pe_{\rm o}(\gamma^{\max}) = \prod_{t=1}^{\tau} \xi_{\rm o}(\gamma^{\max} F_\tau(t))(1 - \gamma^{\max} F_\tau(t)). \quad (5)$$

As a result,

$$\begin{aligned} \langle Pe_{\rm o}(\gamma^{\max}) \rangle &= \prod_{t=1}^{\tau} (1 - \gamma^{\max} F_\tau(t)) \\ &\quad \times \left\langle \prod_{t=1}^{\tau} \xi_{\rm o}(\gamma^{\max} F_\tau(t)) \right\rangle \\ &\leq \langle \overline{Pe_{\rm o}(\gamma^{\max})} \rangle, \end{aligned}$$

where $\langle \overline{Pe_{\rm o}(\gamma^{\max})} \rangle$ is

$$\begin{aligned} \langle \overline{Pe_{\rm o}(\gamma^{\max})} \rangle &= \prod_{t=1}^{\tau} (1 - \gamma^{\max} F_\tau(t)) \\ &\quad \times \prod_{t=1}^{\tau} \langle \xi_{\rm o}(\gamma^{\max} F_\tau(t) \rangle. \end{aligned} \quad (6)$$

From (4) it is easy to obtain

$$\langle \xi_{\rm o}(\gamma^{\max} F_\tau(t) \rangle = \frac{1}{N-M} \sum_{i=M+1}^{N} \langle s_{f_{\sigma(i)}} \rangle. \quad (7)$$

The critical parameter $\langle \gamma_{\rm c}^{\max} \rangle$ can be calculated by

$$\langle Pe_{\rm o} \gamma^{\max} \rangle = 1.$$

However, it is not easy to calculate the value of $\langle \prod_{t=1}^{\tau} \xi_{\rm o}(\gamma^{\max} F_\tau(t) \rangle$. Suppose $\overline{\gamma}_{\rm c}^{\max}$ is the root of

$$\langle \overline{Pe_{\rm o}(\gamma^{\max})} \rangle = 1. \quad (8)$$

As both $\langle Pe_{\rm o}(\gamma^{\max}) \rangle$ and $\langle \overline{Pe_{\rm o}(\gamma^{\max})} \rangle$ are strictly monotone decreasing with respect to $\gamma^{\max}$, we have $\gamma_{\rm c}^{\max} \leq \overline{\gamma}_{\rm c}^{\max}$.

Consequently, if we can calculate $\langle s_{f_{\sigma(i)}} \rangle, i = 1, \ldots, N$, then by equations (6), (7) and (8), we can derive the upper bound of $\gamma_{\rm c}^{\max}$.

With the help of the theory of order statistics, the analytical expression of $\langle s_{f_\sigma(i)}\rangle, i = 1, \ldots, N$ can be obtained.[15] However, for large N and K, it is very hard to compute these values directly. It is known that by using appropriate population quantile, the expected value of an order statistics can be approximated in large numbers. Therefore, we will use an approximation approach to determine $\langle s_{f_\sigma(i)}\rangle$ in the sequel.

Recalling $\mathcal{F}_K$ which is the set of vector representations of all possible Boolean functions of connectivity K, it can be rewritten as:

$$\mathcal{F}_K = \{g_i | g_i = \psi_K^{-1}(i-1), i = 1, \ldots, 2^{2^K}\}.$$

Thus, for the random Set $\mathcal{B}$, the probability that g_i belongs to $\mathcal{B}$ is $P(F_{K,p} = g_i)$.

Define a set $\mathcal{S}_{K,p} = \{h_i | i = 1, \ldots, l_K\}$, which includes all the possible sensitivities of the functions in $\mathcal{F}_K$, and define a series of sets $\mathcal{F}_{K,i}$ as:

$$\mathcal{F}_{K,i} = \{g_j | g_j \in \mathcal{F}_K, s_{g_j} = h_i\}, i = 1, \ldots, l_K.$$

Denote $n(g_i)$ as the number of g_i in $\mathcal{B}$ and $n(h_i)$ as the number of functions whose sensitivity is h_i. Then we have

$$n(h_i) = \sum_{g_j \in \mathcal{F}_{K,i}} n(g_j).$$

The expected value is

$$\begin{aligned}\langle n(h_i)\rangle &= \sum_{g_j \in \mathcal{F}_{K,i}} \langle n(g_j)\rangle \\ &= \sum_{g_j \in \mathcal{F}_{K,i}} NP(F_{K,p} = g_j). \end{aligned} \tag{9}$$

Define another permutation $\hat{\sigma}$ satisfying that, for any $i < j$, $h_{\hat{\sigma}(i)} > h_{\hat{\sigma}(j)}$. If we visualize $g_i \in \mathcal{F}_{K,j}$ as a rectangle with width $NP(F_{K,p} = g_j)$ and height h_j. Equation (9) indicates that $\langle n(h_i)\rangle$ can be viewed as a length which is the sum of all the width of rectangles which have the same height h_j. Then we construct l rectangles with width $\langle n(h_i)\rangle$ and height h_i. Place these rectangles from high to low, as shown in Fig. 1.

The generation of the elements in the random set $\mathcal{B}$ of a RBN can be regarded as N independent experiments with identical conditions(the same bias p). Therefore, according to the Borel's law of large numbers, we have that, for a sufficiently large N, $\frac{n(h_i)}{N}$ approximates $\sum_{g_j \in \mathcal{F}_{K,i}} P(F_K = g_j)$; the larger N is, the better the approximation will be. Thus, after reordering, the distribution of the average sensitivity of the Boolean functions of a RBN is close to the configuration shown in Fig. 1 for large N.

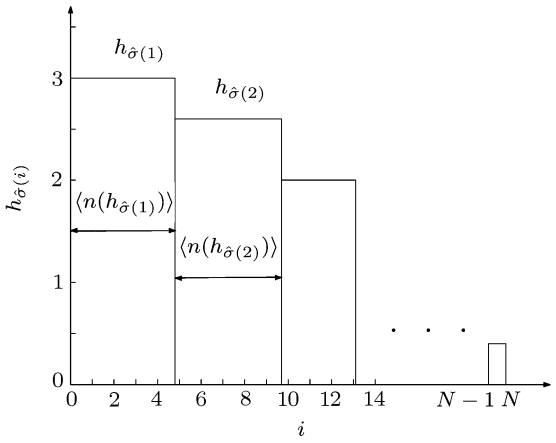


Fig. 1. Distribution of $n(h_{\hat{\sigma}(i)})$.

Then it is reasonable that the expected sensitivity $\langle s_{f_\sigma(i)}\rangle$ can be approximated by the area covered by the rectangles from $i-1$ to i. In order to calculate this value, we define

$$A(t) = \int_{t=0}^{N} H(t)\mathrm{d}t, \tag{10}$$

where $H(t) = h_{\hat{\sigma}(i)}$, if $t \in [\sum_{j=0}^{i-1}\langle n(h_{\hat{\sigma}(j)})\rangle, \sum_{j=i}^{l_K}\langle n(h_{\hat{\sigma}(j)})\rangle]$, $i = 1, \ldots, l_K$, and we set $n(h_{\hat{\sigma}(0)}) = 0$. Therefore,

$$\langle s_{f_{\sigma(i)}}\rangle = A(i) - A(i-1), i = 1, \ldots, N. \tag{11}$$

In this way, we propose an approximating method to compute the EAS which serves to the computation of $\langle \overline{Pe_o(\gamma^{\max})}\rangle$. It should be noticed that for large N and K, numerical calculation is needed for our paper involving equation (8), which will be further discussed in the next section.

4. Numerical case study

As we know, it is hard to give an explicit result for $\langle \gamma_c^{\max}\rangle$ for general RBNs. In this section, we compute its upper bound numerically. All these computation depends on the average sensitivity of each Boolean function and its probability. Here we consider a RBN of size 1000 and $K = 3$, and Table 1 is for the Boolean functions of $K = 3$.

Table 1. Table for computing the average sensitivity of a Boolean function.

i	inputs	outputs	σ_i^j	q_i
1	0 0 0	x_1	$\sigma_1^1=2,\sigma_1^2=3,\sigma_1^3=5$	3
2	0 0 1	x_2	$\sigma_1^1=4,\sigma_1^2=6$	2
3	0 1 0	x_3	$\sigma_1^1=4,\sigma_1^2=7$	2
4	0 1 1	x_4	$\sigma_4^1=8$	1
5	1 0 0	x_5	$\sigma_5^1=6,\sigma_5^1=7$	2
6	1 0 1	x_6	$\sigma_6^1=8$	1
7	1 1 0	x_7	$\sigma_7^1=8$	1
8	1 1 1	x_8	–	–

It is not hard to obtain the following formula to calculate the average sensitivity

$$s_f=\frac{1}{4}\sum_{i=1}^{7}\sum_{j=1}^{q_i}x_i\oplus x_{\sigma_i^j}. \tag{12}$$

The EAS in this paper is basically EAS of order one, which means the sensitivity to just one flip. Note that the EAS of order two (three) is the average change of the output if two (three) of the inputs are flipped, but the probabilities of these events are low. In fact, if the value of one node in the Boolean network is flipped, the possibility that this change reaches to a certain node is 1. If two arbitrarily selected nodes are flipped, the possibility that there exist a node whose inputs include all the flipped nodes is $N/N^2=1/N$. If three arbitrarily selected nodes are flipped, the possibility is even lower, that is, $N/N^3=1/N^2$. Therefore, for a sufficient large N, the effect of the EAS of order two and three can be omitted in a comparison with the EAS of order one. Thus, it will be good enough to consider only EAS of order one for a sufficiently large N.

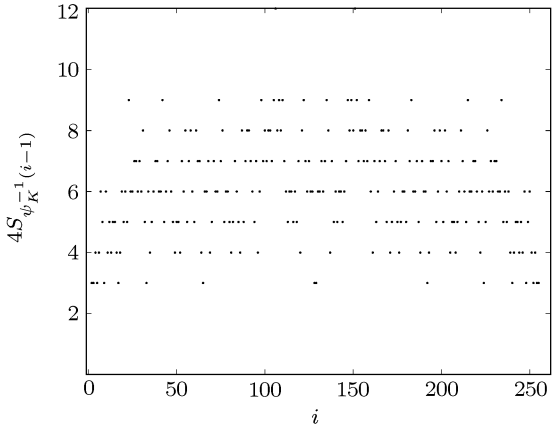


Fig. 2. Average sensitivity distribution.

Therefore, the average sensitivity of all possible Boolean functions has been computed (see Fig. 2). The distribution of these average sensitivities and the related properties of the Boolean functions are shown in Table 2. At first, we find that there is no Boolean function with average sensitivity $\frac{1}{4}$, $\frac{2}{4}$, $\frac{10}{4}$ and $\frac{11}{4}$. Also, Table 2 shows that the most possible average sensitivity is $\frac{6}{4}$, and the least possible average sensitivity is either 0 or $\frac{1}{4}$. For each average sensitivity the Boolean functions categorized by the number of '1' entries in their lookup table have a kind of symmetry. For instance, there are two kinds of functions with the same average sensitivity, $\frac{9}{4}$, and moreover, the number of the two functions is 8. One kind have 3 '1' entries in its lookup table and the other kind have 5. If we randomly select two functions of the two kinds, respectively, and then form a pair of the two functions. From Table 2, it is clear that this pair is complementary, that is, the sum of these two vectors that represent the two functions is $(1,1,\ldots,1)^{\mathrm{T}}$. The symmetry and complementariness can be seen below in Fig. 4.

Table 2. Characteristics of Boolean functions with $K=3$.

$4s_f$	number of functions	number of '1' entries
12	2	4
9	8	3
	8	5
8	30	4
7	24	3
	24	5
6	16	2
	32	4
	16	6
5	24	3
	24	5
4	12	2
	6	4
	12	6
3	8	3
	8	5
0	1	8
	1	0

Additionally, for each pair, the average of number of '1' entries is just the half of the length of the lookup table.

Then we will calculate the upper bound of $\langle\gamma_{\mathrm{c}}^{\max}\rangle$ for a RBN of size 1000 and $K=3$ based on the methods proposed in the preceding sections. In fact, this can be done by combining (11), (6), (7) and (8). The calculation of $\langle s_{f_{\sigma(i)}}\rangle$ is a key step to obtain the upper bound.

It should be noticed that the value of $\langle s_{f_{\sigma(i)}} \rangle$ changes only when the interval $[i-1, i]$ contains at least one border of two rectangles in Fig. 1 when i increases from 1 to N. As $L_3 = 9$, for $K = 3$ there exist at most 8 borderlines. This means that the number of these changes is at most 8. This can be verified in Table 2. Therefore, this number is relatively small enough to be neglected when N is large enough. Thus, we obtain an estimation of $\langle s_{f_{\sigma(i)}} \rangle$ as

$$\langle s_{f_{\sigma(i)}} \rangle = H(i), \tag{13}$$

where $H(t)$ was defined in Eq. (10). In Fig. 3, the distributions of $\langle s_{f_{\sigma(i)}} \rangle$ for $p = 0.7$ with this approximation is shown.

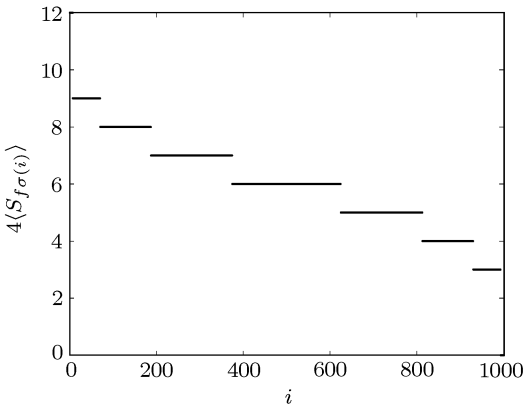


Fig. 3. Ordered EAS distribution for $K = 3$.

With the help of the estimation (13), we can numerically determine the relationship between P and $\langle \bar{\gamma}_{\rm c}^{\max} \rangle$ for a certain $F_\tau(t)$.

In Fig. 4, the dots show the relationship between P and the $\langle \bar{\gamma}_{\rm c}^{\max} \rangle$ using our method, while the stars show that of randomly freezing nodes. It can be seen that theoretically the new method should reduce the number of nodes in the control of disorder.

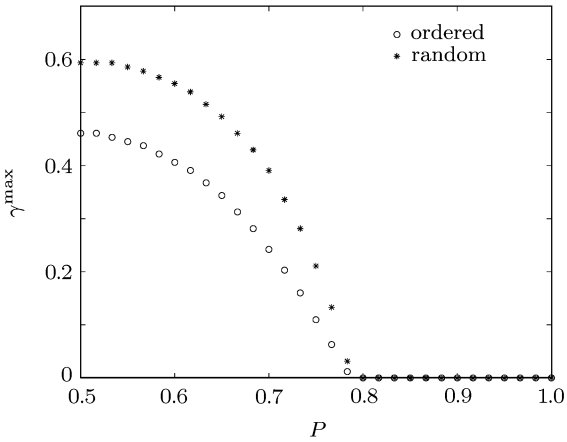


Fig. 4. $P - \langle \bar{\gamma}_{\rm c}^{\max} \rangle$ relationship.

Define $C(t)$ to be the distance of two Boolean systems, whose states are $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$, respectively, namely $C(t) = \boldsymbol{W}_{\rm H}(X(t) \oplus Y(t))$. Define $Z(t)$ to describe whether the damage is amplified at time t as:

$$Z(t) = \begin{cases} 0, & \text{if } C(t) > 1, \\ 1, & \text{if } C(t) \leq 1. \end{cases}$$

In other words, if $Z(t) = 0$, damage is amplified at time t; if $Z(t) = 1$, the damage is not amplified at time t.

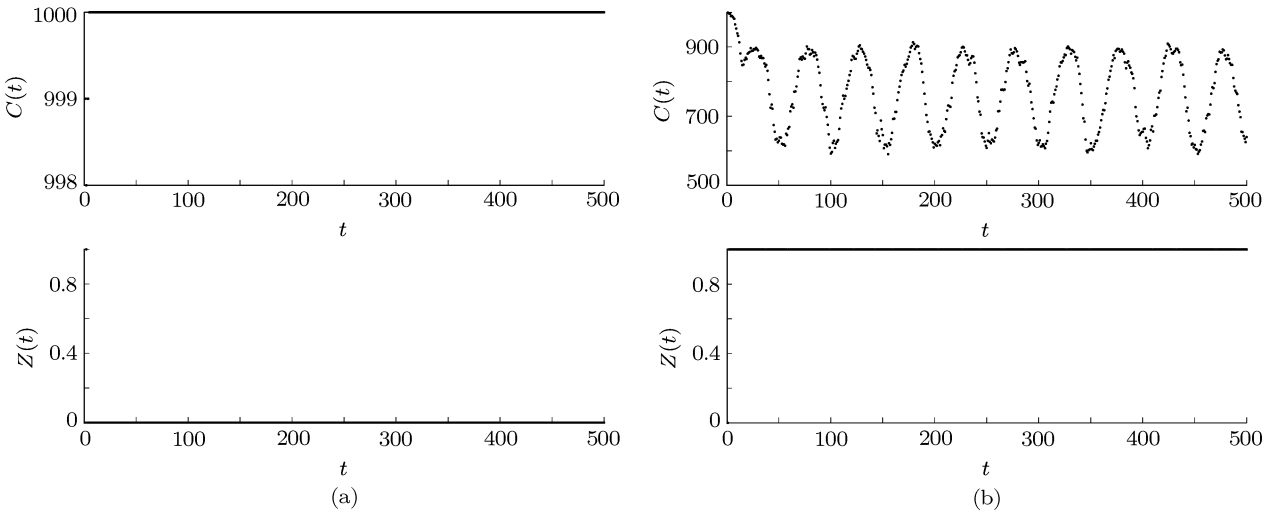


Fig. 5. The evolution of $C(t)$ and $Z(t)$ for $(P, \gamma^{\max}) = (0.5, 0.2)$. $Z(t)$ becomes constant when t is quite large in two cases: (a) $Z(t) = 0$; (b) $Z(t) = 0$.

To verify our analysis, a simulation has been carried out. For a comparison with the result of Ref. [12], we set the same configuration: RBN of 1000 nodes and the connectivity $K = 3$, the periodic signal as $\sin(wt)^2$, and the period $\tau = 50$. In Ref. [12], the controlled state was defined to be reaching a periodic behaviour. The period can be equal to an integer multiple of the external period. Here we do not use the existing methods since how large the integer should be is unknown. Instead, we provide a way to check the phase of the RBN.

Figure 5 is given to show that $Z(t)$ is insensitive to t when t is large. Figures 5(a) and 5(b) represent two standard cases of evolution for $(P, \gamma^{\max}) = (0.5, 0.2)$ (the results are similar for other $(P, \gamma^{\max})$ pairs). $Z(t)$ converges to either 0 or 1 in a few steps.

For each pair of values $(P, \gamma^{\max})$ we carried out 100 simulations, and each simulation was run twice with the initial value of a randomly given node flipped in the second run. Since $Z(t)$ is not sensitive to t after a short transient period, each simulation was run for 100 steps without loss of generality. If the state of two trajectories at the 100th step is the same or at most one node is flipped, the simulation is considered to have blocked the damage percolation; otherwise, the network fails to block the damage percolation.

In Fig. 6, each pixel represents the fraction of the 100 simulations for each pair $(P, \gamma^{\max})$. If all the 100 simulations do not block the damage percolation, then the pixel is painted black; if all of them do, it is painted white. According to the intensity scale shown in Fig. 6, the fractions between 0 and 1 are painted in grey tones. Figure 6 shows that the $(P, \langle\bar{\gamma}_{\rm c}^{\max}\rangle)$ curve bounds the possible area of the actual critical curve $(P, \langle\gamma_{\rm c}^{\max}\rangle)$, which is well matched with our analysis given above.

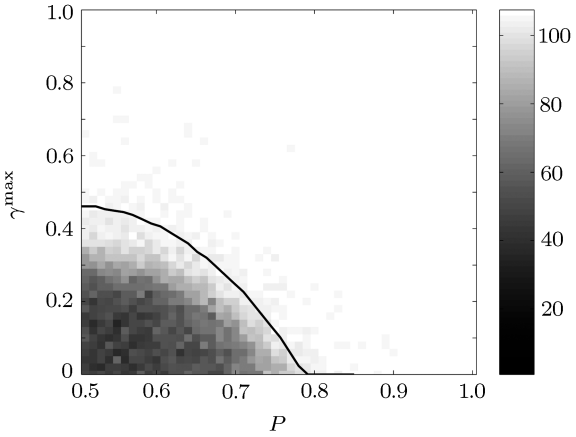


Fig. 6. Simulation.

5. Conclusions

In this paper we proposed a new strategy to control the RBN. The basic idea is to freeze the more sensitive nodes prior to the less sensitive nodes. In fact, the ordering was carried out to make the EAS of the frozen nodes be higher than that of the whole network with reducing the number of frozen nodes.

However, the exact expected percolation during one period for fixed $\gamma^{\max}$ and P is not easy to obtain. For a trade-off between the accuracy and the feasibility, we proposed a method to calculate and derive the upper bound of $\gamma_{\rm c}^{\max}$, and then gave the simulations in the case of $K = 3$ for illustration.

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