The relation between the generalised Eshelby integral and the generalised BCS and DB models*

Fan Tian-You (范天佑)[a)†] and Fan Lei(范 蕾)[b)]

[a)] *School of Science, Beijing Institute of Technology, Beijing* 100081, *China*
[b)] *School of Life Science, Beijing Institute of Technology, Beijing* 100081, *China*

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The generalised BCS dislocation group model and the generalised DB atomic cohesive force zone model have obtained the same results on nonlinear fracture study of some one-, two- and three-dimensional quasicrystals. This work reveals some inherent connection between the two models, and finds that their common basis is the generalised Eshelby integral based on the generalised Eshelby energy–momentum tensor for quasicrystals. Further applications of the theory in solving nonlinear fracture problems of the materials are also discussed.

Keywords: quasicrystals, crack, plasticity, generalised Eshelby integral

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1. Introduction

The study on elasticity including the linear elastic fracture theory of quasicrystals has gained some achievements, see e.g. Ref. [1]. People pay attention to the study on dynamics, the relevant results can refer to Ref. [2]. Another focus on the mechanical properties is the plasticity which is an interesting subject for both theoretical and experimental studies. The crack and fracture problems under plastic deformation of the material attract attention of researchers, but the solution of which presents a great difficulty. In Refs. [3] and [4] Fan *et al.* suggested some phenomenological models to solve the problems of nonlinear fracture of quasicrystals. One of them is the so-called generalised Bilby–Cottrell–Swinden–Schmith (BCS) model (or dislocation pile model), another is the generalised Dugdale–Barenblatt (DB) model (or atomic cohesive force zone or narrow band plastic zone model). In these two approaches the physical background and the adopted mathematical methods are quite different, but the results are similar. All results can be exactly reduced to the linear solutions when the nonlinear behaviour is reduced to a linear one of the material.

In Ref. [4] the authors discussed the mechanism of plastic deformation of quasicrystalline materials based on the experimental observations from different laboratories, subsequently we emphasised and analysed repeatedly the effects of the new degrees of freedom—phason modes in quasicrystals, just the presence of the phason degrees leads to the complexity of the mechanical behaviour including the plasticity of the material, the major difficulty to the development of theory of plasticity for the material is also addressed.

The physical nature of plasticity in quasicrystalline material is essentially unknown so far. Though there is no possibility to obtain the support of plasticity theory in studying the nonlinear fracture, this does not mean that one could not carry out some in-depth investigation. The possible way is to take some simplified models.

The BCS model originated from the Eshelby continuum dislocation hypothesis. Eshelby[5] developed a series of the theory for the defects of crystals, which can be extended to discuss the relevant problems concerning quasicrystals. According to this point of view we here put forward a generalised Eshelby energy–momentum tensor and generalised Eshelby integral for describing nonlinear behaviour of some one- and two-dimensional quasicrystals, whose explicit physical meaning and important applications in the fracture analysis under nonlinear deformation of the material will be exhibited in the following sections. The case for three-dimensional quasicrystals is similar but the

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†Corresponding author. E-mail: tyfan2006@yahoo.com.cn
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description will be much more lengthy, so the discussion is omitted here due to the space limitation.

2. An overview of the relation between the generalised BCS model and generalised DM models for quasicrystals

In Ref. [3] Fan *et al.* suggested the generalised BCS model for some one- and two-dimensional quasicrystals, and in Ref. [4] we proposed a generalised DB model for the same materials. By making use of the two models one can obtain the same results on sizes of plastic zones and the crack tip opening displacements. The mathematical details were provided in the references mentioned above, here only the main results of the work are figured out.

If considering a crack as a dislocation group and the plastic zone at the crack tip as a dislocation pile up in a quasicrystal, then the dislocation density distribution function can be determined based on the corresponding dislocation solutions and singular integral equation method. With the dislocation density distribution function the extent of dislocation pile up (or the size of the plastic zone) and the amount of dislocations motion (or equivalent to the crack tip opening displacement) can be obtained. This is the work of Ref.[3], the scheme is named the generalised BCS model, which originated from classical Bilby, Cottrell, Swinden and Smith model.[6,7]

By taking another scheme in Ref. [4] (i.e., being commensurate with likely elasto-perfect plastic crack model in quasicrystal), and taking linearisation, we can obtain the elasto-perfect plastic crack solution through determining stress potential function or displacement potential function and solving the boundary value problem of partial differential equations of higher order, in which the most effective analytic tool is the complex variable function-conformal mapping method. As direct results of the solution the size of the plastic zone and the crack opening displacement are determined. The root of this scheme originated from the classical Dugdale and Barenblatt work,[8,9] so it is called the generalised DB model. It is evident that the procedure is much more complicated than that of the first scheme, but it is constructive and does not depend upon the known dislocation solutions.

The two schemes mentioned above can be used for Mode I, Mode II and Mode III cracks of one-, two- and three-dimensional quasicrystals, but for simplicity here only the results for Mode I Griffith crack of two-dimensional point group 10 mm decagonal quasicrystals are listed. The size of the plastic zone is

$$d = a\left[\sec\left(\frac{\pi p}{2\sigma_{\rm c}}\right) - 1\right], \tag{1}$$

and the crack tip opening displacement is

$$\delta_I = \frac{2\sigma_c a}{\pi}\Big[\frac{1}{L+M} + \frac{K_1}{MK_1 - R^2}\Big]\Big[\ln\sec\Big(\frac{\pi p}{2\sigma_{\rm c}}\Big)\Big], \tag{2}$$

in which a denotes the half-length of the crack, p the applied tensile stress (the traction), $\sigma_{\rm c}$ the uni-axial tensile plastic strength of the material, and

$$L = C_{12}, \quad M = (C_{11} - C_{12})/2$$

the phonon elastic constants, K_1 and R are the phason and phonon–phason coupling elastic constants for decagonal quasicrystals with point group 10 mm, respectively.

It should be noticed that the above two models have different physical backgrounds and the adopted mathematical tools are also quite different, but the results are the same. People believe there may be an inherent connection between these two models. In Section 4 we would like to demonstrate this.

3. Generalised Eshelby energy–momentum tensor and generalised Eshelby integral

Due to a lack of constitutive equations of plasticity of quasicrystals up to now, we use the following nonlinear elastic constitutive relation, i.e., if defining the free energy (or strain energy density) as

$$F(\varepsilon_{ij}, w_{ij}) = \int_0^{\varepsilon_{ij}} \boldsymbol{\sigma}_{ij}{\rm d}\varepsilon_{ij} + \int_0^{w_{ij}} \boldsymbol{H}_{ij}{\rm d}w_{ij}, \tag{3}$$

then there are

$$\boldsymbol{\sigma}_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}}, \quad \boldsymbol{H}_{ij} = \frac{\partial F}{\partial w_{ij}}. \tag{4}$$

So formulae (3) and (4) can be viewed as constitutive law for the nonlinear (including linear) elasticity of quasicrystals, but they, in general, cannot describe plastic deformation. If there is a proportional loading but no unloading, the relationship can give an appropriate description of the plastic deformation.

In the above formulae ε_{ij} and w_{ij} are phonon and phason strain tensors given by

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial \boldsymbol{u}_i}{\partial x_j} + \frac{\partial \boldsymbol{u}_j}{\partial x_i}), \quad w_{ij} = \frac{\partial w_i}{\partial x_j}, \tag{5}$$

with $\boldsymbol{u}_i$ and $\boldsymbol{w}_i$ denoting the phonon and phason displacement vectors; $\boldsymbol{\sigma}_{ij}$ and $\boldsymbol{H}_{ij}$ being the phonon and phason stress tensors, respectively, which satisfy the equilibrium equations (if the body forces and generalised body forces are omitted)

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \frac{\partial H_{ij}}{\partial x_j} = 0. \tag{6}$$

Equations (3)–(6) are the basic equations describing nonlinear deformation of a quasicrystal.

By extending the Eshelby work[5] for crystals to quasicrystals, Fan[10] (can also refer to Ref. [11]) defined the generalised Eshelby energy–momentum tensor for a plane (i.e., two-dimensional) sample of the material as

$$G = Fn_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} - H_{ij} n_j \frac{\partial w_i}{\partial x_1}, \tag{7}$$

where F is defined by Eq. (3), and n_i the unit vector of outward normal at any point of an arc in the quasicrystal, and at which there are

$$\boldsymbol{\sigma}_{ij}\boldsymbol{n}_j = \boldsymbol{T}_i, \quad \boldsymbol{H}_{ij}\boldsymbol{n}_j = \boldsymbol{h}_i,$$

where $\boldsymbol{T}_i$ denotes the traction vector, and $\boldsymbol{h}_i$ the generalised traction vector.

Furthermore, we define an integral as

$$E = \int_{\Gamma} G \mathrm{d}\Gamma, \tag{8}$$

where Γ is an integration path enclosing a crack tip in the material. In memory of Eshelby the integral is named the generalised Eshelby integral. The integral gives the value which is independent of the path. In Appendix A, the proof of the path-independency of the integral can be found.

Because the integral presents this important character, it has some applications in the fracture analysis of quasicrystals.

Another important character is that the value of the integral is equal to the energy release rate g_{III} (under Mode III loading), or g_{II} (under Mode II loading), or g_{I} (under Mode I loading) of one-, two- and three-dimensional quasicrystals respectively, when the material is in linear elastic deformation state. The mathematical proof for this will be found in Appendix B. A proof of more general physical sense of the integral will be given also in the same appendix.

The features of the generalised Eshelby integral (8) described above imply that it may be significant for the analysis of nonlinear fracture of quasicrystals. In the following some applications for this purpose will be discussed.

4. Relation between crack tip opening displacement and the generalised Eshelby integral

Using the path-independency of E-integral (8), we take the integration path Γ shown in Fig. 1, and

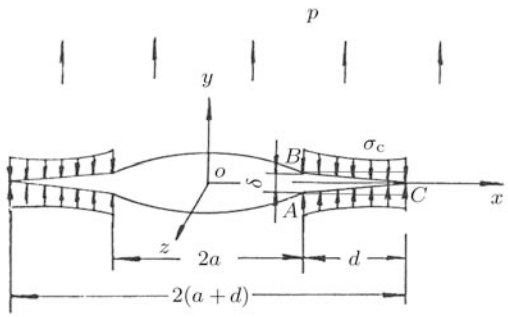


Fig. 1. The integration path for evaluating crack tip opening displacement.

let the path close to the surface of the plastic zone as near as possible. Now the integral is $\int_{ACB}$, and the segment set ACB is the integration path. Along segments AC and CB

$$\mathrm{d}y = 0,\ T_1 = 0,\ T_2 = \sigma_{\mathrm{c}},\ h_1 = 0,\ h_2 = H_{\mathrm{c}},$$

so that

$$\begin{aligned} E &= \int_{\Gamma} G \mathrm{d}\Gamma = \int_{ACB} G \mathrm{d}\Gamma \\ &= \int_A^B \left(\sigma_{\mathrm{c}} \frac{\partial u_y}{\partial x} \mathrm{d}x + H_{\mathrm{c}} \frac{\partial w_y}{\partial x} \mathrm{d}x \right) \\ &= \sigma_{\mathrm{c}} \left[(u_y)_B - (u_y)_A \right] + H_{\mathrm{c}} \left[(w_y)_B - (w_y)_A \right] \\ &\approx \sigma_{\mathrm{c}} \left[(u_y)_B - (u_y)_A \right] \\ &= \sigma_{\mathrm{c}} \delta_{\mathrm{I}}. \end{aligned} \tag{9}$$

This proves the relation between the generalised Eshelby integral and crack tip opening displacement; it shows that the E-integral is equivalent to the crack tip opening displacement, where σ_{c} is the atomic cohesive force (or the plastic yield strength) and H_{c} the 'generalised atomic cohesive force' of the quasicrystal material. Microscopically the quantity H_{c} is meaningful, but it has not been measured macroscopically at

present, thus the effect of which is omitted in formula (9).

For Mode II and Mode III crack the proof is similar.

Since from the generalised BCS model and generalised DB model one can obtain the same result for crack tip opening displacement, which also serves as a direct result of generalised Eshelby integral for quasicrystals, we realise that the generalised Eshelby energy–momentum tensor theory is the united physical basis of the two models.

5. Some further interpretation on the application of *E*-integral in the nonlinear fracture analysis of quasicrystals

The above sections and the appendixes attached in the following exhibit that the generalised Eshelby integral can be as a united basis of generalised BCS model and generalised DB model for nonlinear fracture analysis of some one-, two- and three-dimensional quasicrystals. We suggest taking the E-integral as a fracture parameter, and

$$E = E_{\rm c} \tag{10}$$

as fracture criterion, where $E_{\rm c}$ is the critical value of the integral, a material constant, which can be measured through some conventional specimens, the discussion of which will be given in Appendix C.

Though the measurement of the critical value of generalised Eshelby integral may be easier, the evaluation of values of E-integral of quasicrystals under plastic deformation is very difficult (it is even impossible at present). Therefore the implementation of the criterion (10) is not so convenient in practice. Considering the equivalency between the crack tip opening displacement $\delta(\delta_{\rm I}, \delta_{\rm II}, \delta_{\rm III})$ and E-integral, we suggest taking the crack tip opening displacement as a fracture parameter. Elasto-plastic crack solutions for some one- and two-dimensional quasicrystals can be evaluated based on the generalised DB model (e.g. refer to Ref. [4]), in which the size of the plastic zone and crack tip opening displacement are determined, from which an equivalent plastic fracture criterion is suggested to be

$$\delta = \delta_{\rm c}, \tag{11}$$

in which $\delta_{\rm c}$ represents the critical value of the crack tip opening displacement, a material constant. The shortcoming of this criterion lies in the difficulties of the experimental determination of $\delta_{\rm c}$, but we can obtain through the following relation

$$\delta_{\rm c} = \frac{E_{\rm c}}{\sigma_{\rm c}}. \tag{12}$$

Combining Eq. (10) and Eq. (11) makes the nonlinear fracture analysis for quasicrystals available. Some details will be given in the Appendix C of this article.

6. Discussion and conclusion

Apart from recent developments of elasticity and elastodynamics of quasicrystals, e.g. Refs. [12]–[14] etc, the work in the nonlinearity of the material is remarkable. The interpretation of the subject given above is a preliminary description, in which we showed the importance of generalised Eshelby energy–momentum tensor and generalised Eshelby integral theoretically and practically for nonlinear fracture analysis of quasicrystalline materials. The theory combines the generalised BCS model and generalised DB model, in application, it provides effective fracture criterion and effective procedure to measure the relevant material constants, and thus is helpful to set up a nonlinear fracture theory of the materials. This work is worth carrying out for the development of theory of plastic fracture of quasicrystals; it may also be considered as a development of nonlinear fracture theory for conventional structural materials including crystalline materials.

In the evaluation of $\delta(\delta_{\rm I}, \delta_{\rm II}, \delta_{\rm III})$ the contribution of generalised traction (corresponding to the phason field) can be obtained, but it is omitted in the final expression for simplicity, because the explicit physical meaning of the quantity is not so clear and the measurement of which is not available at present. In connection with this, the term $H_{\rm c}$ in the derivation of Eq. (9) is ignored at last. The relevant simplification is correct since it is commensurate with the development level to date of quasicrystallography.

Appendix A : Proof on path-independency of *E*-integral

Consider a single-connected region D with a closed boundary C. From the Green formula the first, second and third terms of the left-hand side of the integral (8) will be

$$\int_C F n_1 \mathrm{d}\Gamma = \int_C F \mathrm{d}x_2 = \iint\limits_D \frac{\partial F}{\partial x_1}\mathrm{d}x_1\mathrm{d}x_2$$
$$\int_C \Big(\sigma_{ij}n_j\frac{\partial u_i}{\partial x_1} + H_{ij}n_j\frac{\partial w_i}{\partial x_1}\Big)\mathrm{d}\Gamma = \iint\limits_D \frac{\partial}{\partial x_j}\Big(\sigma_{ij}\frac{\partial u_i}{\partial x_1} + H_{ij}\frac{\partial w_i}{\partial x_1}\Big)\mathrm{d}x_1\mathrm{d}x_2.$$

So we have

$$\int_C G\mathrm{d}\Gamma = \iint\limits_D \left[\frac{\partial F}{\partial x_1} - \frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_1} + H_{ij}\frac{\partial w_i}{\partial x_1}\right)\right]\mathrm{d}x_1\mathrm{d}x_2. \tag{A1}$$

Meanwhile, Eq. (6) reads

$$\begin{aligned}\frac{\partial F}{\partial x_1} &= \frac{\partial F}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial x_1} + \frac{\partial F}{\partial w_{ij}}\frac{\partial w_{ij}}{\partial x_1} = \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1} + H_{ij}\frac{\partial w_{ij}}{\partial x_1} = \sigma_{ij}\frac{\partial}{\partial x_1}\Big[\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big] + H_{ij}\frac{\partial}{\partial x_1}\Big(\frac{\partial w_i}{\partial x_j}\Big)\\ &= \sigma_{ij}\frac{\partial}{\partial x_1}\Big(\frac{\partial u_i}{\partial x_j}\Big) + H_{ij}\frac{\partial}{\partial x_1}\Big(\frac{\partial w_i}{\partial x_j}\Big),\end{aligned}$$

here the symmetry of tensor $\boldsymbol{\varepsilon}_{ij}$ (given by the first equation of (5)) is used. The above result can be rewritten as

$$\frac{\partial F}{\partial x_1} = \frac{\partial}{\partial x_j}\Big(\sigma_{ij}\frac{\partial u_i}{\partial x_1}\Big) - \frac{\partial \sigma_{ij}}{\partial x_j}\frac{\partial u_i}{\partial x_1} + \frac{\partial}{\partial x_j}\Big(H_{ij}\frac{\partial w_i}{\partial x_1}\Big) - \frac{\partial H_{ij}}{\partial x_j}\frac{\partial w_i}{\partial x_1} = \frac{\partial}{\partial x_j}\Big(\sigma_{ij}\frac{\partial u_i}{\partial x_1} + H_{ij}\frac{\partial w_i}{\partial x_1}\Big), \tag{A2}$$

in the last step in deriving (A2) the equilibrium equations (6) are used, and in the above description $x = x_1$, $y = x_2$.

Substituting Eq. (A2) into Eq. (A1) we have

$$\int_C G\mathrm{d}\Gamma = 0.$$

If taking $C = \Gamma + BB' - \Gamma' + A'A$ (see Fig. A1) in which Γ' is a path similar to Γ, these two curves intersect the upper face of the crack at points B and B', and the lower face at points A and A', respectively, due to

$$\mathrm{d}x_2 = \mathrm{d}y = 0,\ T_i = 0,\ h_i = 0,$$

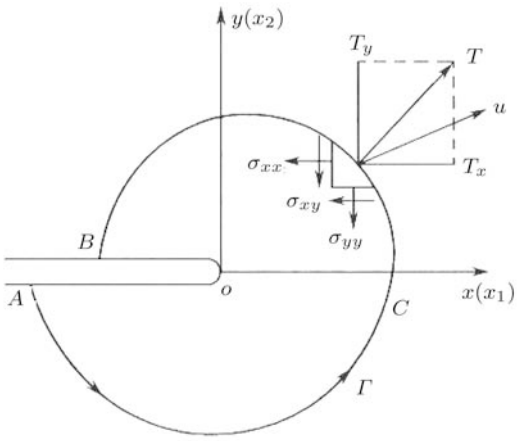


Fig. A1. Integration path of E-integral.

at the segments BB' and $A'A$, there exist

$$\int_{BB'+A'A} G\mathrm{d}\Gamma = 0, \quad \int_\Gamma G\mathrm{d}\Gamma = \int_{\Gamma'} G\mathrm{d}\Gamma.$$

Because Γ and Γ' are arbitrary paths satisfying the requirements of E-integral definition, the last equality proves its path-independency.

Appendix B : Proof of the equivalency of *E*-integral to energy release rate under linear elastic case for quasicrystals

In the text of this paper we argue that the E-integral is equivalent to the energy release rate when the quasicrystals are in linear elastic deformation state, the mathematical proof about this is given here. The proof can be done for one-, two- and three-dimensional quasicrystals and for Mode I, Mode II and Mode III cracks. For simplicity here only discuss Mode III crack for one-dimensional hexagonal quasicrystals.

At linear elastic case, the generalised Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl},$$
$$H_{ij} = K_{ijkl}w_{kl} + R_{klij}\varepsilon_{kl},$$

is reduced to

$$\sigma_{yz} = \sigma_{yz} = 2C_{44}\varepsilon_{yz} + R_3 w_{zy},$$
$$\sigma_{zx} = \sigma_{xz} = 2C_{44}\varepsilon_{zx} + R_3 w_{zx},$$
$$H_{zy} = K_2 w_{zy} + 2R_3\varepsilon_{zy},$$
$$H_{zx} = K_2 w_{zx} + 2R_3\varepsilon_{zx},$$

for the phonon–phason coupling anti-plane shearing (or longitudinal shearing) state in elasticity of one-dimensional hexagonal quasicrystals, one finds the free energy (or strain energy density) to be

$$F = C_{44}(\varepsilon_{xz}^2 + \varepsilon_{yz}^2) + \frac{1}{2}K_2(w_{zx}^2 + w_{zy}^2) + R_3(\varepsilon_{zx}w_{zx} + \varepsilon_{yz}w_{zy}). \tag{B1}$$

Substituting the crack solution [6] into (B1) we have

$$F = \frac{1}{4\Big(C_{44}K_2 - R_3^2\Big)}\Big\{C_{44}\Big[K_2K_{\rm III}^{\parallel} - R_3K_{\rm III}^{\perp}\Big]^2 + 2K_2\Big[C_{44}K_{\rm III}^{\perp} - R_3K_{\rm III}^{\parallel}\Big]^2 + 2R_3\Big[K_2K_{\rm III}^{\parallel} - R_3K_{\rm III}^{\perp}\Big]\Big[C_{44}K_{\rm III}^{\perp} - R_3K_{\rm III}^{\parallel}\Big]\Big\}, \tag{B2}$$

where $K_{\rm III}^{\parallel} = \sqrt{\pi a}\tau_1$ and $K_{\rm III}^{\perp} = \sqrt{\pi a}\tau_2$ are stress intensity factors associated with phonon and phason fields respectively.

Integrating the quantity given in Eq. (B2) around a path surrounding the crack tip, then one can obtain the first term of the E-integral. Because of the path-independency of the integral, we can take the half-circle with the crack tip as its origin and with radius r, so that

$$\int_{\pi}^{-\pi} F{\rm d}y = \int_{\pi}^{-\pi} Fr\cos\theta{\rm d}\theta = 0, \tag{B3}$$

where (r, θ) denotes the crack tip polar coordinates.

Now we calculate the second and third terms of the integral.

According to the definition in Section 3 it is known that between the traction, generalised traction and the phonon, phason stresses, we have

$$T_z = \sigma_{zx}n_x + \sigma_{zy}n_y = \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta,$$
$$h_z = H_{zx}n_x + H_{zy}n_y = H_{zx}\cos\theta + H_{zy}\sin\theta.$$

By substituting the crack solution[6] into the above formulae, we find

$$T_z = \frac{1}{\sqrt{2\pi r}}K_{\rm III}^{\parallel}\sin(\theta/2),$$
$$h_z = \frac{1}{\sqrt{2\pi r}}K_{\rm III}^{\perp}\sin(\theta/2). \tag{B4}$$

Based on the crack solution[6] we obtain

$$u_z = \frac{K_2\tau_1 - R_3\tau_2}{C_{44}K_2 - R_3^2}\sqrt{2ar}\sin(\theta/2),$$
$$w_z = \frac{C_{44}\tau_2 - R_3\tau_1}{C_{44}K_2 - R_3^2}\sqrt{2ar}\sin(\theta/2). \tag{B5}$$

In addition, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial\theta}.$$

Substituting these results and relations into the second and third terms of the integral, we obtain

$$-\int_{\pi}^{-\pi}\Big(\sigma_{ij}n_j\frac{\partial u_i}{\partial x_1} + H_{ij}n_j\frac{\partial w_i}{\partial x_1}\Big){\rm d}\Gamma = \frac{K_2(K_{\rm III}^{\parallel})^2 + C_{44}(K_{\rm III}^{\perp})^2 - 2R_3K_{\rm III}^{\parallel}K_{\rm III}^{\perp}}{C_{44}K_2 - R_3^2}. \tag{B6}$$

From (B 3) and (B 6) we find that

$$E_{\rm III} = \frac{K_2(K_{\rm III}^{\parallel})^2 + C_{44}(K_{\rm III}^{\perp})^2 - 2R_3K_{\rm III}^{\parallel}K_{\rm III}^{\perp}}{C_{44}K_2 - R_3^2} = g_{\rm III}, \tag{B7}$$

in which the evaluation of $g_{\rm III}$ is given in Refs. [4], [8], and [9]. Similarly one can obtain

$$E_{\rm I} = g_{\rm I}, \tag{B8}$$

and

$$E_{\rm II} = g_{\rm II}, \tag{B9}$$

where the subscripts mean the Mode I, Mode II and Mode III of the cracks, and the expressions of $g_{\rm I}$ and $g_{\rm II}$ are not listed any more (which can be found in Refs. [8], [1], [12] etc). The above demonstration indicates that the generalised Eshelby integral is equivalent to energy release rate of crack in linear elasticity of quasicrystals.

Under the plastic deformation of quasicrystals, in general, the generalised Eshelby integral does not represent the energy release rate, the reason of this is that the unloading can occur due to crack extension,

and stress–strain relations described by Eqs. (3) and (4) do not remain one to one correspondence between stresses and strains, the physical background of the E-integral will not be held. But if the potential energy per unit thickness for a plane (i.e., a two-dimensional) region Ω occupied by a quasicryslal, with the boundary Γ, is defined as

$$V = \int_{\Omega} F\mathrm{d}x\mathrm{d}y - \int_{\Gamma} (T_i u_i + h_i w_i)\mathrm{d}\Gamma, \tag{B10}$$

then the total potential energy is

$$\Pi = BV, \tag{B11}$$

where B denotes the thickness of the material, and there is a relation

$$E = -\frac{\delta \Pi}{\delta A} = -\frac{1}{B}\frac{\delta \Pi}{\delta a}. \tag{B12}$$

Here E-integral can be expressed only by the 'difference quotient' rather than the 'differential quotient'. The difference quotient is not the energy release rate, which only represents the ratio between individual difference value of the energy over difference value of crack length for the same configuration specimen but with different crack lengths (this excludes any crack extension). Considering the situation, the physical meaning of Eq. (B12) is left for further discussion, even if, Eq. (B12) provides useful application for calibrating the measurement data of the integral in experiments. The relevant discussion will be done in the next appendix of the paper.

For linear elastic case unloading does not affect the validity of the constitutive law of Eqs. (3) and (4), the crack growth does not break the path-independency of $E-$ integral. In this case formula (B12) will be replaced by

$$E = -\frac{\partial \Pi}{\partial A} = -\frac{1}{B}\frac{\partial \Pi}{\partial a}. \tag{B13}$$

If $B = 1$ (i.e., the specimen has unit thickness), then we have

$$E = -\frac{\partial \Pi}{\partial a} = g, \tag{B14}$$

i.e., in the case the generalised Eshelby integral is equal to crack energy release rate. In the case of plastic deformation in small scale, e.g. $d/a \ll 1$, formula (B14) will approximately hold. But for plastic deformation in large scale, the integral has to be expressed by Eq. (B12) and does not represent crack energy release rate.

Appendix C : On the evaluation of the critical value of E-integral

The E-integral provides not only the theoretical basis for generalised BCS model as well as generalised DB model, but also an effective tool for measuring the material constants E_c and δ_c. We have mentioned that the measurement of δ_c is difficult, but the measurement of E_c is easier. Due to the connection between these two quantities the value of the former can be obtained from that of the latter.

At low and conventional temperatures, quasicrystals exhibit brittle behaviour, their fracture belongs to linear elastic one, and the measurement of fracture toughness can be carried out by indentation technique, see e.g. Ref. [15]. At high temperature, the materials show large plastic deformation, the measurement can be done by bending experiment. In experiment the three-point bending specimen, see Fig. C1, is one

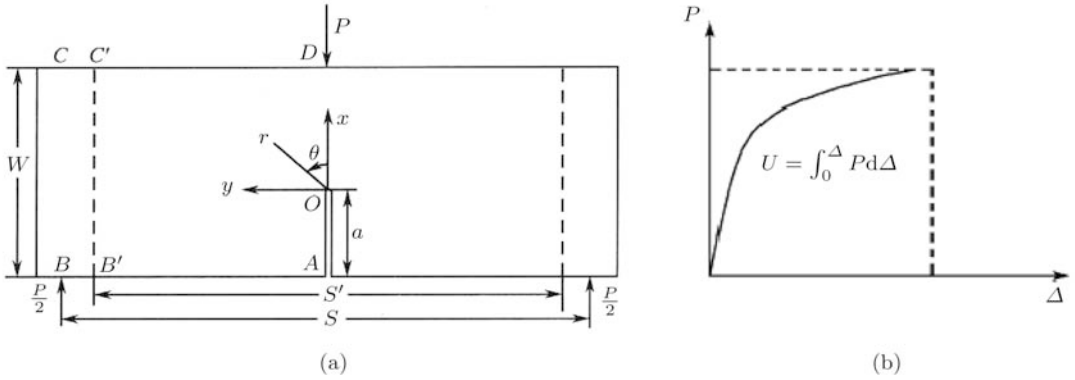


Fig. C 1. Three point bending specimen and calibration of relevant energy.

of most useful specimens. There are multi-specimens testing procedure or single specimen testing procedure. For simplicity we here discuss the single specimen testing only.

For the three-point bending specimen shown in Fig. C1(a), based on formula (B12) the value of E-integral is evaluated by

$$E = \frac{2U}{B(W-a)}, \tag{C1}$$

where W represents the width of the specimen, and U the area under P–Δ curve, i.e.,

$$U = \int_0^{\Delta} P \mathrm{d}\Delta \tag{C2}$$

see Fig. C1(b), in which P is the load (force per unit thickness) at the loading point, and Δ the displacement of the same point. When one observes the initiation of crack growth, then the value of E-integral is marked by E_{c}, the nonlinear fracture toughness of the quasicrystal..

If the phason field is absent, the material reduces to crystalline material or conventional structural material. In this case, the E-integral is reduced to conventional Eshelby integral, which was independently observed again by Rice[16] and Cherepanov [17] after 11 years compared with the Eshelby early publication,[5] and named as J-integral widely in engineering circle. The work of Rice has been systematically developed, especially Begley and Landes[18,19] put forward the experimental study on J-integral, further promotes the development of nonlinear fracture theory and its applications for conventional engineering materials. They found that the measurement can obtain meaningful results in terms of very small size specimens. Those experiences would be helpful for the experimental study of nonlinear fracture of quasicrystalline materials.

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