Symmetry of Hamiltonian and conserved quantity for a system of generalized classical mechanics*

Zhang Yi(张 毅)†

College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, *China*

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This paper focuses on a new symmetry of Hamiltonian and its conserved quantity for a system of generalized classical mechanics. The differential equations of motion of the system are established. The definition and the criterion of the symmetry of Hamiltonian of the system are given. A conserved quantity directly derived from the symmetry of Hamiltonian of the generalized classical mechanical system is given. Since a Hamilton system is a special case of the generalized classical mechanics, the results above are equally applicable to the Hamilton system. The results of the paper are the generalization of a theorem known for the existing nonsingular equivalent Lagrangian. Finally, two examples are given to illustrate the application of the results.

Keywords: symmetry of Hamiltonian, generalized classical mechanics, conserved quantity

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1. Introduction

A system of generalized classical mechanics is a system whose Lagrangian contains high-order derivatives of generalized coordinates with respect to time. The theory of the generalized classical mechanics, has been developing increasingly since the middle of 20th century.[1,2] The Lagrange equations with high-order derivatives have been used in physics, especially in mechanics and field theory, for example, in the electromagnetic theory with second-order derivatives. In mathematics, some good results have been obtained in the descriptions of modern geometric methods, for example, in Ref. [2]. In recent twenty years, many important results on the theory of symmetries and conserved quantities of generalized classical mechanics have been obtained.[3−13]

In 1966, Currie and Saletan[14] studied the problem of the equivalent Lagrangian for the single degree of freedom mechanical system and pointed out that there exist conserved quantities in this case. Hojman and Harleston[15] extended the result to general multi-degree-of-freedom systems. In Refs. [16] and [17], the symmetry under which two Lagrangians can lead to the same differential equations, i.e., when the solutions to the equations of motion corresponding to one of the Lagrangians each satisfy the equations of motion obtained from the other Lagrangian and conversely, is called a symmetry of Lagrangian. Mei and Wu presented the symmetry of Lagrangian of nonholonomic system.[18−20] In the present paper, we further generalize the results to the system of generalized classical mechanics, and present a new symmetry of Hamiltonian for the system of generalized classical mechanics. The conserved quantity corresponding to the symmetry of Hamiltonian of the system is deduced. Since a Hamilton system is a special case of the generalized classical mechanics, the results above are equally applicable to the Hamilton system. Finally, two examples are given to illustrate the application of the results.

2. Differential equations of motion for the system of generalized classical mechanics

Consider a system of generalized classical mechanics whose configuration is determined by n generalized coordinates q^i $(i = 1, \ldots, n)$. The Lagrangian of the system is

$$L = L\left(t, q^i_{(0)}, q^i_{(1)}, \ldots, q^i_{(\omega)}\right), \quad (i = 1, \ldots, n), \quad (1)$$

where

$$q^i_{(j)} = \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, q^i\,(t)\,, (i = 1, \ldots, n;\ j = 0, 1, \ldots, \omega)\,. \quad (2)$$

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†Corresponding author. E-mail: weidiezh@pub.sz.jsinfo.net

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The generalized Euler–Lagrange equations of the system are[2−4]

$$\sum_{j=0}^{\omega}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial q^i_{(j)}}=0,\quad (i=1,\ldots,n)\,. \tag{3}$$

Introduce the generalized momenta and the generalized Hamiltonian

$$p_i^{(s)}=\sum_{j=0}^{\omega-s-1}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial q^i_{(j+s+1)}},\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1)\,, \tag{4}$$

and

$$H\left(t,\boldsymbol{q}_{(s)},\boldsymbol{p}^{(s)}\right)=p_i^{(s)}\dot{q}^i_{(s)}-L,\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1)\,, \tag{5}$$

then the differential equations of motion of the system, Eq. (3) can be expressed in the canonical form as follows:[3]

$$\dot{q}^i_{(s)}=\frac{\partial H}{\partial p_i^{(s)}},\quad \dot{p}_i^{(s)}=-\frac{\partial H}{\partial q^i_{(s)}},\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1)\,. \tag{6}$$

Let

$$a^\mu=\begin{cases} q^i_{(s)}, & (\mu=2sn+i;s=0,1,\ldots,\omega-1;\ i=1,\ldots,n),\\ p_i^{(s)}, & (\mu=(2s+1)\,n+i;\quad s=0,1,\ldots,\omega-1;\ i=1,\ldots,n),\end{cases} \tag{7}$$

and

$$R_\mu=\begin{cases} p_i^{(s)}, & (\mu=2sn+i;\quad s=0,1,\ldots,\omega-1;\ i=1,\ldots,n),\\ 0, & (\mu=(2s+1)\,n+i;s=0,1,\ldots,\omega-1;\ i=1,\ldots,n),\end{cases} \tag{8}$$

then Eq. (6) can be further expressed in the covariant geometry form[8]

$$\Omega_{\mu\nu}\dot{a}^\nu-\frac{\partial H}{\partial a^\mu}=0,\quad (\mu=1,\ldots,2\omega n) \tag{9}$$

and the contravariant algebraic form

$$\dot{a}^\mu-\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}=0,\quad (\mu=1,\ldots,2\omega n)\,, \tag{10}$$

where

$$\Omega_{\mu\nu}=\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu},\quad (\Omega_{\mu\nu})=(\Omega^{\mu\nu})^{-1}\,,$$

$$\Omega^{\mu\nu}\Omega_{\nu\tau}=\delta_{\mu\tau},\quad (\Omega^{\mu\nu})=\begin{bmatrix}\boldsymbol{\Omega}_0 & & \\ & \ddots & \\ & & \boldsymbol{\Omega}_{\omega-1}\end{bmatrix},$$

and

$$\boldsymbol{\Omega}_0=\cdots=\boldsymbol{\Omega}_{\omega-1}=\begin{bmatrix}\mathbf{0_{n\times n}} & \boldsymbol{I}_{n\times n}\\ -\boldsymbol{I}_{n\times n} & \mathbf{0_{n\times n}}\end{bmatrix}, \tag{11}$$

where $0_{n\times n}$ is the zero matrix with order n and $\boldsymbol{I}_{n\times n}$ is the unit matrix with order n.

3. Definition and criterion of the symmetry of Hamiltonian for the system of generalized classical mechanics

Define

$$S_\mu=\Omega_{\mu\nu}\dot{a}^\nu-\frac{\partial H}{\partial a^\mu}, \tag{12}$$

and let $H=H\,(t,\boldsymbol{a})$, $R_\mu=R_\mu\,(\boldsymbol{a})$ be replaced by the dynamical functions $\bar{H}=\bar{H}\,(t,\boldsymbol{a})$, $\bar{R}_\mu=\bar{R}_\mu\,(t,\boldsymbol{a})$, then we will have

$$\bar{S}_\mu=\bar{\Omega}_{\mu\nu}\dot{a}^\nu-\frac{\partial\bar{H}}{\partial a^\mu}-\frac{\partial\bar{R}_\mu}{\partial t}, \tag{13}$$

where

$$\bar{\Omega}_{\mu\nu}=\frac{\partial\bar{R}_\nu}{\partial a^\mu}-\frac{\partial\bar{R}_\mu}{\partial a^\nu},\quad \text{with}\quad \det\left(\bar{\Omega}_{\mu\nu}\right)\neq 0. \tag{14}$$

Definition For a generalized classical system under consideration, if the solutions to the following equations of motion corresponding to the dynamical functions $H=H\,(t,\boldsymbol{a})$, $R_\mu=R_\mu\,(\boldsymbol{a})$

$$S_\mu=0 \tag{15}$$

each satisfy the equations of motion obtained by other dynamical functions $\bar{H} = \bar{H}(t, \boldsymbol{a})$, $\bar{R}_\mu = \bar{R}_\mu(t, \boldsymbol{a})$ as follows:

$$\bar{S}_\mu = 0 \tag{16}$$

and vice versa, then the corresponding invariance is called a symmetry of Hamiltonians of the system.

From Eqs. (16) and (13), we have

$$\dot{a}^\mu = \bar{\Omega}^{\mu\nu}\left(\frac{\partial \bar{H}}{\partial a^\nu} + \frac{\partial \bar{R}_\nu}{\partial t}\right). \tag{17}$$

Substituting Eq. (17) into expression (12), and from Eq. (15), we obtain

$$\Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\left(\frac{\partial \bar{H}}{\partial a^\rho} + \frac{\partial \bar{R}_\rho}{\partial t}\right) = \frac{\partial H}{\partial a^\mu}, \quad (\mu = 1, \ldots, 2\omega n). \tag{18}$$

Therefore we have

Criterion For a generalized classical system under consideration, if two groups of dynamical functions H, R_μ and $\bar{H}$, $\bar{R}_\mu$ satisfy condition (18), then corresponding invariance is a symmetry of Hamiltonian of the system.

4. Conserved quantity deduced by the symmetry of Hamiltonian for the system of generalized classical mechanics

We now deduce a conserved quantity by using the symmetry of Hamiltonian of the generalized classical mechanics.

Substituting Eq. (18) into expression (12), we have

$$\begin{aligned} S_\mu &= \Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\left(\bar{\Omega}_{\rho l}\dot{a}^l - \frac{\partial \bar{H}}{\partial a^\rho} - \frac{\partial \bar{R}_\rho}{\partial t}\right) \\ &= \Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\bar{S}_\rho. \end{aligned} \tag{19}$$

That is,

$$\bar{S}_\mu = \Lambda^\nu_\mu S_\nu, \tag{20}$$

where

$$\Lambda^\nu_\mu = \bar{\Omega}_{\mu\rho}\Omega^{\rho\nu}, \tag{21}$$

and $\det \boldsymbol{\Lambda} = \det\left(\Lambda^\nu_\mu\right) \neq 0$.

According to Eq. (21), Eq. (18) can be written as

$$\frac{\partial \bar{H}}{\partial a^\mu} + \frac{\partial \bar{R}_\mu}{\partial t} = \Lambda^\nu_\mu \frac{\partial H}{\partial a^\nu}. \tag{22}$$

Differentiating Eq. (22) with respect to a^ρ, we obtain

$$\frac{\partial^2 \bar{H}}{\partial a^\mu \partial a^\rho} + \frac{\partial^2 \bar{R}_\mu}{\partial t \partial a^\rho} = \frac{\partial \Lambda^\nu_\mu}{\partial a^\rho}\frac{\partial H}{\partial a^\nu} + \Lambda^\nu_\mu \frac{\partial^2 H}{\partial a^\nu \partial a^\rho}. \tag{23}$$

From Eq. (21), we have

$$\Lambda^\nu_\mu \Omega_{\nu l} = \bar{\Omega}_{\mu l}. \tag{24}$$

Differentiating Eq. (24) with respect to a^ρ, we obtain

$$\frac{\partial \Lambda^\nu_\mu}{\partial a^\rho}\Omega_{\nu l} = \frac{\partial \bar{\Omega}_{\mu l}}{\partial a^\rho} - \Lambda^\nu_\mu \frac{\partial \Omega_{\nu l}}{\partial a^\rho}. \tag{25}$$

From Eq. (25), and using the formulae (11) and (14), we can obtain

$$\frac{\partial \Lambda^\nu_\mu}{\partial a^\rho}\Omega_{\nu l} = \frac{\partial \Lambda^\nu_\mu}{\partial a^l}\Omega_{\nu\rho} + \frac{\partial \bar{\Omega}_{\rho l}}{\partial a^\mu} - \Lambda^\nu_\mu \frac{\partial \Omega_{\rho l}}{\partial a^\nu}. \tag{26}$$

From Eq. (10) and expression (12), Eq. (15) can be written as follows:

$$\Omega_{\mu\nu}\Omega^{\nu\rho}\frac{\partial H}{\partial a^\rho} = \frac{\partial H}{\partial a^\mu}. \tag{27}$$

Substituting Eq. (27) into Eq. (23), we have

$$\begin{aligned} &\frac{\partial^2 \bar{H}}{\partial a^\mu \partial a^\rho} + \frac{\partial^2 \bar{R}_\mu}{\partial t \partial a^\rho} \\ &= \frac{\partial \Lambda^\nu_\mu}{\partial a^\rho}\Omega_{\nu l}\Omega^{l\tau}\frac{\partial H}{\partial a^\tau} + \Lambda^\nu_\mu \frac{\partial^2 H}{\partial a^\nu \partial a^\rho}. \end{aligned} \tag{28}$$

Substituting Eq. (26) into Eq. (28), we have

$$\begin{aligned} &\frac{\partial \Lambda^\nu_\mu}{\partial a^l}\Omega_{\nu\rho}\Omega^{l\tau}\frac{\partial H}{\partial a^\tau} + \frac{\partial \bar{\Omega}_{\rho l}}{\partial a^\mu}\Omega^{l\tau}\frac{\partial H}{\partial a^\tau} - \Lambda^\nu_\mu \frac{\partial \Omega_{\rho l}}{\partial a^\nu}\Omega^{l\tau}\frac{\partial H}{\partial a^\tau} \\ &+ \Lambda^\nu_\mu \frac{\partial^2 H}{\partial a^\nu \partial a^\rho} - \frac{\partial^2 \bar{H}}{\partial a^\mu \partial a^\rho} - \frac{\partial^2 \bar{R}_\mu}{\partial t \partial a^\rho} = 0. \end{aligned} \tag{29}$$

Taking the partial derivative of Eq. (24) with respect to t, we obtain

$$\frac{\partial \Lambda^\nu_\mu}{\partial t}\Omega_{\nu\rho} = \frac{\partial \bar{\Omega}_{\mu\rho}}{\partial t}, \tag{30}$$

i.e.,

$$\frac{\partial \Lambda^\nu_\mu}{\partial t}\Omega_{\nu\rho} - \frac{\partial^2 \bar{R}_\rho}{\partial a^\mu \partial t} + \frac{\partial^2 \bar{R}_\mu}{\partial a^\rho \partial t} = 0. \tag{31}$$

Taking the partial derivative of Eq. (27) with respect to a^ν, we have

$$\begin{aligned} &\frac{\partial \Omega_{\rho l}}{\partial a^\nu}\Omega^{l\tau}\frac{\partial H}{\partial a^\tau} + \Omega_{\rho l}\frac{\partial}{\partial a^\nu}\left(\Omega^{l\tau}\frac{\partial H}{\partial a^\tau}\right) \\ &- \frac{\partial^2 H}{\partial a^\rho \partial a^\nu} = 0. \end{aligned} \tag{32}$$

Similarly, we obtain

$$\begin{aligned} &\frac{\partial \bar{\Omega}_{\rho l}}{\partial a^\mu}\Omega^{l\tau}\frac{\partial H}{\partial a^\tau} + \bar{\Omega}_{\rho l}\frac{\partial}{\partial a^\mu}\left(\Omega^{l\tau}\frac{\partial H}{\partial a^\tau}\right) \\ &- \frac{\partial^2 \bar{H}}{\partial a^\rho \partial a^\mu} - \frac{\partial^2 \bar{R}_\rho}{\partial t \partial a^\mu} = 0. \end{aligned} \tag{33}$$

Summing up Eqs. (29) and (31), and using Eqs. (32) and (33), we can obtain

$$\frac{\tilde{\mathrm{d}}\Lambda_\mu^\nu}{\mathrm{d}t}\Omega_{\nu\rho} - \bar{\Omega}_{\rho l}\frac{\partial}{\partial a^\mu}\left(\Omega^{l\tau}\frac{\partial H}{\partial a^\tau}\right) + \Lambda_\mu^\nu\Omega_{\rho l}\frac{\partial}{\partial a^\nu}\left(\Omega^{l\tau}\frac{\partial H}{\partial a^\tau}\right) = 0, \tag{34}$$

where

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\frac{\partial}{\partial a^\mu}. \tag{35}$$

Equation (34) can be expressed in the matrix form as follows:

$$\frac{\tilde{\mathrm{d}}\Lambda}{\mathrm{d}t} = -\boldsymbol{A}\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\boldsymbol{A}, \tag{36}$$

where $\boldsymbol{A} = (A_{\mu\nu})$ is a $2\omega n \times 2\omega n$ matrix, with $A_{\mu\nu} = (\partial/\partial a^\mu)\left(\Omega^{\nu\tau}(\partial H/\partial a^\tau)\right)$. Therefore, using the properties of the trace of matrix, we can obtain

$$\begin{aligned}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left(\mathrm{tr}\boldsymbol{\Lambda}\right) &= \mathrm{tr}\left(\frac{\tilde{\mathrm{d}}\boldsymbol{\Lambda}}{\mathrm{d}t}\right)\\ &= -\mathrm{tr}\left(\boldsymbol{A}\boldsymbol{\Lambda}\right) + \mathrm{tr}\left(\boldsymbol{\Lambda}\boldsymbol{A}\right)\\ &= 0.\end{aligned} \tag{37}$$

Using Eqs. (36)and (37), we can easily have

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left(\mathrm{tr}\boldsymbol{\Lambda}^m\right) = 0 \tag{38}$$

for an arbitrary integer m, and then we have a conserved quantity

$$\mathrm{tr}\boldsymbol{\Lambda}^m = \mathrm{const.} \tag{39}$$

Therefore we have

Proposition 1 For a generalized classical system under consideration, a symmetry of Hamiltonian of the system will directly lead to a conserved quantity (39).

Next, we consider an important special case. If we choose $\omega = 1$, then the equations in expression (6) become

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad (i = 1, \ldots, n). \tag{40}$$

Equations in expression (40) are the equations of motion of a Hamilton system. Therefore we have

Proposition 2 For a Hamilton system (40), a symmetry of Hamiltonian of the system will directly lead to a conserved quantity (39).

5. Example

In order to illustrate the above results, we give two examples.

Example 1 We study the problem of symmetry of Hamiltonian and the conserved quantity of a generalized classical mechanical system, whose Lagrangian is[2−4]

$$L = \frac{1}{2}\alpha_1\dot{q}^2 + \frac{1}{2}\alpha_2\ddot{q}^2, \quad (\alpha_1 > 0, \quad \alpha_2 > 0). \tag{41}$$

According to formulae (4), (5), (7) and (8), we can easily obtain

$$\begin{aligned}&H = a^2a^3 - \frac{1}{2}\alpha_1\left(a^3\right)^2 + \frac{1}{2\alpha_2}\left(a^4\right)^2,\\ &R_1 = a^2, \quad R_2 = 0, \quad R_3 = a^4, \text{ and } R_4 = 0.\end{aligned} \tag{42}$$

The differential equations of motion in expression (9) can be written as follows:

$$-\dot{a}^2 = 0, \quad \dot{a}^1 - a^3 = 0, \quad -\dot{a}^4 - a^2 + \alpha_1 a^3 = 0,$$

and

$$\dot{a}^3 - \frac{1}{\alpha_2}a^4 = 0. \tag{43}$$

Hence, expression (12) gives

$$S_1 = -\dot{a}^2, \quad S_2 = \dot{a}^1 - a^3, \quad S_3 = -\dot{a}^4 - a^2 + \alpha_1 a^3,$$

and

$$S_4 = \dot{a}^3 - \frac{1}{\alpha_2}a^4. \tag{44}$$

Choosing the dynamical functions $\bar{H}$ and $\bar{R}_\mu$ as

$$\begin{aligned}&\bar{H} = a^2a^3 - \frac{1}{3\alpha_1}\left(a^2\right)^3 - \frac{1}{2}\alpha_1\left(a^3\right)^2 + \frac{1}{2\alpha_2}\left(a^4\right)^2,\\ &\bar{R}_1 = a^2 - \frac{1}{2}\left(a^2\right)^2, \quad \bar{R}_2 = 0, \quad \bar{R}_3 = a^4,\end{aligned}$$

and

$$\bar{R}_4 = \frac{1}{2\alpha_1}\left(a^2\right)^2, \tag{45}$$

obviously condition (18) is satisfied, and functions (45) and (42) correspond to a symmetry of Hamiltonian of the system. From expression (13), we have

$$\begin{aligned}&\bar{S}_1 = \left(1 - a^2\right)S_1, \quad \bar{S}_2 = \left(1 - a^2\right)S_2 - \frac{1}{\alpha_1}a^2S_3,\\ &\bar{S}_3 = S_3, \quad \bar{S}_4 = \frac{1}{\alpha_1}a^2S_1 + S_4,\end{aligned} \tag{46}$$

and

$$\boldsymbol{\Lambda}_1^1 = \boldsymbol{\Lambda}_2^2 = 1 - a^2, \quad \boldsymbol{\Lambda}_3^3 = \boldsymbol{\Lambda}_4^4 = 1. \tag{47}$$

Therefore we obtain

$$\mathrm{tr}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_1^1 + \boldsymbol{\Lambda}_2^2 + \boldsymbol{\Lambda}_3^3 + \boldsymbol{\Lambda}_4^4 = -2a^2 + 4 = \mathrm{const.} \tag{48}$$

The formula (48) is a conserved quantity induced by the symmetry of Hamiltonian corresponding to functions (45) and (42).

If we choose

$$\begin{aligned}\bar{H} &= a^2a^3 - \frac{1}{3\alpha_1}\left(a^2\right)^3 - \frac{1}{4\alpha_1}\left(a^2\right)^4 \\ &\quad - \frac{1}{2}\alpha_1\left(a^3\right)^2 + \frac{1}{2\alpha_2}\left(a^4\right)^2, \\ \bar{R}_1 &= a^2 - \frac{1}{2}\left(a^2\right)^2 - \frac{1}{3}\left(a^2\right)^3, \\ \bar{R}_2 &= 0, \\ \bar{R}_3 &= a^4,\end{aligned}$$

and

$$\bar{R}_4 = \frac{1}{2\alpha_1}\left(a^2\right)^2 + \frac{1}{3\alpha_1}\left(a^2\right)^3, \tag{49}$$

then we obtain

$$\mathrm{tr}\boldsymbol{\Lambda} = -2\left[\left(a^2\right)^2 + a^2 - 2\right] = \text{const.} \tag{50}$$

Formula (50) is a conserved quantity induced by the symmetry of Hamiltonian corresponding to functions (49) and (42).

And if we choose

$$\begin{aligned}\bar{H} &= a^2a^3 - \frac{1}{3\alpha_1}t\left(a^2\right)^3 - \frac{1}{2}\alpha_1\left(a^3\right)^2 \\ &\quad - \frac{1}{2\alpha_1}\left(a^2\right)^2a^4 + \frac{1}{2\alpha_2}\left(a^4\right)^2, \\ \bar{R}_1 &= a^2 - a^2a^4 + \alpha_1a^1a^2, \\ \bar{R}_2 &= ta^1a^2, \quad \bar{R}_3 = a^4,\end{aligned}$$

and

$$\bar{R}_4 = \frac{1}{\alpha_1}a^2a^4 + \frac{1}{2\alpha_1}\left(a^2\right)^2t - a^1a^2, \tag{51}$$

then we obtain

$$\mathrm{tr}\boldsymbol{\Lambda} = -2\left(a^2t + a^4 - \alpha_1a^1 - 2\right) = \text{const.} \tag{52}$$

Formula (52) is a conserved quantity induced by the symmetry of Hamiltonian corresponding to functions (51) and (42).

Example 2 Now we study the problem of symmetry of Hamiltonian and the conserved quantity of a Hamilton system as follows:

$$H = \frac{1}{2}\left[\left(a^1\right)^2 + \left(a^2\right)^2\right],$$

and

$$R_1 = a^2, \quad R_2 = 0. \tag{53}$$

The differential equations of motion in expression (9) give

$$-\dot{a}^2 - a^1 = 0, \quad \text{and} \quad \dot{a}^1 - a^2 = 0. \tag{54}$$

Expression (12) can be written as

$$S_1 = -\dot{a}^2 - a^1, \quad \text{and} \quad S_2 = \dot{a}^1 - a^2. \tag{55}$$

Choose

$$\bar{H} = -\frac{1}{4}\left[\left(a^1\right)^2 + \left(a^2\right)^2\right]^2, \quad \bar{R}_1 = -\left(a^1\right)^2a^2,$$

and

$$\bar{R}_2 = a^1\left(a^2\right)^2. \tag{56}$$

Obviously condition (18) is satisfied, and functions (56) and (53) correspond to a symmetry of Hamiltonian of the system. From the expression (13), we have

$$\begin{aligned}\bar{S}_1 &= -\left[\left(a^1\right)^2 + \left(a^2\right)^2\right]S_1, \\ \bar{S}_2 &= -\left[\left(a^1\right)^2 + \left(a^2\right)^2\right]S_2,\end{aligned} \tag{57}$$

and

$$\boldsymbol{\Lambda}^1_1 = \boldsymbol{\Lambda}^2_2 = -\left[\left(a^1\right)^2 + \left(a^2\right)^2\right]. \tag{58}$$

Therefore we have

$$\mathrm{tr}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^1_1 + \boldsymbol{\Lambda}^2_2 = -2\left[\left(a^1\right)^2 + \left(a^2\right)^2\right] = \text{const.} \tag{59}$$

Formula (59) is a conserved quantity induced by the symmetry of Hamiltonian corresponding to the functions (56) and (53).

If we choose

$$\begin{aligned}\bar{H} &= \frac{1}{3}\left(a^1\right)^3\cos t - \frac{1}{3}\left(a^2\right)^3\sin t, \\ \bar{R}_1 &= a^1a^2\cos t,\end{aligned}$$

and

$$\bar{R}_2 = a^1a^2\sin t, \tag{60}$$

then we have

$$\mathrm{tr}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^1_1 + \boldsymbol{\Lambda}^2_2 = 2\left(a^1\cos t - a^2\sin t\right) = \text{const.} \tag{61}$$

Formula (61) is a conserved quantity induced by the symmetry of Hamiltonian corresponding to the functions (60) and (53).

6. Conclusion

In this paper a new symmetry is presented of a system of generalized classical mechanics, i.e., the symmetry of Hamiltonian, under which two groups of dynamical functions H, R_μ and $\bar{H}$, $\bar{R}_\mu$ are expressed by the same differential equations of motion. The symmetry of Hamiltonian is different from the Noether symmetry, the Lie symmetry and the Mei symmetry. A symmetry of Hamiltonian can lead to a conserved quantity. Since a Hamilton system is a special case of the generalized classical mechanics, the results of the paper are equally applicable to the Hamilton system.

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