Geometric formulations and variational integrators of discrete autonomous Birkhoff systems*

Liu Shi-Xing(刘世兴)[a)b)], Liu Chang(刘 畅)[a)], and Guo Yong-Xin(郭永新)[b)†]

[a)] *Department of Mechanics, Beijing Institute of Technology, Beijing* 100081, *China*
[b)] *College of Physics, Liaoning University, Shenyang* 110036, *China*

(Received 22 June 2010; revised manuscript received 28 September 2010)

The variational integrators of autonomous Birkhoff systems are obtained by the discrete variational principle. The geometric structure of the discrete autonomous Birkhoff system is formulated. The discretization of mathematical pendulum shows that the discrete variational method is as effective as symplectic scheme for the autonomous Birkhoff systems.

Keywords: autonomous Birkhoff syetem, discrete variational principle, variational integrators
PACS: 45.20.Jj **DOI:** 10.1088/1674-1056/20/3/034501

1. Introduction

Birkhoff equations[1−3] are more general than the Hamilton equation and applied to hadron physics, statistical mechanics, space mechanics, engineering, biophysics, electromagnetic[3−5] and so on. Theoretically, all conservative and nonconservative, self-adjoint and non-self-adjoint, holonomicand nonholonomic constrained systems can always be represented in the form of Birkhoff equation.[6] The research of Birkhoff dynamics is an important field in modern analytical mechanics. A number of issues regarding the Birkhoff system have been studied, such as the Birkhoff equations of a nonholonomic constrained system, the symmetry theory, global analysis, stability of motion, geometrical description, variational principle.[6−19] Birkhoff equations are more complicated than Hamilton equations, leading to the complicacy of its computational method. In the past years, there were only some results about the numerical algorithms for Birkhoff systems, such as the symplectic schemes,[20,21] and the first integrals of discrete equation of motion for Birkhoff systems.[14]

In the present paper, the geometric formulations and variational integrators for the discrete autonomous Birkhoff systems are studied in detail. First, the geometric formulation of autonomous Birkhoff equation and the variational principle of Pfaff–Birkhoff are reviewed. Secondly, the discrete variational principle and the geometric formulation of autonomous discrete Birkhoff equation are discussed. Thirdly, the variational integrator for the autonomous Birkhoff system is constructed. Finally, one example is given to illustrate the applied results.

Throughout the paper, Einstein's summation convention is used and $\mu, \nu = 1, 2, \ldots, 2n$.

2. Autonomous Birkhoff equation and geometric formulation

The differential equations of motion of autonomous Birkhoff system are given as

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} = 0, \tag{1}$$

where $B = B(a)$ is Birkhoffian, $R_\mu = R_\mu(a)$ are the Birkhoff functions, where $\boldsymbol{a} = (a^1, a^2, \ldots, a^{2n})$, and $\boldsymbol{a}^\mu$ are the independent variables.

Let 2n-dimensional manifold M^{2n} denote the configuration space with coordinate $\boldsymbol{a}^\mu$, and R_μ be the C^r function. Define a symplectic structure Ω on M^{2n} as

$$\Omega = \frac{1}{2}\Omega_{\mu\nu}\mathrm{d}a^\mu \wedge \mathrm{d}a^\nu, \quad \Omega_{\mu\nu} = \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right), \tag{2}$$

where Ω is usually called the Birkhoff's tensor. They are called regular, if the functional determinant is not

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10872084 and 10932002), the Research Program of Higher Education of Liaoning Province, China (Grant No. 2008S098), the Program of Supporting Elitists of Higher Education of Liaoning Province, China (Grant No. 2008RC20), the Program of Constructing Liaoning Provincial Key Laboratory, China (Grant No. 2008403009), the Foundation Research Plan of Liaoning educational Bureau, China (Grant No. L2010147), and the Youth fund of Liaoning University, China (Grant No. 2008LDQN04).
†Corresponding author. E-mail: yxguo@lnu.edu.cn
© 2011 Chinese Physical Society and IOP Publishing Ltd

null in the region considered, i.e.,

$$\det\left(\Omega_{\mu\nu}\right) \neq 0, \tag{3}$$

otherwise, being degenerate. Suppose that $B \in C^r\left(M^{2n}\right)$. Define the vector field $\boldsymbol{X}_B$ on M^{2n} as

$$\boldsymbol{X}_B = -\Omega^{-1}\mathrm{d}B, \tag{4}$$

where $\boldsymbol{X}_B$ is called the Birkhoff vector field. $\left(M^{2n}, \Omega, X_B\right)$ is called the autonomous Birkhoff dynamic system with respect to the functions R_μ.

It is easy to know that Birkhoff equations characterize the most general possible, exact symplectic form in local coordinates. In fact, the exact character of the two-form implies the structure on T^*M that is the cotangent space of manifold M^{2n}, which is characterized by Eq. (1). The Ω is the geometrical structure of the autonomous Birkhoff equation.

Consider a Birkhoffian $B : M^{2n} \to R$ and Birkhoff functions $R_\nu : M^{2n} \to R$. The Pfaffian $P\left(a, \dot{a}\right)$ is

$$P\left(a, \dot{a}\right) = R_\nu\left(a\right)\dot{a}^\nu - B\left(a\right). \tag{5}$$

The Pfaff action is defined as

$$A = \int_{t_1}^{t_2} P\left(a, \dot{a}\right)\mathrm{d}t, \tag{6}$$

where $a^\nu = a^\nu\left(\right)$ shows a C^2 curve in configuration space. Next, the variation of A at $a^\nu\left(t\right)$ with fixed end points is calculated. If $\delta a^\nu\left(t_1\right) = \delta a^\nu\left(t_2\right) = 0$ and $\delta A = 0$, the autonomous Birkhoff equation (1) can be obtained.

3. Geometric formulation of discrete autonomous Birkhoff system

Take again a configuration manifold R^{2n} and define the discrete state space also as R^{2n}, then a discrete Pfaffian will be a function $P_\mathrm{d} : R^{2n} \to R$. To relate discrete mechanics to continuous mechanics it is necessary to introduce a time-step $h \in R$, and to take P_d to be dependent on this time-step. For the moment, we will take $P_\mathrm{d} : R^{2n} \times R \to R$, and neglect the h dependence except where it is important.

Construct the increasing sequence of times $\left\{t_k = kh \middle| k = 0, \ldots, N\right\} = \left\{t_k\right\}_{k=0}^N \subset R$ from the time-step h, and define the discrete path space as

$$C_\mathrm{d}\left(R^{2n}\right) = C_\mathrm{d}\left(\left\{t_k\right\}_{k=0}^N, R^{2n}\right) = \left\{a_\mathrm{d}^\nu : \left\{t_k\right\}_{k=0}^N \to R^{2n}\right\},$$

then we will identify a discrete trajectory $a_\mathrm{d}^\nu \in C_\mathrm{d}\left(R^{2n}\right)$ with its image $a_\mathrm{d}^\nu = \left\{a_k^\nu\right\}_{k=0}^N$, where $a_k^\nu = a_\mathrm{d}^\nu\left(t_k\right)$ and define the discrete action map $A_\mathrm{d} : C_\mathrm{d}\left(R^{2n}\right) \to R^{2n}$ as

$$A_\mathrm{d}\left(a_\mathrm{d}^\nu\right) = \sum_{k=0}^{N-1} P_\mathrm{d}\left(a_k^\nu, a_{k+1}^\nu\right). \tag{7}$$

Thus the discrete Pfaff–Birkhoff principle for the autonomous system is given as

$$\begin{aligned} &\delta A_\mathrm{d}\left(a_\mathrm{d}^\nu\right) = \delta\sum_{k=0}^{N-1} P_\mathrm{d}\left(a_k^\nu, a_{k+1}^\nu\right) = 0, \\ &\delta a_0^\nu = \delta a_N^\nu = 0. \end{aligned} \tag{8}$$

When $h \to 0$, it is equivalent to the continuous counterpart.

Theorem 1 In configuration space R^{2n}, for the autonomous Birkhoff system with the discrete Pfaffian $P_\mathrm{d}\left(a_k^\nu, a_{k+1}^\nu\right)$, the discrete equation can be derived from Eq. (8) as follows:

$$D_2P_\mathrm{d}\left(a_{k-1}^\nu, a_k^\nu\right) + D_1P_\mathrm{d}\left(a_k^\nu, a_{k+1}^\nu\right) = 0, \tag{9}$$

where D_iP_d denotes the partial derivative of P_d with respect to the ith argument, where $i = 1, 2$.

There are $2n \times N$ unknown quantities that are the same in number of equations in expression (9), and the equations in expression (9) can be solved.

Proof From expression (8), we have

$$\begin{aligned} &\delta A_\mathrm{d}\left(a_\mathrm{d}^\nu\right) \\ &= \delta\sum_{k=0}^{N-1} P_\mathrm{d}\left(a_k^\nu, a_{k+1}^\nu\right) \\ &= \sum_{k=0}^{N-1}\left[D_1P_\mathrm{d}\left(a_k^\nu, a_{k+1}^\nu\right)\delta a_k^\nu\right. \\ &\quad \left. + D_2P_\mathrm{d}\left(a_k^\nu, a_{k+1}^\nu\right)\delta a_{k+1}^\nu\right] \\ &= \sum_{k=1}^{N-1}\left[D_2P_\mathrm{d}\left(a_{k-1}^\nu, a_k^\nu\right) + D_1P_\mathrm{d}\left(a_k^\nu, a_{k+1}^\nu\right)\right]\delta a_k^\nu \\ &\quad + D_1P_\mathrm{d}\left(a_0^\nu, a_1^\nu\right)\cdot\delta a_0^\nu + D_2P_\mathrm{d}\left(a_{N-1}^\nu, a_N^\nu\right)\cdot\delta a_N^\nu \\ &= 0, \end{aligned} \tag{10}$$

where a discrete integration by parts and the $\delta a_0^\nu = \delta a_N^\nu = 0$ have been used. If the variation of the action is required to be zero for an arbitrary δa_k^ν, then the discrete autonomous Birkhoff equation (9) can be obtained. □

A discrete evolution operator X_Bd plays the same role as a continuous vector $\boldsymbol{X}_B$, and is defined as a map $\boldsymbol{X}_\mathrm{Bd} : R^{2n} \times R^{2n} \to \left(R^{2n} \times R^{2n}\right) \times \left(R^{2n} \times R^{2n}\right)$. The discrete object corresponding to the flow is the

discrete map $\boldsymbol{F}_{\mathrm{P_d}} : R^{2n} \times R^{2n} \to R^{2n} \times R^{2n}$. In coordinates, if the discrete evolution operator maps $\boldsymbol{X}_{\mathrm{Bd}} : \left(a_{k-1}^{\nu}, a_k^{\nu}\right) \mapsto \left(a_{k-1}^{\nu}, a_k^{\nu}, a_k^{\nu}, a_{k+1}^{\nu}\right)$, the discrete map will be $\boldsymbol{F}_{P_{\mathrm{d}}} : \left(a_{k-1}^{\nu}, a_k^{\nu}\right) \mapsto \left(a_k^{\nu}, a_{k+1}^{\nu}\right)$. Now we consider a particular case of a discrete Birkhoff system.

A discrete path $a_{\mathrm{d}}^{\nu} \in C_{\mathrm{d}}\left(R^{2n}\right)$ is said to be a solution of the discrete Birkhoff equation if the first term on the right-hand side of Eq. (10) vanishes for all variations $\delta a_k^{\nu} \in T_{a_k^{\nu}} C_{\mathrm{d}}\left(R^{2n}\right)$. This means that the points $\{a_k^{\nu}\}$ satisfy $F_{P_{\mathrm{d}}}\left(a_{k-1}^{\nu}, a_k^{\nu}\right) = \left(a_k^{\nu}, a_{k+1}^{\nu}\right)$, or, equivalently, they satisfy the discrete Birkhoff equations in expression (9) for all $k = 1, \ldots, N-1$.

Let the second and third terms on the right-hand side of Eq. (10) take the following forms:

$$\Theta_{P_{\mathrm{d}}}^{+}\left(a_k^{\nu}, a_{k+1}^{\nu}\right) = D_2 P_{\mathrm{d}}\left(a_k^{\nu}, a_{k+1}^{\nu}\right) \mathrm{d}a_{k+1}^{\nu}, \tag{11a}$$

$$\Theta_{P_{\mathrm{d}}}^{-}\left(a_k^{\nu}, a_{k+1}^{\nu}\right) = -D_1 P_{\mathrm{d}}\left(a_k^{\nu}, a_{k+1}^{\nu}\right) \mathrm{d}a_k^{\nu}. \tag{11b}$$

The one-forms $\Theta_{P_{\mathrm{d}}}^{+}$ and $\Theta_{P_{\mathrm{d}}}^{-}$ are called the discrete Birkhoff one–forms.

Unlike the continuous case, there are two one-forms that arise from the boundary terms in the discrete case. By calculating, we have

$$\mathrm{d}\Theta_{P_{\mathrm{d}}}^{+} = \mathrm{d}\Theta_{P_{\mathrm{d}}}^{-}. \tag{12}$$

This will be reflected below in the fact that there is only single discrete two-form, which is the same as the continuous case and it is important for symplecticity.

Theorem 2 If the discrete map $F_{P_{\mathrm{d}}}$ satisfies the following expression:

$$F_{P_{\mathrm{d}}}^{*}\left(\Omega_{P_{\mathrm{d}}}\right) = \Omega_{P_{\mathrm{d}}},$$

where $F_{P_{\mathrm{d}}}^{*}$ is the operation of pull back and $\Omega_{P_{\mathrm{d}}} = \mathrm{d}\Theta_{P_{\mathrm{d}}}^{+} = \mathrm{d}\Theta_{P_{\mathrm{d}}}^{-}$ is the discrete Birkhoff symplectic form, with coordinate expression

$$\Omega_{P_{\mathrm{d}}} = \frac{\partial^2 P_{\mathrm{d}}}{\partial a_0^{\mu} \partial a_1^{\nu}} \mathrm{d}a_0^{\mu} \wedge \mathrm{d}a_1^{\nu},$$

the discrete Birkhoff map $F_{P_{\mathrm{d}}}$ is discretely symplectic.

Proof Define the discrete solution space $C_{P_{\mathrm{d}}}\left(R^{2n}\right) \subset C_{\mathrm{d}}\left(R^{2n}\right)$ as a set of solutions of the discrete Birkhoff equation since element $a_{\mathrm{d}}^{\nu} \in C_{P_{\mathrm{d}}}\left(R^{2n}\right)$ is formed by the iteration of map $F_{P_{\mathrm{d}}}$, it can be uniquely determined by the initial condition $(a_0^{\nu}, a_1^{\nu}) \in R^{2n} \times R^{2n}$.

Define $\nu_{\mathrm{d}} = (a_0^{\nu}, a_1^{\nu}) \in R^{2n} \times R^{2n}$ and $\omega_{\nu_{\mathrm{d}}} \in T_{\nu_{\mathrm{d}}}\left(R^{2n} \times R^{2n}\right)$ and use the discrete autonomous Birkhoff equations (9), (11a) and (11b), changing the variation of Eq. (10) into differentiation, then Eq. (10) will be reduced into

$$\begin{aligned}
&\mathrm{d}A_{\mathrm{d}} \cdot \delta a_{\mathrm{d}}^{\nu} \\
&= \Theta_{P_{\mathrm{d}}}^{+}\left(F_{P_{\mathrm{d}}}^{N-1}\left(\nu_{\mathrm{d}}\right)\right)\left(\left(F_{P_{\mathrm{d}}}^{N-1}\right)_{*}\left(\omega_{\nu_{\mathrm{d}}}\right)\right) \\
&\quad - \Theta_{P_{\mathrm{d}}}^{-}\left(\nu_{\mathrm{d}}\right)\left(\omega_{\nu_{\mathrm{d}}}\right) \\
&= \left(\left(F_{P_{\mathrm{d}}}^{N-1}\right)^{*}\left(\Theta_{P_{\mathrm{d}}}^{+}\right)\right)\left(\nu_{\mathrm{d}}\right)\left(\omega_{\nu_{\mathrm{d}}}\right) - \Theta_{P_{\mathrm{d}}}^{-}\left(\nu_{\mathrm{d}}\right)\left(\omega_{\nu_{\mathrm{d}}}\right),
\end{aligned}$$

where $\left(F_{P_{\mathrm{d}}}\right)_{*} = \left(F_{P_{\mathrm{d}}}^{-1}\right)^{*}$. Due to the fact that $\mathrm{d}^2 A_{\mathrm{d}} = 0$, we obtain

$$\left(F_{P_{\mathrm{d}}}^{N-1}\right)^{*}\left(\Omega_{P_{\mathrm{d}}}\right) = \Omega_{P_{\mathrm{d}}},$$

For any subinterval of $0, \ldots, N$, this argument also holds and the statement is true for any number of steps of F_P. For a single step, there is $F_{P_{\mathrm{d}}}^{*}\left(\Omega_{P_{\mathrm{d}}}\right) = \Omega_{P_{\mathrm{d}}}$. So the discrete Birkhoff map F_P is discretely symplectic, just as the continuous case. □

4. Variational integrators

Now considering a discrete Birkhoff system as an approximation to a given continuous system, the discrete system is an integrator for the continuous case. As we have seen, discrete Birkhoff map preserves the symplectic structure, regarded as an integrator, they are necessarily symplectic. Then how can we implement the variational integrator for the discrete autonomous Birkhoff system?

Now, with a trajectory $a_0^{\nu}, a_1^{\nu}, a_2^{\nu}, \ldots, a_{N-1}^{\nu}, a_N^{\nu}$ given, we calculate the map $F_{P_{\mathrm{d}}} : R^{2n} \times R^{2n} \times R \to R^{2n} \times R^{2n}$ according to

$$D_2 P_{\mathrm{d}}\left(a_{k-1}^{\nu}, a_k^{\nu}, h\right) = -D_1 P_{\mathrm{d}}\left(a_k^{\nu}, a_{k+1}^{\nu}, h\right). \tag{13}$$

Taking $b_k^{\nu} = D_2 P_{\mathrm{d}}\left(a_{k-1}^{\nu}, a_k^{\nu}, h\right)$ for each k, this equation is simply

$$b_k^{\nu} = -D_1 P_{\mathrm{d}}\left(a_k^{\nu}, a_{k+1}^{\nu}, h\right), \tag{14}$$

which together with the next updated

$$b_{k+1}^{\nu} = D_2 P_{\mathrm{d}}\left(a_k^{\nu}, a_{k+1}^{\nu}, h\right). \tag{15}$$

By solving the implicit equation (14) for a_{k+1}^{ν} and then evaluating the explicit equation (15) to give b_{k+1}^{ν}, we update $\left(a_k^{\nu}, b_k^{\nu}\right) \mapsto \left(a_{k+1}^{\nu}, b_{k+1}^{\nu}\right)$.

To solve the implicit equation (14), an iterative technique such as Newton's method must be adapted. But for the explicit equation (15), there are many standard implementation techniques available, for example, an Euler rule or other integrator type, and so

on. In the present paper, the Störmer–Verlet scheme is used to solve Eqs. (14) and (15).

Use the Störmer–Verlet rule,[22−24] define the discrete Pfaffian as

$$\begin{aligned} & P_{\rm d}\left(a_k^\nu, a_{k+1}^\nu, h\right) \\ = & \, hP\left(a_k^\nu, (a_{k+1}^\nu - a_k^\nu)/h\right)/2 \\ & + hP\left(a_{k+1}^\nu, (a_{k+1}^\nu - a_k^\nu)/h\right)/2, \end{aligned} \tag{16}$$

and compute Eqs. (14) and (15), then we will obtain b_k^ν, b_{k+1}^ν.

5. Example

Given that a mathematical pendulum (mass $m = 1$, massless rod of length $l = 1$, and gravitational acceleration $g = 1$, q is the position coordinate and p is the momentum) is a system having the Pfaffian,

$$P = a^2\dot{a}^1 - \left(a^2\right)^2/2 + \cos\left(a^1\right), \tag{17}$$

where we take $a^1 = q, a^2 = p$, $B = (1/2)p^2 - \cos q$, $R_1 = a^2$ and $R_2 = 0$. The total energy of the system is then given as

$$E = (a^2)^2/2 - \cos(a^1),$$

which is conserved during motion.

By using Eq. (16), the discrete Pfaffian can be obtained as

$$\begin{aligned} P_{\rm d} = & \, h[a_k^2(a_{k+1}^1 - a_k^1)/h - \left(a_k^2\right)^2/2 + \cos\left(a_k^1\right)]/2 \\ & + h[a_{k+1}^2(a_{k+1}^1 - a_k^1)/h - \left(a_{k+1}^2\right)^2/2 \\ & + \cos\left(a_{k+1}^1\right)]/2. \end{aligned} \tag{18}$$

From Eqs. (14) and (15) it follows that

$$b_k^1 = (a_k^2 + a_{k+1}^2)/2 + h\sin(a_k^1)/2, \tag{19}$$

$$b_{k+1}^1 = (a_k^2 + a_{k+1}^2)/2 - h\sin(a_{k+1}^1)/2, \tag{20}$$

$$b_k^2 = -(a_{k+1}^1 - a_k^1)/2 + ha_k^2/2, \tag{21}$$

$$b_{k+1}^2 = (a_{k+1}^1 - a_k^1)/2 - ha_{k+1}^2/2. \tag{22}$$

Now we subtract Eq. (19) from Eq. (20), add Eqs. (21) and (22) and solve Eqs. (19) and (21) for a_{k+1}^ν, and we obtain

$$\begin{aligned} & a_{k+1}^1 = a_k^1 - 2b_k^2 + ha_k^2, \\ & b_{k+1}^1 = b_k^1 - h[\sin(a_{k+1}^1) + \sin(a_k^1)]/2; \\ & a_{k+1}^2 = -a_k^2 + 2b_k^1 - h\sin(a_k^1), \\ & b_{k+1}^2 = -b_k^2 + h(a_k^2 - a_{k+1}^2)/2. \end{aligned} \tag{23}$$

From the results of the above example it can follow that the discrete variational method is as effective as symplectic scheme. Figure 1 shows the preservation of energy as a key point of comparison among the calculation implementations of three methods. The discrete variational method and the symplectic scheme can present comparatively stable solution but Runge–Kutta method is a dissipative scheme. Figure 2 shows that the computational result of variational Verlet is still periodically orbital, but that from the Runge–Kutta method is unstable.

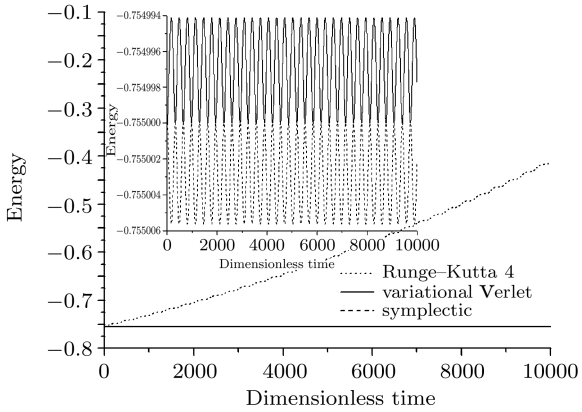


Fig. 1. Energies of mathematical pendulum computed with variational Verlet (solid line), symplectic (dash line) and Runge–Kutta (dot line) methods.

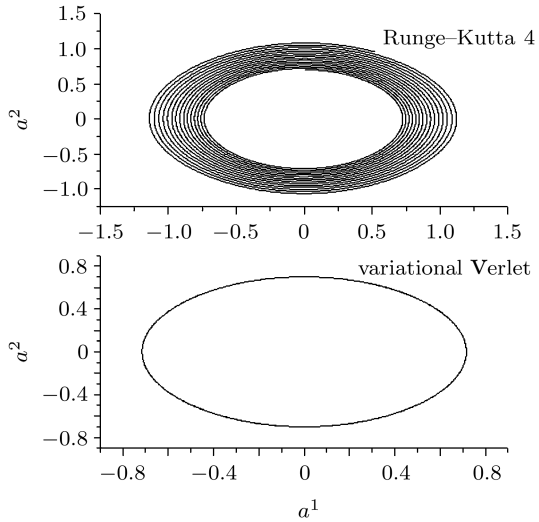


Fig. 2. Trajectories of phase space computed with variational Verlet (lower) and Runge–Kutta (upper) methods.

6. Conclusion

In this paper we have performed the numerical computation of autonomous Birkhoff system by using the discrete variational method and constructed discrete variational integrators. The method is then applied to the mathematical pendulum. A similar computational result to that from the symplectic scheme is obtained. In fact, the discrete map is in the form of discrete Birkhoffian symplectic scheme. The discrete Birkhoffian equation shares similarities to the continuous equations of motion and preserves a symplectic form and an invariant quantity.

References

[1] Birkhoff G D 1927 *Dynamical Systems* (Providence RI: AMS College Publications)

[2] Santilli R M 1978 *Foundations of Theoretical Mechanics I: The Inverse Problem in Newtonian Mechanics* (New York: Springer)

[3] Santilli R M 1983 *Foundations of Theoretical Mechanics II: Birkhoffian Generalization of Hamiltonian Mechanics* (New York: Springer)

[4] Badía-Majós A, Carińena J F and López C 2006 *J. Phys.* A **39** 14699

[5] Delia Ionescu 2006 *Journal of Geometry and Physics* **56** 2545

[6] Mei F X, Shi R C, Zhang Y F and Wu H B 1996 *Dynamics of Birkhoff Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[7] Chen X W and Mei F X 2000 *Mech. Res. Commun.* **27** 365

[8] Wu H B and Mei F X 1995 *Chin. Sin. Bull.* **40** 885

[9] Mei F X and Levesque B 1995 *Trans. CSME.* **19** 59

[10] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)

[11] Guo Y X, Luo S K, Shang M and Mei F X 2001 *Rep. Math. Phys.* **47** 313

[12] Fu J L and Chen L Q 2004 *Phys. Lett.* A **323** 95

[13] Luo S K 2002 *Chin. Phys. Lett.* **19** 449

[14] Zhang H B, Chen L Q, Gu S L and Liu C C 2007 *Chin. Phys.* **16** 582

[15] Hu C L and Xie J F 2008 *Chin. Phys.* B **17** 1153

[16] Ding N, Fang J H and Chen X X 2008 *Chin. Phys.* B **17** 1967

[17] Zhang Y 2008 *Chin. Phys.* B **17** 4365

[18] Wang P, Fang J H and Wang X M 2009 *Chin. Phys.* B **18** 1312

[19] Ding G T 2010 *Acta. Phys. Sin.* **59** 3643 (in Chinese)

[20] Su H L and Qin M Z 2003 arXiv: Math-ph/0301001, v1 3 Jan.

[21] Zhang X W, Wu J K, Zhu H P and Huang K F 2002 *Applied Mathematics and Mechanics* **23** 1029

[22] Marsden J E and West M 2001 *Acta. Numerica.* 357

[23] Hairer E, Lubich C and Wanner G 2002 *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations* vol 31, (Berlin: Springer)

[24] Cortes J 2002 *Geometric, Control and Numerical Aspects of Nonholonomic Systems* (Berlin: Springer)