Global exponential stability of reaction–diffusion neural networks with discrete and distributed time-varying delays*

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This paper investigates the global exponential stability of reaction–diffusion neural networks with discrete and distributed time-varying delays. By constructing a more general type of Lyapunov–Krasovskii functional combined with a free-weighting matrix approach and analysis techniques, delay-dependent exponential stability criteria are derived in the form of linear matrix inequalities. The obtained results are dependent on the size of the time-varying delays and the measure of the space, which are usually less conservative than delay-independent and space-independent ones. These results are easy to check, and improve upon the existing stability results. Some remarks are given to show the advantages of the obtained results over the previous results. A numerical example has been presented to show the usefulness of the derived linear matrix inequality (LMI)-based stability conditions.

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1. Introduction

In recent years, neural networks (NNs) have been widely studied (see Refs. [1]–[11]) due to their immense potential and prospective application in a variety of areas, such as signal processing, pattern recognition, static image processing, associative memory, and combinatorial optimisation. It is well known that these applications heavily depend on the dynamic behaviour of NNs. Since exponential stability is one of the most important issues related to such dynamic behaviour, particularly when the exponential convergence rate is used to determine the speed of neural computations, considerable attention has been devoted to research in this area. Therefore, a great number of results on the exponential stability of NNs have been proposed in the literatures (see, e.g., Refs.[1]–[6] and the references therein). However, strictly speaking, diffusion effects cannot be avoided in NNs when electrons are moving in asymmetric electromagnetic fields. Therefore, we should consider that the activations vary in space as well as in time.

Because of the finite speed of information processing, time delays are common in practical control systems, especially in many biological and artificial NNs. Time delays may decrease the quality of the system and even lead to oscillation, divergence and instability.[12] In the case of distributed parameter systems, even arbitrarily small delays in feedback may destabilize the system (see e.g. Refs. [13]–[15] and the references therein). The stability issue of systems with delay is, therefore, of theoretical and practical value. Recently, many authors have considered the stability of NNs in reaction–diffusion terms, which are expressed by partial differential equations. Also, the stability of reaction–diffusion neural networks (RDNNs) with a time delay has been extensively studied and various stability conditions have been obtained for these NN models.[16−25] For instance, in Ref. [24], by constructing a Lyapunov–Krasovskii functional and using linear matrix inequalitys (LMIs), the sufficient conditions are presented for a class of reaction–diffusion-uncertain NNs with time-varying delays, where the discrete delay $\tau(t)$ must be differentiable and $\dot{\tau}(t) \leq \eta < 1$. Qiu and Cao[25] proposed an exponential stability criterion for a class of delayed NNs by employing Lyapunov functionals, where delay τ is a non-negative constant. Clearly,

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such constraints on the delay term $\tau(t)$ were relatively strong. However, unlike the existing works, delay-dependent exponential stability criteria for reaction–diffusion with discrete and distributed time-varying delays have not been fully investigated, and remain important and challenging.

In this paper, by constructing an improved Lyapunov–Krasovskii functional (LKF) combined with free-weighting matrix approach and analysis techniques, we obtain delay-dependent and space-dependent exponential stability criteria for RDNNs with discrete and distributed time-varying delays and formulate them in the form of LMIs. Although the idea of a free-weighting matrix has appeared in Ref. [26], to the best of our knowledge, no effort has been made towards applying the idea for RDNNs with discrete and distributed time-varying delays. According to the novel LKF, a less conservative criterion is obtained in the form of LMIs. A theorem and corollary were obtained in which the boundedness, monotonicity and differentiability conditions on the activation functions are not required and have no restriction on the magnitude of the change rate of time-varying delays. Undoubtedly, the newly proposed stability criteria will be a wider adaptive range since the activation functions and time-varying delays are allowed to be more general.

2. Problem formulation

In this paper, the RDNN with discrete and distributed time-varying delays is described as follows:

$$\begin{aligned}
\frac{\partial u_i}{\partial t} &= \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}\left(t,x,u\right)\frac{\partial u_i}{\partial x_k}\right) - a_i u_i + \sum_{j=1}^{n}\left(b_{ij} f_j\left(u_j\left(t,x\right)\right)\right) + \sum_{j=1}^{n}\left(c_{ij} f_j\left(u_j\left(t-d\left(t\right),x\right)\right)\right) \\
&\quad + \sum_{j=1}^{n}\left(e_{ij}\int_{t-\tau(t)}^{t} f_j\left(u_j\left(s,x\right)\right)\mathrm{d}s\right) + J_i\left(t\right), \quad t \geq t_0 \geq 0, \quad x \in \Omega, \\
\frac{\partial u_i}{\partial \bar{v}} &:= \mathrm{col}\left(\frac{\partial u_i}{\partial x_1},\ldots,\frac{\partial u_i}{\partial x_m}\right) = 0, \quad t \geq t_0 \geq 0, \quad x \in \partial\Omega, \\
u_i\left(t_0+s,x\right) &= \phi_i\left(s,x\right), \quad \frac{\partial u_i\left(t_0+s,x\right)}{\partial t} = \frac{\partial \phi_i\left(s,x\right)}{\partial t}, \quad -\tilde{\tau} \leq s \leq 0, \quad i = 1,2,\ldots,n.
\end{aligned} \tag{1}$$

where $x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}} \in \Omega$, Ω is a compact set with smooth boundary $\partial\Omega$ and mes$\Omega > 0$ in space $\mathbb{R}^m$; $u_i(t,x)$ denotes the state of the ith neuron at time t and in space x; f_j is the activation functions of the jth neuron; the scalar a_i is the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external input at time t and in space x; b_{ij}, c_{ij} and e_{ij}, $i,j = 1,2,\ldots,n$ are known constants denoting the strength of the ith neurons on the jth neurons; smooth functions $D_{ik} = D_{ik}(t,x,u) \geq 0$ stand for transmission diffusion operator along the ith neurons; J_i denotes the ith component of an external input source introduced from outside the network into the cell i at time t and in space x; $d(t)$ and $\tau(t)$ denote the discrete time-varying delay and the distributed time-varying delay, respectively, and are assumed to satisfy $0 \leq \tau(t) \leq \tau_0$, $0 \leq d(t) \leq d_0$, $\max\{\tau_0, d_0\} = \tilde{\tau}$, $\dot{\tau}(t) \leq \mu$, $\dot{d}(t) \leq d$, $\phi_i(s,x)$ is the bounded and first order continuous differential; $\bar{v}$ is the outer normal vector of $\partial\Omega$.

Also, the activation functions satisfy the Lipschitz condition, i.e. there exist positive diagonal matrices $L = \mathrm{diag}(L_1, L_2, \ldots, L_n) > 0$, such that

$$\begin{aligned}
&\left|f_j\left(\xi_1\right) - f_j\left(\xi_2\right)\right| \leq L_j\left|\xi_1 - \xi_2\right|, \\
&\forall \xi_1, \quad \xi_2 \in \mathbb{R}, \quad j = 1,2,\ldots,n.
\end{aligned} \tag{2}$$

Remark 1 The above assumption on discrete and distributed time-varying delays have no restriction on the magnitude of the derivative of time-varying delays, the structure of time delays are more general than existing ones.[17–24] By condition (2), it can be seen that there exists an equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^{\mathrm{T}}$ for system (1).[27,28]

Definition 1 The equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^{\mathrm{T}}$ of system (1) is said to be globally exponentially stable, if there exist constants $\alpha > 0$ and $\beta \geq 1$ such that

$$\|u(t,x)-u^*\|_2 \le \beta e^{-2\alpha(t-t_0)} \sup_{s\in[-\tilde{\tau},0]} \left(\|\phi(s,x)-u^*\|_2, \left\| \frac{\partial(\phi(s,x)-u^*)}{\partial s} \right\|_2 \right), \tag{3}$$

for all $t \ge t_0 \ge 0$.

Now, we let $z_i(t,x) = u_i(t,x) - u_i^*, i = 1,2,\ldots,n$. Therefore, system (1) can be transformed into the following form:

$$\begin{aligned} \frac{\partial z(t,x)}{\partial t} = & \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_k^* \frac{\partial z(t,x)}{\partial x_k} \right) \\ & - Az(t,x) + B\bar{f}(z(t,x)) \\ & + C\bar{f}(z(t-d(t),x)) \\ & + E\int_{t-\tau(t)}^{t} \bar{f}(z(s,x))\mathrm{d}s, \end{aligned} \tag{4}$$

$$\left.\frac{\partial z}{\partial \bar{v}}\right|_{\partial\Omega} = 0, \quad z(t_0+s,x) = \psi(s,x),$$
$$\frac{\partial z(t_0+s,x)}{\partial t} = \frac{\partial \psi(s,x)}{\partial t}, -\tilde{\tau} \le s \le 0, x \in \Omega,$$

where $z = (z_1,\ldots,z_n)^{\mathrm{T}}, D_k^* = \mathrm{diag}(D_{1k},\ldots,D_{nk})^{\mathrm{T}}$, $A = \mathrm{diag}(a_1,\ldots,a_n)$, $B = (b_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, $E = (e_{ij})_{n\times n}$, $\psi = (\psi_1,\ldots,\psi_n)^{\mathrm{T}}, \bar{f}(z) = \left(\bar{f}_1(z_1),\ldots,\bar{f}_n(z_n)\right)^{\mathrm{T}}$, $\psi_j = \phi_j - u_j^*$, $\bar{f}_j(z_j) = f_j\left(z_j+u_j^*\right) - f_j\left(u_j^*\right)$ with $\bar{f}_j(0) = 0$ and $\bar{f}_j(\cdot)$ satisfy condition (2).

Lemma 1[29] (Jensen inequality) (i) For any constant matrix $\boldsymbol{\Xi} \in \mathbb{R}^{n\times n}$, $\boldsymbol{\Xi} > 0$ scalar function $d(t) : 0 < d(t) < d$, and vector function $\omega : [0,d] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then

$$\left(\int_0^{d(t)} \omega(s)\,\mathrm{d}s\right)^{\mathrm{T}} \boldsymbol{\Xi} \left(\int_0^{d(t)} \omega(s)\,\mathrm{d}s\right) \le d(t)\int_0^{d(t)} \omega(s)^{\mathrm{T}} \boldsymbol{\Xi}\omega(s)\,\mathrm{d}s. \tag{5}$$

(ii) For any constant matrices $\boldsymbol{\Xi} \in \mathbb{R}^{n\times n}$, $\boldsymbol{\Xi} > 0$, $\Omega \subset \mathbb{R}^n$, $\mathrm{mes}\Omega > 0$, if $\omega : \Omega \to \mathbb{R}^n$ is vector function such that the integration is well defined, then

$$\left(\int_\Omega \omega(s)\,\mathrm{d}s\right)^{\mathrm{T}} \boldsymbol{\Xi} \left(\int_\Omega \omega(s)\,\mathrm{d}s\right) \le |\Omega|\int_\Omega \omega(s)^{\mathrm{T}} \boldsymbol{\Xi}\omega(s)\,\mathrm{d}s. \tag{6}$$

3. Main results

Theorem 1 Given $\alpha > 0$, if there exist positive definite matrix $\boldsymbol{P} > 0$, $Q > 0$, $G > 0$, $H > 0$,

$$\boldsymbol{\Xi}_0 = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ * & X_{22} & X_{23} & X_{24} \\ * & * & X_{33} & X_{34} \\ * & * & * & X_{44} \end{bmatrix} > 0$$

and positive definite diagonal matrix $\boldsymbol{R} > 0$, compatibly dimensional arbitrary matrices N_1, N_2, such that the following linear matrix inequalities hold:

$$\boldsymbol{\Xi}_1 = \begin{bmatrix} \tilde{\tau}X_{11}+\alpha_{11} & \tilde{\tau}X_{12}+\alpha_{12} & \tilde{\tau}X_{13}+\alpha_{13} & \tilde{\tau}X_{14}+\alpha_{14} \\ * & \tilde{\tau}X_{22}+\alpha_{22} & \tilde{\tau}X_{23} & \tilde{\tau}X_{24} \\ * & * & \tilde{\tau}X_{33}+\alpha_{33} & \tilde{\tau}X_{34}+\alpha_{34} \\ * & * & * & \tilde{\tau}X_{44}+\alpha_{44} \end{bmatrix} < 0, \tag{7}$$

$$\boldsymbol{\Xi}_2 = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & \beta_{15} \\ * & X_{22} & X_{23} & X_{24} & \beta_{25} \\ * & * & X_{33} & X_{34} & 0 \\ * & * & * & X_{44} & 0 \\ * & * & * & * & \beta_{55} \end{bmatrix} > 0, \tag{8}$$

then equilibrium point $\boldsymbol{u}^*$ of system (1) is unique and globally exponentially stable. In which

$$\begin{aligned} \alpha_{11} = & -2\boldsymbol{PA} + 2\alpha\boldsymbol{P} + 2\boldsymbol{PBL} + \boldsymbol{Q} + 2\boldsymbol{N}_1 + \boldsymbol{LGL} + \tilde{\tau}\boldsymbol{LHL} + 4\tilde{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{RA} \\ & - 8\tilde{\tau}\boldsymbol{ARB}^{+}L + 4\tilde{\tau}|\Omega|\boldsymbol{LB}^{+\mathrm{T}}\boldsymbol{RB}^{+}L, \\ \alpha_{12} = & -\boldsymbol{N}_1 + \boldsymbol{N}_2^{\mathrm{T}}, \quad \alpha_{13} = \boldsymbol{PC} - 4\tilde{\tau}\boldsymbol{ARC}^{+} + 4\tilde{\tau}|\Omega|\boldsymbol{LB}^{+\mathrm{T}}\boldsymbol{RC}, \end{aligned}$$

$$\alpha_{14} = \boldsymbol{PE} - 4\tilde{\tau}\boldsymbol{ARE}^{+} + 4\tilde{\tau}\left|\Omega\right|\boldsymbol{LB}^{+\mathrm{T}}\boldsymbol{RE}^{+}, \quad \alpha_{22} = -\left(\mathrm{e}^{-2\alpha\tau_0} - \mu\right)\boldsymbol{Q} - 2\boldsymbol{N}_2,$$
$$\alpha_{33} = 4\tilde{\tau}\left|\Omega\right|\boldsymbol{C}^{+\mathrm{T}}\boldsymbol{RC}^{+} - \left(\mathrm{e}^{-2\alpha d_0} - d\right)\boldsymbol{G}, \quad \alpha_{34} = 4\tilde{\tau}\left|\Omega\right|\boldsymbol{C}^{+\mathrm{T}}\boldsymbol{RE}^{+}, \quad \alpha_{44} = 4\tilde{\tau}\left|\Omega\right|\boldsymbol{E}^{+\mathrm{T}}\boldsymbol{RE}^{+} - \mathrm{e}^{-2\alpha\tilde{\tau}}\tilde{\tau}^{-1}\boldsymbol{H},$$
$$\beta_{15} = N_1, \beta_{25} = N_2, \quad \beta_{55} = \mathrm{e}^{-2\alpha\tilde{\tau}}\boldsymbol{R}, \quad \boldsymbol{R} = \mathrm{diag}\left(r_1, r_2, \ldots, r_n\right).$$

Here, an arbitrarily real matrix $\boldsymbol{E} = (e_{ij})_{n\times n}$, $\boldsymbol{E}^{+} = \left(e_{ij}^{+}\right)_{n\times n}$, where $e_{ij}^{+} = \max\left(e_{ij}, 0\right)$.

Proof Choose an LKF candidate as

$$\begin{aligned} V\left(t\right) = & \int_{\Omega} z\left(t,x\right)^{\mathrm{T}} P z\left(t,x\right)\mathrm{d}x + \int_{\Omega}\int_{t-\tau(t)}^{t} \mathrm{e}^{2\alpha(s-t)} z\left(s,x\right)^{\mathrm{T}} Q z\left(s,x\right)\mathrm{d}s\mathrm{d}x \\ & + \int_{\Omega}\int_{-\tilde{\tau}}^{0}\int_{t+\theta}^{t} \mathrm{e}^{2\alpha(s-t)}\left(\frac{\partial z\left(s,x\right)}{\partial s}\right)^{\mathrm{T}} R\frac{\partial z\left(s,x\right)}{\partial s}\mathrm{d}s\mathrm{d}\theta\mathrm{d}x \\ & + \int_{\Omega}\int_{t-d(t)}^{t} \mathrm{e}^{2\alpha(s-t)}\bar{f}\left(z\left(s,x\right)\right)^{\mathrm{T}} G\bar{f}\left(z\left(s,x\right)\right)\mathrm{d}s\mathrm{d}x \\ & + \int_{\Omega}\int_{-\tilde{\tau}}^{0}\int_{t+\theta}^{t} \mathrm{e}^{2\alpha(s-t)}\bar{f}\left(s,x\right)^{\mathrm{T}} H\bar{f}\left(z(s,x)\right)\mathrm{d}s\mathrm{d}\theta\mathrm{d}x. \end{aligned} \tag{9}$$

Calculating the derivative of $V\left(t\right)$ along the system (4), we find

$$\begin{aligned} \dot{V}\left(t\right) + 2\alpha V\left(t\right) \leq & \int_{\Omega} \boldsymbol{z}\left(t,x\right)^{\mathrm{T}}\left(-2\boldsymbol{PA} + 2\alpha\boldsymbol{P} + 2\boldsymbol{PBL} + \boldsymbol{Q} + 2\boldsymbol{N}_1 + \boldsymbol{LGL} + \tilde{\tau}\boldsymbol{LHL}\right)\boldsymbol{z}\left(t,x\right)\mathrm{d}x \\ & + 2\int_{\Omega} \boldsymbol{z}\left(t,x\right)^{\mathrm{T}}\boldsymbol{PC}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(t-d\left(t\right),x\right)\right)\mathrm{d}x + 2\int_{\Omega} \boldsymbol{z}\left(t,x\right)^{\mathrm{T}}\boldsymbol{PE}\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s\mathrm{d}x \\ & - \left(\mathrm{e}^{-2\alpha\tau_0} - \mu\right)\int_{\Omega} \boldsymbol{z}\left(t-\tau\left(t\right),x\right)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{z}\left(t-\tau\left(t\right),x\right)\mathrm{d}x \\ & + \int_{\Omega}\tilde{\tau}\Big[\boldsymbol{z}\left(t,x\right)^{\mathrm{T}}\left(4A^{\mathrm{T}}\boldsymbol{RA} - 8\boldsymbol{ARB}^{+}L + 4\left|\Omega\right|\boldsymbol{LB}^{+\mathrm{T}}\boldsymbol{RB}^{+}L\right)\boldsymbol{z}\left(t,x\right) \\ & + \boldsymbol{z}\left(t,x\right)^{\mathrm{T}}\left(-8\boldsymbol{ARC}^{+} + 8\left|\Omega\right|\boldsymbol{LB}^{+\mathrm{T}}\boldsymbol{RC}\right)\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(t-d\left(t\right),x\right)\right) \\ & + \boldsymbol{z}\left(t,x\right)^{\mathrm{T}}\left(-8\boldsymbol{ARE}^{+} + 8\left|\Omega\right|\boldsymbol{LB}^{+\mathrm{T}}\boldsymbol{RE}^{+}\right)\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s \\ & + \bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(t-d\left(t\right),x\right)\right)^{\mathrm{T}}\left(4\left|\Omega\right|\boldsymbol{C}^{+\mathrm{T}}\boldsymbol{RC}^{+}\right)\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(t-d\left(t\right),x\right)\right) \\ & + \bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(t-d\left(t\right),x\right)\right)^{\mathrm{T}}\left(8\left|\Omega\right|\boldsymbol{C}^{+\mathrm{T}}\boldsymbol{RE}^{+}\right)\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s \\ & + \left(\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s\right)^{\mathrm{T}}\left(4\left|\Omega\right|\boldsymbol{E}^{+\mathrm{T}}\boldsymbol{RE}^{+}\right)\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s\Big]\mathrm{d}x \\ & - \mathrm{e}^{-2\alpha\tilde{\tau}}\int_{\Omega}\int_{t-\tau(t)}^{t}\left(\frac{\partial\boldsymbol{z}\left(s,x\right)}{\partial s}\right)^{\mathrm{T}}\boldsymbol{R}\frac{\partial\boldsymbol{z}\left(s,x\right)}{\partial s}\mathrm{d}s\mathrm{d}x \\ & - \left(\mathrm{e}^{-2\alpha d_0} - d\right)\int_{\Omega}\bar{\boldsymbol{f}}\left(t-d\left(t\right),x\right)^{\mathrm{T}}\boldsymbol{G}\bar{\boldsymbol{f}}\left(t-d\left(t\right),x\right)\mathrm{d}x \\ & - \mathrm{e}^{-2\alpha\tilde{\tau}}\tilde{\tau}^{-1}\int_{\Omega}\left(\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s\right)^{\mathrm{T}}\boldsymbol{H}\left(\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s\right)\mathrm{d}x \\ & + \int_{\Omega} 2\left[\boldsymbol{z}\left(t,x\right)^{\mathrm{T}}\boldsymbol{N}_1 + \boldsymbol{z}\left(t-\tau\left(t\right),x\right)^{\mathrm{T}}\boldsymbol{N}_2\right] \\ & \times\left(\boldsymbol{z}\left(t,x\right) - \boldsymbol{z}\left(t-\tau\left(t\right),x\right) - \int_{t-\tau(t)}^{t}\frac{\partial\boldsymbol{z}\left(s,x\right)}{\partial s}\mathrm{d}s\right)\mathrm{d}x \\ & + \int_{\Omega}\left(\tilde{\tau}\varsigma_1^{\mathrm{T}}\boldsymbol{\Xi}_0\varsigma_1 - \int_{t-\tau(t)}^{t}\varsigma_1^{\mathrm{T}}\boldsymbol{\Xi}_0\varsigma_1\mathrm{d}s\right)\mathrm{d}x = \int_{\Omega}\left(\varsigma_1^{\mathrm{T}}\boldsymbol{\Xi}_1\varsigma_1 - \int_{t-\tau(t)}^{t}\varsigma_2^{\mathrm{T}}\boldsymbol{\Xi}_2\varsigma_2\mathrm{d}s\right)\mathrm{d}x, \end{aligned} \tag{10}$$

where

$$\varsigma_1 = \left(\begin{array}{cccc} \boldsymbol{z}\left(t,x\right)^{\mathrm{T}} & \boldsymbol{z}\left(t-\tau\left(t\right),x\right)^{\mathrm{T}} & \bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(t-d\left(t\right),x\right)\right)^{\mathrm{T}} & \left(\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s\right)^{\mathrm{T}} \end{array}\right)^{\mathrm{T}},$$

$$\varsigma_2 = \left(\boldsymbol{z}(t,x)^{\mathrm{T}} \ \boldsymbol{z}(t-\tau(t),x)^{\mathrm{T}} \ \bar{\boldsymbol{f}}(\boldsymbol{z}(t-d(t),x))^{\mathrm{T}} \ \left(\int_{t-\tau(t)}^{t} \bar{\boldsymbol{f}}(\boldsymbol{z}(s,x))\mathrm{d}s\right)^{\mathrm{T}} \ \left(\frac{\partial \boldsymbol{z}(s,x)}{\partial s}\right)^{\mathrm{T}} \right)^{\mathrm{T}}.$$

Thus, it follows from inequalities (7) and (8), we can conclude

$$\dot{V}(t) + 2\alpha V(t) < 0, t \geq t_0 \geq 0.$$

Therefore, we can derive

$$\|\boldsymbol{z}(t,x)\|_2 \leq \beta \mathrm{e}^{-2\alpha(t-t_0)} \sup_{s\in[-\tilde{\tau},0]} \left(\|\boldsymbol{\phi}(s,x) - \boldsymbol{u}^*\|_2, \left\| \frac{\partial(\boldsymbol{\phi}(s,x) - \boldsymbol{u}^*)}{\partial s} \right\|_2 \right), \quad t \geq t_0 \geq 0, \tag{11}$$

where $\beta \geq 1$ is a constant. This means that the unique equilibrium point $\boldsymbol{u}^*$ of system (1) is globally exponentially stable. Thus, the proof is completed.

Remark 2 In Theorem 1, above , the new LKF is proposed to derive new delay-dependent and space-dependent exponential stability criterion for system (1). The idea of a free-weighting matrix is introduced, the method mentioned above is not considered in other literatures. Furthermore, the method may be extended for more complex RDNNs. The well-known Leibniz–Newton formula, classical inequalities and the techniques in Refs. [25] and [34] are utilized in the proof. Our results expanded the model in Refs. [16] and [17].

Remark 3 The assumptions on reaction–diffusion terms in this paper are almost the same as the ones in Refs. [16], [17], and [30]–[32], and its physical significance was shown in Ref. [17].

When the smooth operator $D_{ik} = 0$, model (1) becomes the following model:

$$\begin{aligned} \dot{u}_i = & -a_i u_i + \sum_{j=1}^{n} b_{ij} f_j(u_j(t)) \\ & + \sum_{j=1}^{n} c_{ij} f_j(u_j(t - d(t))) \\ & + \sum_{j=1}^{n} e_{ij} \int_{t-\tau(t)}^{t} f_j(u_j(s))\mathrm{d}s + J_i. \end{aligned} \tag{12}$$

From Eq. (12), we have the following simple result:

Corollary 1 Given $\alpha > 0$ if there exist positive definite matrices $P > 0, Q > 0, G > 0, H > 0, \boldsymbol{\Xi}_0 > 0$ and positive definite diagonal matrix $\boldsymbol{R} > 0$, compatibly dimensional arbitrary matrices $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, such that the following LMIs hold:

$$\widehat{\boldsymbol{\Xi}}_1 = \begin{bmatrix} \tilde{\tau}X_{11} + \alpha'_{11} & \tilde{\tau}X_{12} + \alpha_{12} & \tilde{\tau}X_{13} + \alpha'_{13} & \tilde{\tau}X_{14} + \alpha'_{14} \\ * & \tilde{\tau}X_{22} + \alpha_{22} & \tilde{\tau}X_{23} & \tilde{\tau}X_{24} \\ * & * & \tilde{\tau}X_{33} + \alpha'_{33} & \tilde{\tau}X_{34} + \alpha'_{34} \\ * & * & * & \tilde{\tau}X_{44} + \alpha'_{44} \end{bmatrix} < 0, \tag{13}$$

$$\boldsymbol{\Xi}_2 = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} + \beta_{14} & \beta_{15} \\ * & X_{22} & X_{23} & X_{24} & \beta_{25} \\ * & * & X_{33} & X_{34} & 0 \\ * & * & * & X_{44} & 0 \\ * & * & * & * & \beta_{55} \end{bmatrix} > 0, \tag{14}$$

then the system (12) is globally exponentially stable. In which

$$\begin{aligned} &\alpha'_{11} = -2\boldsymbol{P}\boldsymbol{A} + 2\alpha\boldsymbol{P} - 2\boldsymbol{P}\boldsymbol{B}\boldsymbol{L} + \boldsymbol{Q} + 2\boldsymbol{N}_1 + \boldsymbol{L}\boldsymbol{G}\boldsymbol{L} + \tilde{\tau}\boldsymbol{L}\boldsymbol{H}\boldsymbol{L} + \tilde{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A} - 2\tilde{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}\boldsymbol{L} + \tilde{\tau}\boldsymbol{L}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{B}\boldsymbol{L}, \\ &\alpha'_{13} = \boldsymbol{P}\boldsymbol{C} - \tilde{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{C} + \tilde{\tau}\boldsymbol{L}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{C}, \quad \alpha'_{14} = -\tilde{\tau}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E} + \tilde{\tau}\boldsymbol{L}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}, \\ &\alpha'_{33} = \tilde{\tau}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{C} - \left(\mathrm{e}^{-2\alpha d_0} - d\right)\boldsymbol{G}, \quad \alpha'_{34} = \tilde{\tau}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}, \quad \alpha'_{44} = \tilde{\tau}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E} - \mathrm{e}^{-2\alpha\tilde{\tau}}\tilde{\tau}^{-1}\boldsymbol{H}, \end{aligned}$$

and the other notations are the same as those in Theroem 1.

Remark 4 System (12) in this paper is a general NN model. In Refs. [9], [26]–[29] and [33], its special cases were studied by the authors and global stability criteria were given under the case of the boundedness of the activation functions. Moreover, the differentiability on the time-varying delays was also required in Refs. [33]–[35]. However, the assumptions on the boundedness and the differentiability of the activation functions and on the magnitude of the change rate of the time varying delays have been removed in Corollary 1.

4. A simple example

In this section, we provide a simple example to show the effectiveness of the proposed stability criteria.

Example 1 Consider a RDNN with discrete and distributed time-varying delays as

$$\begin{aligned}\frac{\partial z}{\partial t} = {} & \frac{\partial(D\partial z/\partial x)}{\partial x} - \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(t,x\right)\right) \\ & + \boldsymbol{C}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(t-d\left(t\right),x\right)\right) \\ & + \boldsymbol{E}\int_{t-\tau(t)}^{t}\bar{\boldsymbol{f}}\left(\boldsymbol{z}\left(s,x\right)\right)\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}&\left.\frac{\partial z}{\partial x}\right|_{x=0} = \left.\frac{\partial z}{\partial x}\right|_{x=1/8} = 0, \quad z\left(t_0+s,x\right) = \psi\left(s,x\right), \\ &\frac{\partial z\left(t_0+s,x\right)}{\partial t} = \frac{\partial\psi\left(s,x\right)}{\partial t},\end{aligned} \tag{15}$$

where $\bar{\boldsymbol{f}}\left(z\left(t,x\right)\right) = \left|z\left(t,x\right)\right|$. The system (15) with the following parameters: $\alpha = 1.5$, $\tau_0 = d_0 = \tilde{\tau} = 2$, $\mu = d = 2$, $|\Omega| = 1/8$, $A = B = C = 1$, $E = -1$, $P = Q = 0.1$, $G = H = 0.2$, $N_1 = -0.5$, $N_2 = 0.5$, $R = 0.1$, $\boldsymbol{\Xi}_0 = \mathrm{diag}\left(1, 0.1, 0.1, 0.1\right)$, $L = 1$, obviously, satisfies

$$\boldsymbol{\Xi}_1 < 0, \boldsymbol{\Xi}_2 > 0.$$

Therefore, it follows from Theorem 1 that the null solution of system (15) is globally exponentially stable.

5. Conclusions

This paper has proposed an interesting global exponential stability criterion for RDNNs with discrete and distributed time-varying delays. The criterion is expressed in the form of LMIs. The obtained results are dependent on delay and the measure of the space, and therefore, have less conservativeness than delay-independent and space-independent ones. To reduce the conservatism of the stability conditions, we introduce an improved LKF that employs free-weighting matrices. The proposed methods are applicable to a more general class of mixed time-varying delays distributed parameter systems because the delays are allowed to be time-varying and the derivatives of the delays can be any size or even non-existent. Meanwhile, the boundedness, monotonicity and differentiability conditions on the activation functions have no restrictions. Hence, it is shown that the newly obtained results are less conservative and more applicable than the existing corresponding ones.

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