Impulsive synchronisation of a class of fractional-order hyperchaotic systems*

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(Received 18 October 2010; revised manuscript received 10 November 2010)

In this paper, an impulsive synchronisation scheme for a class of fractional-order hyperchaotic systems is proposed. The sufficient conditions of a class of integral-order hyperchaotic systems' impulsive synchronisation are illustrated. Furthermore, we apply the sufficient conditions to a class of fractional-order hyperchaotic systems and well achieve impulsive synchronisation of these fractional-order hyperchaotic systems, thereby extending the applicable scope of impulsive synchronisation. Numerical simulations further demonstrate the feasibility and effectiveness of the proposed scheme.

Keywords: hyperchaotic systems, fractional-order hyperchaotic systems, impulsive synchronization
PACS: 05.45.Jn, 05.45.Xt **DOI:** 10.1088/1674-1056/20/3/030506

1. Introduction

Fractional differential calculus dates from the 17th century, but has only been applied to physics and engineering during the recent decade.[1,2] Many systems are known to display fractional-order dynamics, such as viscoelastic systems, quantitative finance and the quantum evolution of complex systems.[3−5] A feature of fractional-order systems is that one or more equations which represent the system change from an integral order into a fractional order. Fractional order is supposed to be the generalisation of an integral order, for example, an integral-order circuit is the special case of a fractional-order circuit. Compared to classical integral-order models, the fractional-order derivative provides an excellent instrument to describe the memory and hereditary properties of various materials and processes. So research of fractional order has a more universal meaning. Since Pecora and Carroll proposed the concept of 'chaotic synchronisation' and presented a synchronisation method in 1990,[6] different types of chaotic control and synchronisation methods have been presented.[7−10] In recent years, fractional-order chaotic system control and synchronisation is becoming the research hotspot of nonlinear science,[11−13] due to its wide application in secure communication and process control. Although research on integral-order chaotic control and synchronisation has achieved some preliminary results, not all integral-order chaotic synchronisation methods are suitable for fractional-order chaotic systems because of their complexity. Some synchronisation methods (active control method, generalised projective synchronisation method, phase synchronisation method, etc.)[14−18] have been successfully applied to fractional-order chaotic systems. Based on the above research, we applied the impulsive synchronisation method to a new hyperchaotic Chen system[19] and its corresponding fractional-order hyperchaotic system, and achieved the self-synchronisation of these systems. Furthermore, we applied this scheme to a class of fractional-order hyperchaotic systems (Lorenz, Rösslor and Chen), and demonstrated the universality of this scheme.

2. Definition of a fractional-order system

At present, there are several definitions of a fractional-order differential system. Now we introduce the Jumarie's derivative[20] as follows:

$$D^{\alpha}f(t) = \lim_{h\to 0}\frac{\Delta^{\alpha}f(t)}{h^{\alpha}} = \lim_{h\to 0}\frac{\sum_{0}^{\infty}(-1)^{k}\begin{pmatrix}\alpha\\k\end{pmatrix}f[t+(a-k)h]}{h^{\alpha}}, \quad (1)$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 60573172 and 60973152), the Doctoral Program Foundation of the Institution of Higher Education of China (Grant No. 20070141014) and the Natural Science Foundation of Liaoning Province, China (No. 20082165).

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where $f : R \to R$, $x \to f(x)$ denote a continuous (but not necessarily differentiable) function, h is a constant discretisation span, and $0 < \alpha \leq 1$ is the order of the fractional-order differential system.

3. Description of impulsive synchronisation

In the impulsive synchronisation,[21] the form of the drive system is as follows:

$$\dot{\boldsymbol{x}} = f(t, \boldsymbol{x}), \tag{2}$$

$f : R_+ \times R^n \to R^n$ is a continuous function, $\boldsymbol{x} \in R^n$ represents the state variables.

The response system is as follows:

$$\begin{cases} \dot{\boldsymbol{y}} = f(t, \boldsymbol{y}), t \neq t_k; \\ \Delta \boldsymbol{y} = \boldsymbol{y}(t_k^+) - \boldsymbol{y}(t_k^-) = \boldsymbol{B}_k \boldsymbol{e}, \quad t = t_k, \\ k = 1, 2, \ldots; \\ \boldsymbol{y}(t_0^+) = \boldsymbol{y}_0. \end{cases} \tag{3}$$

$\boldsymbol{B}_k$ are $n \times n$ matrices, $\boldsymbol{e} = [e_1, e_2, \ldots, e_n]^{\mathrm{T}} = [y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n]^{\mathrm{T}}$. Define a discrete instant set $\{t_k\}$, $t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$, $t_k \to \infty$ as $k \to \infty$. And $t_k (k = 1, 2, \ldots)$ are the discrete instants at which the drive signals are transmitted to the response system. The states of the response system are changed at these instants according to the synchronisation errors.

Based on drive system (2) and response system (3), we can obtain the error dynamic system as follows:

$$\begin{cases} \dot{\boldsymbol{e}} = f(t, \boldsymbol{y}) - f(t, \boldsymbol{x}), \quad t \neq t_k; \\ \Delta \boldsymbol{e} = \boldsymbol{B}_k \boldsymbol{e}, t = t_k. \end{cases} \tag{4}$$

Because the states of the drive system are continuous in time, $\Delta \boldsymbol{x}$ will be zero at time instants t_k.

In order to achieve asymptotical synchronisation of drive system (2) and response system (3) for any initial condition, i.e., the asymptotical stabilisation of error dynamic system (4), we need to find the appropriate control gain matrices $\boldsymbol{B}_k$ and impulsive distances $\tau_k = t_k - t_{k-1} (k = 1, 2, \ldots)$.

4. Impulsive synchronisation scheme

4.1. A new fractional-order hyperchaotic Chen system

In 2007, Gao *et al.* proposed a new hyperchaotic system[19] from the Chen system, which is described as follows:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = bx_1 + cx_2 - x_1 x_3 - x_4, \\ \dot{x}_3 = x_1 x_2 - dx_3, \\ \dot{x}_4 = x_1 + g, \end{cases} \tag{5}$$

where a, b, c, d and g are the system instants. System (5) is hyperchaotic when $a = 36$, $b = -16$, $c = 28$, $d = 3$ and $g = 0.5$. Figure 1 displays the corresponding hyperchaotic attractors of the new hyperchaotic system.

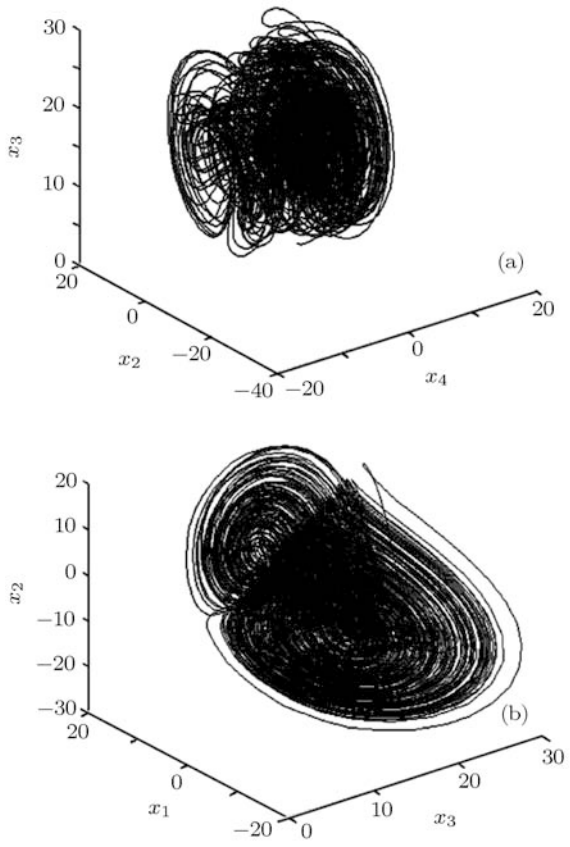


Fig. 1. Hyperchaotic attractor trimetric projection of system (5).

The corresponding fractional-order system of system (5) is as follows:

$$\begin{cases} \dfrac{\mathrm{d}^\alpha x_1}{\mathrm{d}t^\alpha} = a(x_2 - x_1), \\ \dfrac{\mathrm{d}^\alpha x_2}{\mathrm{d}t^\alpha} = bx_1 + cx_2 - x_1x_3 - x_4, \\ \dfrac{\mathrm{d}^\alpha x_3}{\mathrm{d}t^\alpha} = x_1x_2 - dx_3, \\ \dfrac{\mathrm{d}^\alpha x_4}{\mathrm{d}t^\alpha} = x_1 + g, \end{cases} \tag{6}$$

where $0 < \alpha \leq 1$ is the order of the fractional-order system. The experimental results indicate that system (6) is hyperchaotic when $0.72 < \alpha \leq 1$. Figure 2 displays the hyperchaotic attractor trimetric projection of system (6) when $\alpha = 0.72$ and $\alpha = 0.9$.

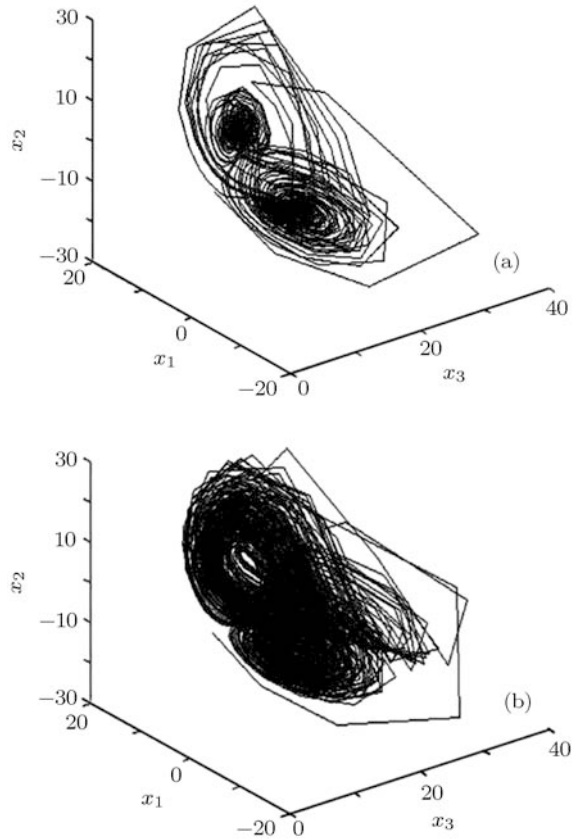


Fig. 2. Hyperchaotic attractor trimetric projection of system (6). (a) $\alpha = 0.72$, (b) $\alpha = 0.9$

Two of the biggest Lyapunov indexes are $\lambda_1 = 8.2130$, and $\lambda_2 = 0.1015$ when $\alpha = 0.9$. When $\alpha = 0.72$, we have $\lambda_1 = 12.3014$, and $\lambda_2 = 0.2318$. It is obviously that this fractional-order system is hyperchaotic.

4.2. Design of the synchronisation scheme

Based on impulsive synchronisation theory, we decompose the dynamic system into its linear and nonlinear parts. Thus Eq. (1) is rewritten as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \Phi(\boldsymbol{x}), \tag{7}$$

where $\boldsymbol{A}\boldsymbol{x}$ represents the linear part of the system and $\Phi(\boldsymbol{x})$ represents the nonlinear part,

$$\boldsymbol{A} = \begin{bmatrix} -a & a & 0 & 0 \\ b & c & 0 & -1 \\ 0 & 0 & -d & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi(\boldsymbol{x}) = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \\ g \end{bmatrix}. \tag{8}$$

The response system is written as

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1), \\ \dot{y}_2 = by_1 + cy_2 - y_1y_3 - y_4, \\ \dot{y}_3 = y_1y_2 - dy_3, \\ \dot{y}_4 = y_1 + g, \quad t \neq t_k \\ \Delta\boldsymbol{y} = \boldsymbol{y}(t_k^+) - \boldsymbol{y}(t_k^-) = \boldsymbol{B}_k e, \quad t = t_k, \\ k = 1, 2, \ldots, \\ \boldsymbol{y}(t_0^+) = \boldsymbol{y}_0. \end{cases} \tag{9}$$

The error dynamic system is written as:

$$\begin{cases} \dot{\boldsymbol{e}} = \boldsymbol{A}\boldsymbol{e} + \varphi(\boldsymbol{x}, \boldsymbol{y}), \quad t \neq t_k, \\ \Delta\boldsymbol{e} = \boldsymbol{B}_k\boldsymbol{e}, \quad t = t_k, \end{cases} \tag{10}$$

where

$$\varphi(\boldsymbol{x},\boldsymbol{y}) = \Phi(\boldsymbol{y}) - \Phi(\boldsymbol{x}) = \begin{bmatrix} 0 \\ -y_1y_3 + x_1x_3 \\ y_1y_2 - x_1x_2 \\ 0 \end{bmatrix}. \tag{11}$$

Next, we will analyse the synchronisation condition of response system (9) and drive system (5).

Because chaotic systems have bounded states regardless of the initial states, we can find a positive number M such that

$$|x_j(t)| \leq M \quad \text{and} \quad |y_j(t)| \leq M$$

for any initial states after a time.

Let β_k and λ be the largest eigenvalues of $(\boldsymbol{I} + \boldsymbol{B}_k)^{\mathrm{T}}(\boldsymbol{I} + \boldsymbol{B}_k)$ $(k = 1, 2, \ldots)$ and $0.5(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})$, respectively, and let $V(\boldsymbol{e}) = 0.5\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$ be the Lyapunov function, so we have

$$\dot{V}(\boldsymbol{e}) = 0.5\dot{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{e} + 0.5\boldsymbol{e}^{\mathrm{T}}\dot{\boldsymbol{e}}. \tag{12}$$

Combine Eq. (10) and Eq. (12) yields

$$\dot{V}(\boldsymbol{e}) = 0.5\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A})\boldsymbol{e} - x_3e_1e_2 + x_1e_2e_3$$

$$\leq \lambda \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} + M(e_1e_2 + e_2e_3)$$
$$\leq 2(\lambda + M)V(\boldsymbol{e}(t)), \quad t \in (t_{k-1}, t_k],$$
$$(k = 1, 2, \ldots). \tag{13}$$

From Eq. (13), we have

$$V(\boldsymbol{e}(t)) \leq V(\boldsymbol{e}(t_{k-1}^+))e^{2(\lambda+M)(t-t_{k-1})},$$
$$t \in (t_{k-1}, t_k], \quad (k = 1, 2, \ldots). \tag{14}$$

According to Eqs. (10) and (14), we have

$$V(e(t_k^+)) \leq \beta_k \boldsymbol{e}(t_k)^{\mathrm{T}}\boldsymbol{e}(t_k) = \beta_k V(\boldsymbol{e}(t_k)),$$
$$(k = 1, 2, \ldots). \tag{15}$$

According to Eqs. (14) and (15), we have

$$V(\boldsymbol{e}(t_1^+)) \leq \beta_1 V(\boldsymbol{e}(t_0^+))e^{2(\lambda+M)(t_1-t_0)}; \tag{16}$$

so, when $t \in (t_1, t_2]$, we obtain

$$V(\boldsymbol{e}(t)) \leq \beta_1 V(\boldsymbol{e}(t_0^+))e^{2(\lambda+M)(t-t_0)}. \tag{17}$$

In the same way, when $t \in (t_k, t_{k+1}]$, we obtain

$$V(\boldsymbol{e}(t)) \leq \beta_1\beta_2 \ldots \beta_k V(\boldsymbol{e}(t_0^+))e^{2(\lambda+M)(t-t_0)}$$
$$= V(\boldsymbol{e}(t_0^+))[\gamma\beta_1 e^{2(\lambda+M)\tau_1}][\gamma\beta_2 e^{2(\lambda+M)\tau_2}]\ldots$$
$$[\gamma\beta_k e^{2(\lambda+M)\tau_k}]\frac{1}{\gamma^k}e^{2(\lambda+M)(t-t_k)}, \tag{18}$$

where $\gamma > 1$ is a constant. If

$$\gamma\beta_k e^{2(\lambda+M)\tau_k} \leq 1, \tag{19}$$

then, when $k \to \infty$,

$$V(\boldsymbol{e}(t)) \to 0, \quad \boldsymbol{e} \to 0. \tag{20}$$

It demonstrates that error dynamic system (10) will be stabilised at the origin.

From the above theoryetical analysis we obtain that if there exists a constant $\gamma > 1$, such that

$$\gamma\beta_k e^{2(\lambda+M)\tau_k} \leq 1, \tag{21}$$

then response system (9) will achieve asymptotical synchronisation with drive system (5).

In order to simplify the synchronisation process, τ_k and $\boldsymbol{B}_k$ can be chosen constant, so we let $\tau_k = \tau > 0$ and $\boldsymbol{B}_k = \boldsymbol{B}$ $(k = 1, 2, \ldots)$, then response system (9) will achieve asymptotical synchronisation with drive system (5) if

$$\gamma\beta e^{2(\lambda+M)\tau} \leq 1, \tag{22}$$

where β is the largest eigenvalue of $(\boldsymbol{I} + \boldsymbol{B})^{\mathrm{T}}(\boldsymbol{I} + \boldsymbol{B})$.

5. Numerical simulations

5.1. Integral-order chaotic system

For the integral-order hyperchaotic system (5), let the initial values as follows: $(x_1(t_0), x_2(t_0), x_3(t_0), x_4(t_0)) = (1, 2, 3, 4)$ and $(x_1(t_0), x_2(t_0), x_3(t_0), x_4(t_0)) = (15, 30, 45, 60)$. Figures 3(a) and 3(b) display the time response of the variables, we can obtain the boundary $M \leq 30$. Then we have

$$0.5(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}) = \begin{bmatrix} -36 & 10 & 0 & 0.5 \\ 10 & 28 & 0 & -0.5 \\ 0 & 0 & -3 & 0 \\ 0.5 & -0.5 & 0 & 0 \end{bmatrix}; \tag{23}$$

the eigenvalues of this matrix are –37.5348, –3, 0.0027 and 29.5321, so the largest eigenvalue $\lambda = 29.5321$. Let $\boldsymbol{B}_k = \boldsymbol{B}(k = 1, 2, \ldots)$, and select $\boldsymbol{B}$ as follows:

$$\boldsymbol{B} = \begin{bmatrix} N & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & N \end{bmatrix}, \tag{24}$$

then we obtain that $\beta = (1+N)^2$. Thus from Eq. (22) the boundaries of the stable region can be determined as

$$0 \leq \tau \leq -(\ln\gamma + \ln(1+N)^2)/119.0642. \tag{25}$$

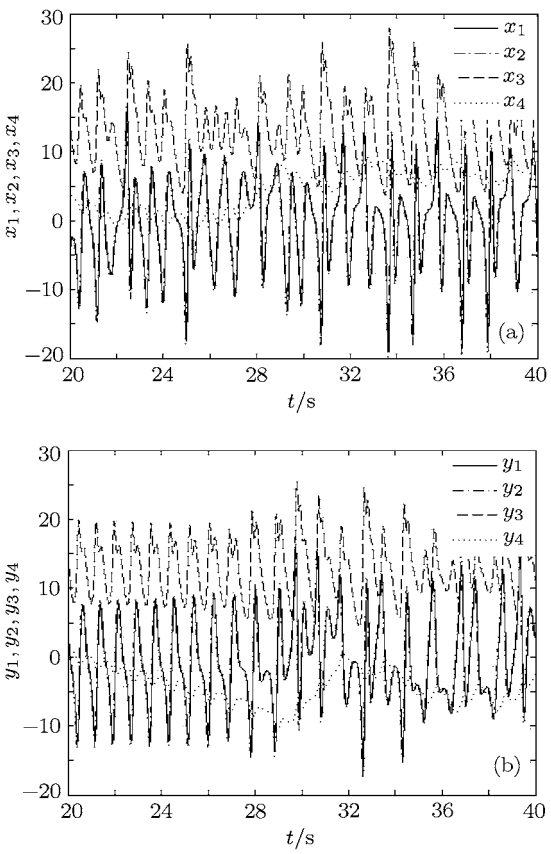


Fig. 3. Time response of variables of the integral-order hyperchaotic system (5).

Figure 4 displays the boundaries of the stable region for different values of γ. According to Fig. 4, we select the parameters $N = -1.2$, $\gamma = 1$, $\tau = 0.01$, and let the initial values as follows: $(x_1(t_0), x_2(t_0), x_3(t_0), x_4(t_0)) = (1, 2, 3, 4)$ and $(y_1(t_0), y_2(t_0), y_3(t_0), y_4(t_0)) = (15, 30, 45, 60)$.

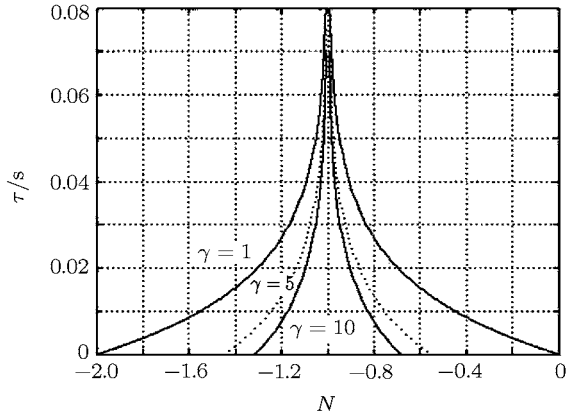


Fig. 4. The boundaries of the stable region for different values of γ.

Figure 5 displays the synchronisation simulation result. We can see that response system (9) and drive system (5) achieve synchronisation first after 0.04s (i.e. four time impulsive signals).

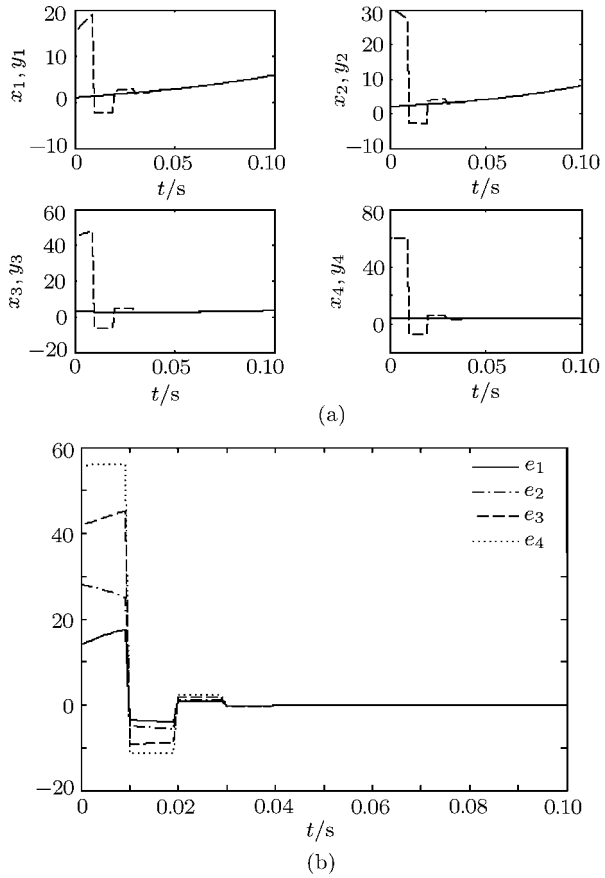


Fig. 5. Impulsive synchronisation result of response system (9) and drive system (5). (a) Time response of the corresponding constants, (b) The synchronisation errors.

5.2. Fractional-order chaotic system

Integral order is a special case of fractional order and we will next apply the impulsive synchronisation method to the new fractional-order Chen chaotic system (6). The order α is selected as 0.72, 0.8 and 0.9. The parameters and the control gain matrix $\boldsymbol{B}$ of system (6) is the same as that of the integral-order system (5), so the largest eigenvalues of $0.5(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})$ and $(\boldsymbol{I}+\boldsymbol{B})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B})$ are the same as those of the integral-order system, which are $\lambda = 29.5321$ and $\beta = (1+N)^2$, respectively.

The response system of the fractional-order system is

$$\begin{cases} \dfrac{\mathrm{d}^\alpha y_1}{\mathrm{d}t^\alpha} = a(y_2 - y_1), \\ \dfrac{\mathrm{d}^\alpha y_2}{\mathrm{d}t^\alpha} = by_1 + cy_2 - y_1y_3 - y_4, \\ \dfrac{\mathrm{d}^\alpha y_3}{\mathrm{d}t^\alpha} = y_1y_2 - dy_3, \\ \dfrac{\mathrm{d}^\alpha y_4}{\mathrm{d}t^\alpha} = y_1 + g, t \neq t_k, \\ \Delta \boldsymbol{y} = \boldsymbol{y}(t_k^+) - \boldsymbol{y}(t_k^-) = \boldsymbol{B}_k\boldsymbol{e}, \quad t = t_k, \\ k = 1, 2, \ldots, \\ \boldsymbol{y}(t_0^+) = \boldsymbol{y}_0. \end{cases} \tag{26}$$

1) $\alpha = 0.72$

Similar to integral-order system (5), the time response of the variables of system (6) indicates that its boundary is $M \leq 30$ when $\alpha = 0.72$. From Eq. (22), we obtain the stable region boundary:

$$0 \leq \tau \leq -(\ln\gamma + \ln(1+N)^2)/119.0642. \tag{27}$$

Based on the stable region, as illustrated by Eq. (27), we select the parameters $N = -1.2$, $\gamma = 1, \tau = 0.01$. Figure 6 displays the synchronisation simulation result. We can see that when $\alpha = 0.72$, response system (26) and drive system (6) achieve synchronisation elementary after 0.4s (i.e. 40 times the impulsive signal).

2) $\alpha = 0.8$

The time response of the variables of system (6) indicates that its boundary is $M \leq 25$ when $\alpha = 0.8$. From Eq. (22), we obtain the stable region boundary

$$0 \leq \tau \leq -(\ln\gamma + \ln(1+N)^2)/109.0642. \tag{28}$$

Based on the stable region, as illustrated by Eq. (28), we select the parameters $N = -1.2$, $\gamma = 1$, $\tau = 0.01$. Figure 7 displays the synchronisation simulation result. We can see when $\alpha = 0.8$ the response system (26) and the drive system (6) achieve synchronisation elementary before 0.3s (i.e. 30 times the impulsive signal).

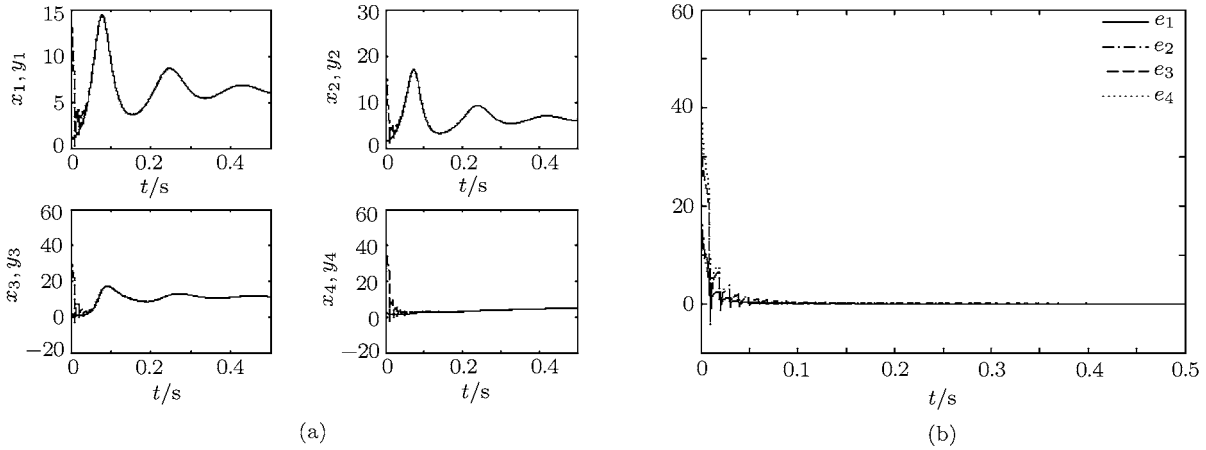


Fig. 6. Impulsive synchronisation result of system (26) and system (6) with $\alpha = 0.72$. (a) Time response of the corresponding constants, (b) The synchronisation errors.

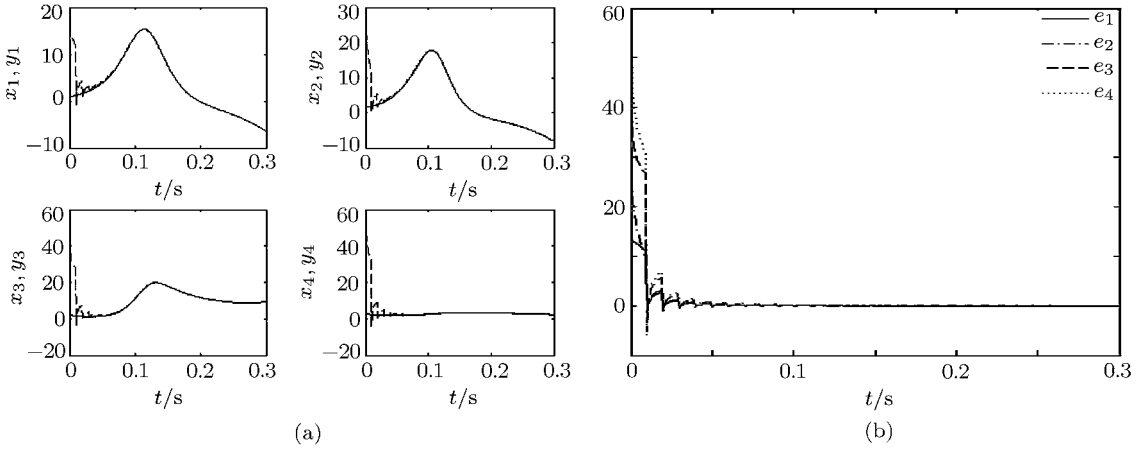


Fig. 7. Impulsive synchronisation result of system (26) and system (6) for $\alpha = 0.8$. (a) Time response of the corresponding constants, (b) The synchronisation errors.

3) $\alpha = 0.9$

The time response of the variables of system (6) indicates that its boundary is $M \leq 25$ when $\alpha = 0.8$. From Eq. (22), we obtain the stable region boundary

$$0 \leq \tau \leq -(\ln \gamma + \ln(1+N)^2)/109.0642. \tag{29}$$

Based on the stable region, as illustrated by Eq. (29), we select the parameters $N = -1.2$, $\gamma = 1$, $\tau = 0.01$. Figure 8 displays the synchronisation simulation result.

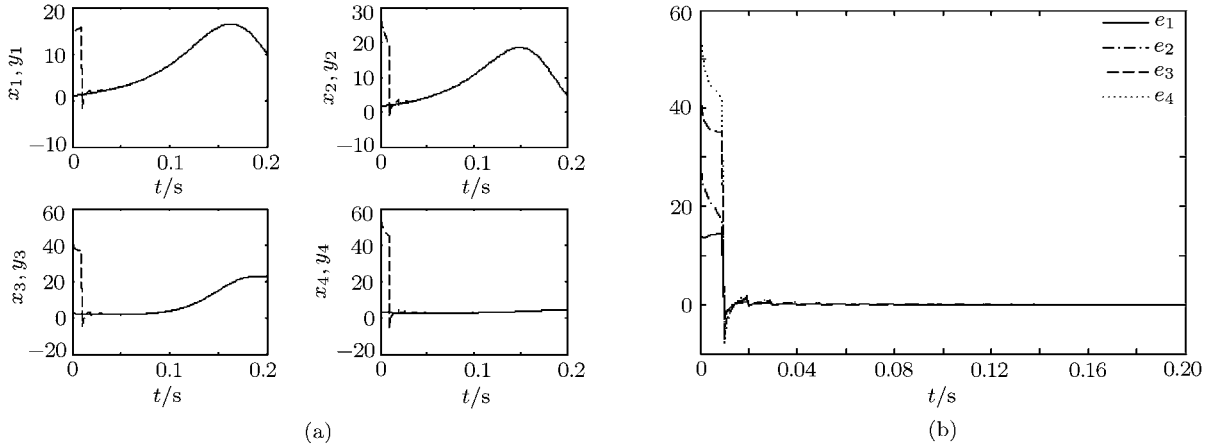


Fig. 8. Impulsive synchronisation result of system (26) and system (6) with $\alpha = 0.9$. (a) Time response of the corresponding constants, (b) The synchronisation errors.

We can see that when $\alpha = 0.8$, response system (26) and drive system (6) achieve synchronisation first after 0.14s (i.e. 14 times the impulsive signal).

6. Conclusion

1) In this paper, we apply the impulsive synchronisation method to a new integral-order hyperchaotic Chen system. Theoretical analysis and numerical simulations further demonstrate the feasibility and effectiveness of the proposed scheme. Based on this, we apply the synchronisation method to the corresponding fractional-order system; and numerical simulation proves that the fractional-order system can achieve self-synchronisation well via this method.

2) Numerical simulations indicate that when the system is hyperchaotic, impulsive synchronisation can be achieved quicker along with the increase of the order. We apply the synchronisation scheme to a class of fractional-order hyperchaotic systems (Lorenz, Rösslor and Chen) and the experimental results indicate that these systems can achieve impulsive synchronisation well via the scheme (for space reasons, we omit it here). It proves this scheme may be universal.

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