The Stochastic stability of a Logistic model with Poisson white noise*

Duan Dong-Hai(段东海)[a)†], Xu Wei(徐 伟)[a)], Su Jun(苏 军)[b)], and Zhou Bing-Chang(周丙常)[a)]

[a)] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an* 710072, *China*
[b)] *School of Science, Xi'an University of Science and Technology, Xi'an* 710054, *China*

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The stochastic stability of a logistic model subjected to the effect of a random natural environment, modeled as Poisson white noise process, is investigated. The properties of the stochastic response are discussed for calculating the Lyapunov exponent, which had proven to be the most useful diagnostic tool for the stability of dynamical systems. The generalised Itô differentiation formula is used to analyse the stochastic stability of the response. The results indicate that the stability of the response is related to the intensity and amplitude distribution of the environment noise and the growth rate of the species.

Keywords: Poisson white noise, Itô formula, Lyapunov exponent, stochastic bifurcation
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1. Introduction

A broad class of random excitations on structures can be modeled as Gaussian white noise processes. The stability of the solution of stochastic differential equations has been extensively studied in cases where the excitations are Gaussian white noise processes. However, when excitations are characterised by stochastic impulsive loading, such as moving loads traveling on a bridge, earthquake shake on structures and sea wave and wind action on ships, these representations may not be satisfactory.[1,2] In these cases Poisson white noise is a more realistic model. Methods for the evaluation of the response of linear and nonlinear systems enforced by the Poisson white noise process have recently been proposed by several authors.[3−8] Some of these are based on the theory of Markov processes and determine the probability density function (p.d.f.) of the response process as a solution of the generalised stochastic differential equations.[9−17]

In this paper, we examine the stochastic stability of a Logist model subjected to the Poisson white noise process, the so-called the Pulse-type Logist model, which describes the interaction between species and their random natural environment, while the effect of the random environment is modeled as a Poisson white noise process. The Poisson white noise process can be viewed as an independent, identically distributed, pulse sequence arriving at the jump times, and depends on a Poisson counting process. The generalised Itô differentiation formula is applied to analyse and calculate the Lyapunov exponent. The objective is to determine the relationship between the environment noise and the exponential growth rate of the species density.

2. Logistic model with Poisson white noise process

Let us consider the stochastic differential equation describing the evolution of the population density for the species with the Poisson white noise process,

$$\mathrm{d}X(t) = rX(t)(1 - X(t))\mathrm{d}t + X(t)\mathrm{d}Y(t), \quad t \geq 0, \tag{1}$$

where the response $X(t)$ represents the population density of the species, the parameter r is rate of growth, $Y(t)$ is a compound Poisson process (in the present paper, capital letters stand for stochastic processes or stochastic variables, while lowercase letters represent deterministic quantities),

$$Y(t) = \begin{cases} 0, & \text{if} \quad N(t) = 0, \\ \sum_{i=1}^{N(t)} Y_i U(t - S_i), & \text{if} \quad N(t) > 0, \end{cases} \tag{2}$$

where $\{Y_i, i = 1, 2, \ldots\}$ is an independent identically distributed (i.i.d.) random vector, $\{N(t), t = 1, 2, \ldots\}$ is a homogeneous Poisson counting process with intensity $\nu > 0$, $U(t)$ is a the unit step function.

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†Corresponding author. E-mail: donghaiduan@yahoo.cn
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2.1. Poisson white noise process

Let the formal derivative of the compound Poisson process, that is, the Poisson white noise process, be

$$W(t) = \frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \sum_{i=1}^{N(t)} Y_i \delta(t - S_i), \quad t \geq 0, \tag{3}$$

where $0 = S_0 < S_1 < \ldots < S_k < \ldots < S_{N(t)} < t$ are the arrival times of the Poisson counting process $\{N(t), t \geq 0\}$, called the Poisson times; $\delta(t - S_i)$ is the Dirac delta function.

Suppose that $\{Y_i, i = 1, 2, \ldots\}$ is in L_2, that is, $E(Y_i^2)$ exists and is finite, the statistical characterisations of $Y(t)$ and $W(t)$ can be given in terms of their correlation functions,[5,18−20]

$$\begin{aligned} &\mathrm{Cor}(Y(t_1), \ldots, Y(t_p)) = \nu E Y_1^p \min(t_1, \ldots, t_p), \\ &\mathrm{Cor}(W(t_1), \ldots, W(t_p)) \\ &= \nu E Y_1^p \delta(t_1 - t_2) \ldots \delta(t_1 - t_p). \end{aligned} \tag{4}$$

This process is a quadratic pure jump process with quadratic variation $[Y, Y](t) = \sum_{k=1}^{N(t)} Y_k^2$. So it can also be given by the Poisson random measure

$$Y(t) = \int_0^t \int_{\mathcal{R}} y M(\mathrm{d}s, \mathrm{d}y), \tag{5}$$

where $M(\mathrm{d}s, \mathrm{d}y)$ is defined in the Itô sense. The Poisson random measure of an arbitrary set of $[0, \infty) \times \mathcal{R}$ is a Poisson random variable counting the number of points in this set. The mean of this Poisson random variable for the set $(t, t + \mathrm{d}t) \times (y, y + \mathrm{d}y)$ is

$$EM(\mathrm{d}t, \mathrm{d}y) = \lambda \mathrm{d}t \mathrm{d}F(y), \tag{6}$$

so that Eq. (1) defining $X(t)$ can be given alternatively by

$$\begin{aligned} \mathrm{d}X(t) = {} & rX(t)(1 - X(t))\mathrm{d}t + \int_{\mathcal{R}} X(t) y M(\mathrm{d}t, \mathrm{d}y), \\ & t \geq 0. \end{aligned} \tag{7}$$

Suppose that $g(X(t), t)$ is a real-valued function with a continuous second-order derivative, $rX(t)(1 - X(t))$ and $X(t)y$ satisfy some growth and Lipschitz conditions. So we have the generalised Itô differentiation formula[18−20]

$$\begin{aligned} \mathrm{d}g(X(t), t) = {} & \left(\frac{\partial g(X(t), t)}{\partial t} + rX(t) \right. \\ & \left. \times (1 - X(t)) \frac{\partial g(X(t), t)}{\partial X(t)} \right) \mathrm{d}t \\ & + \int_{\mathcal{R}} (g(X(t) + X(t)y, t) \\ & - g(X(t), t) M(\mathrm{d}t, \mathrm{d}y), \end{aligned} \tag{8}$$

and can be used to determine the statistical characterisation of the response process.

2.2. Properties of stochastic response $X(t)$

Property 1 The response $X(t)$ has sample paths which are right continuous, with left limits and is a stationary process. That is, $X(t)$ is the continuous deterministic process for $S_{i-1} < t < S_i$ and the quadratic pure jump process at the Poisson time $S_i, i = 1, 2, \ldots, N(t)$.

Proof Let $X_i^- = X(S_i-)$ and $X_i^+ = X(S_i+)$ be the solution $X(t)$ immediately prior to and after the arrival time $S_i, i = 1, 2, \ldots, N(t)$. The function

$$h(X(0), s) = \frac{X(0)\mathrm{e}^{rs}}{1 + X(0)(\mathrm{e}^{rs} - 1)} \tag{9}$$

represents the determined solution of the model at time s for the initial conditions $X(0)$ when process $X(t)$ has no pulse. So we have

$$\begin{aligned} X_i^- &= h(X_{i-1}^+, S_i - S_{i-1}), X(S_{i-1} + s) \\ &= h(X_{i-1}^+, s), \quad 0 \leq s < S_i - S_{i-1}, \end{aligned} \tag{10}$$

$$X_i^+ = X_i^- + X_i^- Y_i = X_i^-(1 + Y_i). \tag{11}$$

These deterministic mappings show that $X(t)$ has sample paths, which are right continuous with left limits, and the mappings can define sequences $\{X_i^+\}$ and $\{X_i^-\}, i = 1, 2, \ldots, N(t)$. Figure 1 shows that samples of solution of the response $X(t)$ are right continuous, with left limits.

Property 2 The response $X(t)$ is a Markov process.

Proof Consider the Euler approximation of Eq. (1)

$$\begin{aligned} X(t + \Delta t) = {} & X(t) + rX(t)(1 - X(t))\Delta t \\ & + X(t)\Delta Y(t), \end{aligned} \tag{12}$$

where $\Delta Y(t) = \mathrm{d}Y(t) = Y(t + \Delta t) - Y(t)$; $\Delta t > 0$ is a small time step. $Y(t)$ is the compound Poisson process and has independent increments. So that, the future and the past value of the vector, $X(t + \Delta t)$ and $\{X(s), s < t\}$, are independent of the current value $X(t)$. The future density has the distribution

$$\begin{aligned} & P(X(t + \Delta t) < x_i \mid X(t) = x) \\ = {} & P(x\Delta Y(t) < x_i - x - rx(1 - x)\Delta t). \end{aligned} \tag{13}$$

The Markov process is essential for the solution of the Eq. (1) by the path integral method and the Itô formula.

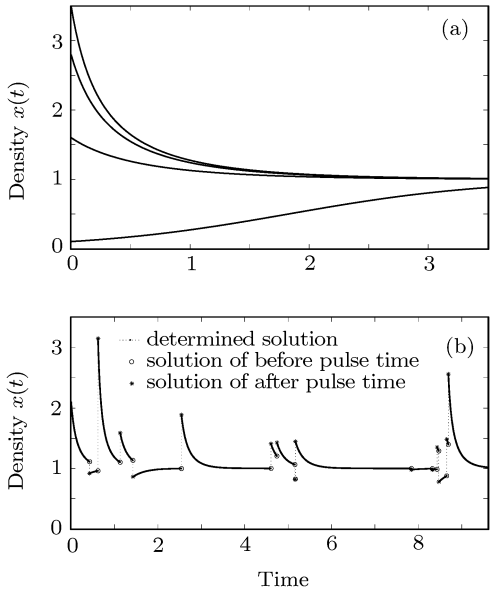


Fig. 1. Samples of solution of the response $X(t)$, (a) initial condition $X(0) = 0.1,\ 1.6,\ 2.8,\ 3.5$, $r = 1.2$, $Y_1 = 0$, and (b) initial condition $X(0) = 2.1$, $r = 2$, $\nu = 5$, $Y_1 \sim U(-\sqrt{3/\nu}, \sqrt{3/\nu})$.

Property 3 The response $X(t)$ is an approximate ergodic process.

Proof Because $\{Y_i, i = 1, 2, \ldots\}$ is an independent, identically distributed random vector, $X(t)$ can be defined based on Eqs. (10) and (11). For any measurable function g, the random variables $g(1+Y_i)$ are independent, by the strong law of large numbers, provided that $E(g(X_i^+))$ exists and is finite, we have

$$\frac{1}{n}\sum_{k=1}^{N(t)} g(X_i^+) \to E(g(X_i^+)), \quad \text{as} \quad n \to \infty, \quad \text{a.s.} \tag{14}$$

where X_i^- converges to a constant, which is an approximate degenerated random variable sequence, so $X(t)$ is an approximate ergodic process.

3. Stochastic stability

Suppose that the stationary solution of Eq. (1), if it exists, is denoted by $X_s(t)$. The objective is to determine a subset D of values r such that the solution $X(t)$ starting from a small vicinity of the stationary solution $X_s(t)$ converges to $X_s(t)$ as time increases indefinitely.

Obviously, $X_s(t) = 0$ is a trivial stationary solution of Eq.(1), which indicates that species die out after long enough time from the biological viewpoint. Because the response $X(t)$ is a Markov process, it preserves its properties if it is restarted at a Poisson time. So the permanence of the species, which is in a harmonious state, can be achieved within a finite time if the nontrivial stationary solution of Eq. (1) exists. Figure 2. shows two samples of $X(t)$ starting from $X(0) = 2.1$, $\nu = 5$, $Y_1 \sim U(-\sqrt{3/\nu}, \sqrt{3/\nu})$. (a) For $r = 2.0$ the sample seems to belong to a stationary process with a non-zero mean. On the other hand, (b) for $r = 0.8$ the sample converges to zero as t increases, indicating species extinction.

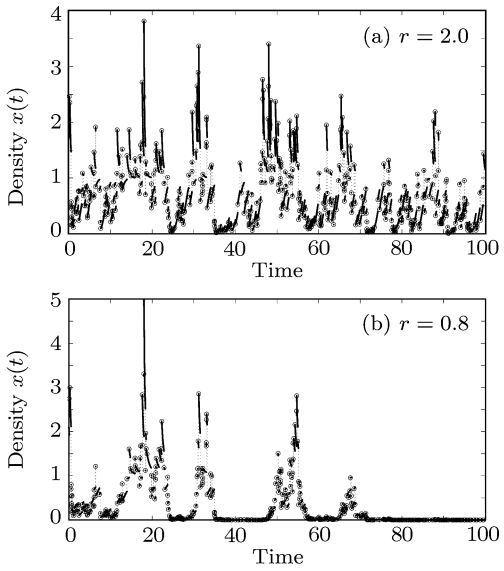


Fig. 2. Samples of $X(t)$ starting from $X(0) = 2.1$, $\nu = 5$, $Y_1 \sim U(-\sqrt{3/\nu}, \sqrt{3/\nu})$.

3.1. Lyapunov exponent

Consider the difference $\tilde{X}(t) = X(t) - X_s(t)$, where $X_s(t)$ and $X(t)$ are stationary and transient solutions of Eq. (1) respectively. So $\tilde{X}(t) = X(t) - X_s(t)$ can be defined as the stochastic differential equation

$$\begin{aligned} \mathrm{d}\tilde{X}(t) &= r\tilde{X}(t)(1 - (X(t) + X_s(t))\mathrm{d}t \\ &\quad + \tilde{X}(t)\int_{\mathcal{R}} yM(\mathrm{d}t, \mathrm{d}y) \end{aligned} \tag{15}$$

with $t \geq 0$. Since $X_s(t)$ and $X(t)$ are stationary and transient solutions of Eq. (1), we have the approximation $X(t) + X_s(t) = 2X_s(t)$. Moreover, the response $X(t)$ is the approximate ergodic process, the exponential growth rates of $\tilde{X}(t)$ are given by the Lyapunov exponent.[3,17]

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \|\tilde{X}(t)\|. \tag{16}$$

The Lyapunov exponent can be obtained based on a generalised Itô differentiation formula, Eq. (8).

Applying the generalised Itô differentiation formula to the function $g(t) = \ln \|\tilde{X}(t)\|$, we have

$$\begin{aligned} \mathrm{d}g(t) &= r(1-2X_s(t))\mathrm{d}t + \int_{\mathcal{R}} (\ln \|\tilde{X}(t) \\ &\quad + \tilde{X}(t)y\| - \ln \|\tilde{X}(t)\|) M(\mathrm{d}t, \mathrm{d}y) \\ &= r(1-2X_s(t))\mathrm{d}t + \int_{\mathcal{R}} \ln \|1+y\| M(\mathrm{d}t, \mathrm{d}y), \end{aligned}$$

so that

$$\begin{aligned} g(t) &= g(0) + \int_0^t r(1-2X_s(t))\mathrm{d}t \\ &\quad + \int_0^t \int_{\mathcal{R}} \ln \|1+y\| M(\mathrm{d}t, \mathrm{d}y). \end{aligned} \tag{17}$$

Therefore, the absolute value of the solution of Eq. (1) is

$$\begin{aligned} \|X(t)\| &= \|X(0)\| \exp\Big(\int_0^t r(1-2X_s(t))\mathrm{d}t \\ &\quad + C(t)\Big), \end{aligned} \tag{18}$$

where $C(t)$ is the compound Poisson process derived by the $\ln \|1+Y_k\|$. So the Lyapunov exponent is given by

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \Big(\int_0^t r(1-2X_s(t))\mathrm{d}t + C(t) \Big). \tag{19}$$

Since the response $X(t)$ is an approximate ergodic process, as $t \to \infty$ we have

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \Big(\int_0^t r(1-2X_s(t))\mathrm{d}t \\ &= r(1-2EX_s(t)), \quad \text{a.s.} \end{aligned} \tag{20}$$

Moreover, because $\{N(t), t > 0\}$ is a homogeneous Poisson counting process, the equality

$$P(N(t) > n) = 1 - \exp(-\nu t) \sum_{k=0}^{n} \frac{(\nu t)^k}{k!} \tag{21}$$

holds for $t \geq 0$ and any n. So the number of Poisson events increases indefinitely as t converges to infinite, that is, $N(t) \to \infty$ almost surely as $t \to \infty$. This result shows that $C(t)$ can be calculated from an infinite sum of independent identically random variables, so we have

$$\lim_{t\to\infty} \frac{C(t)}{t} = \nu E \ln \|1+Y_1\|, \quad \text{a.s.} \tag{22}$$

Therefore, the Lyapunov exponent becomes

$$\lambda = r(1-2EX_s(t)) + \nu E \ln \|1+Y_1\|, \quad \text{a.s.} \tag{23}$$

for any intensity ν of the homogeneous Poisson counting process $N(t)$.

3.2. Stochastic bifurcation

Consider first the stability of the trivial stationary solution $X_s(t) = 0$. Notice that $EX_s(t) = 0$, so the Lyapunov exponent can be written

$$\lambda = r + \nu E \ln \|1+Y_1\|, \quad \text{a.s.} \tag{24}$$

The trivial stationary solution $X_s(t) = 0$ is the stable with probability one provided that $\lambda < 0$, that is

$$r < -\nu E \ln \|1+Y_1\|, \quad \text{a.s.} \tag{25}$$

Consider now the stability of the nontrivial stationary solution $X_s(t) \neq 0$. According to the generalised Itô differentiation formula, $X_s(t)$ satisfies Eq. (15), that is

$$\begin{aligned} \mathrm{d} \ln \|X_s(t)\| &= r(1-X_s(t))\mathrm{d}t \\ &\quad + \int_{\mathcal{R}} \ln |1+y| M(\mathrm{d}t, \mathrm{d}y), \end{aligned} \tag{26}$$

Moreover, by using the stochastic average of both sides of Eq. (26), the following relationship holds,

$$\begin{aligned} \frac{\mathrm{d}E \ln \|X_s(t)\|}{\mathrm{d}t} &= r(1-EX_s(t)) \\ &\quad + \nu E \ln \|1+Y_1\|, \end{aligned} \tag{27}$$

Notice that $X_s(t)$ is a stationary process, the left-hand side of Eq. (27) is zero, so we obtain

$$EX_s(t) = 1 + \frac{\nu E \ln \|1+Y_1\|}{r}, \tag{28}$$

and the Lyapunov exponent of Eq. (23) becomes

$$\lambda = -r - \nu E \ln \|1+Y_1\|, \quad \text{a.s.} \tag{29}$$

Hence we have the relationships between the rate of growth r and the noise average intensity of surroundings, and the nontrivial stationary solution is stable if $\lambda < 0$, that is

$$r > -\nu E \ln \|1+Y_1\|, \quad \text{a.s.} \tag{30}$$

Equations (25) and (30) give 'qualitative changes': the Lyapunov exponent of Eq. (23) increases from a negative to a positive value, this phenomenon is called a stochastic D-bifurcation[9,18] as shown in Fig. 3. The boundary of the stability region is dependent on the

intensity ν and amplitude distribution Y_1 of the noise and growth rate r of the species.

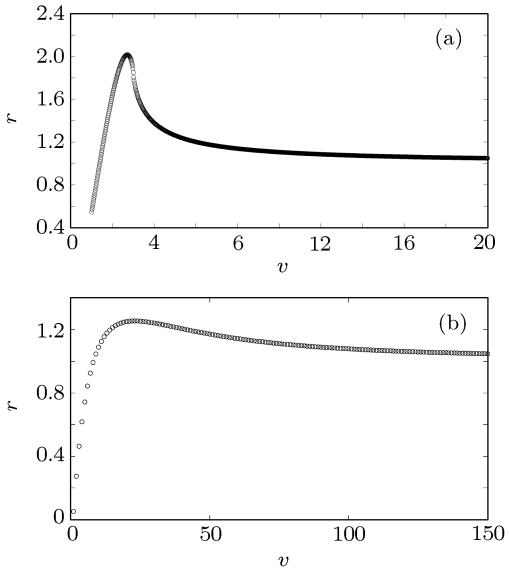


Fig. 3. Boundary of trivial and nontrivial stationary solutions, $X(0) = 2.1$, (a) $Y_1 \sim U(-\sqrt{3/\nu}, \sqrt{3/\nu})$, (b) $Y_1 \sim La(0, \sqrt{1/\nu})$.

4. Conclusions

We have developed a method for studying the stochastic stability of a Logist model with the Poisson white noise process. The effect of a random environment is modeled as a Poisson white noise process, which can be viewed as a sequence of independent, identically distributed pulses arriving in time depending on a Poisson counting process. The properties of the stochastic response are discussed for calculating the Lyapunov exponent. Results indicate that the response stability is related to the intensity and amplitude distribution of the environment noise and rate of growth of the species. It is found that stochastic bifurcation of the stationary solution for Eq.(1) occurs at the boundary of the stability region. From a realistic perspective, we expect the coexistence of the species to be in a harmonious state. For this purpose, effective measures must be designed and applied to avoid the extinction of the species or to lengthen the transition time.

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