Fermion tunneling from squashed black holes in the Gödel universe and charged Kaluza–Klein space–time

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This paper is devoted to the investigation the fermion tunneling radiation of squashed black holes in the Gödel universe and charged Kaluza–Klein space–time. For black holes with different dimensions, establishing a set of appropriate matrices $\boldsymbol{\gamma}^{\mu}$ for the general covariant Dirac equation plays an important role in the semi-classical tunneling method. By constructing two sets of $\boldsymbol{\gamma}^{\mu}$ matrices, we have successfully derived the tunneling probability and Hawking temperature of the black holes.

Keywords: Hawking temperature, squashed black holes, fermions tunneling

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1. Introduction

The tunneling method may provide a new physical picture of Hawking radiation, so the study of Hawking radiation via the quantum tunneling method has been a hot topic in black hole physics. Recently, a semi-classical quantum tunneling method, firstly proposed by Kraus and Wilczek[1] and then elaborated by Parikh and Wilczek[2] and Parikh,[3] was used to investigate the Hawking radiation of black holes. A considerable amount of work has been done on this issue.[4−20] Kerner and Mann[21] recently showed that Hawking radiation could be viewed as a quantum tunneling process and fermion tunneling could correctly recover Hawking temperature in four-dimensional (4D) spacetime. Since then, some researchers, by applying the fermion tunneling method, have adequately investigated the spin 1/2 particle emission effect from typical 4D black holes.[22−31]

Five-dimensional (5D) black holes and their properties have attracted considerable attention in recent years.[32−39] This is because they may present the possibility of direct observation of Hawking radiation and serving as a large spatial extra-dimensional probe in future high-energy collisions. Therefore, the study of fermion tunneling radiation from 5D black holes becomes greatly important.

In this paper, we attempt to generalize the tunneling method to derive the Hawking radiation of a Dirac particle from squashed black holes in the Gödel universe and charged Kaluza–Klein space–time. To our knowledge, this research, following and extending the earlier literature, has not appeared in the other literature, and its applicability for such black holes has not been demonstrated so far, thus it is important to confirm whether the tunneling method actually works for the black holes in this paper. For black holes with different dimensions, establishing a set of appropriate matrices $\boldsymbol{\gamma}^{\mu}$ for the general covariant Dirac equation plays an important role in semi-classical tunneling method. The tunneling probability and the Hawking temperature of the black holes are derived. The result shows that the Hawking temperature of the black holes is not related to the species of emitted particles.

2. Fermion tunneling from a squashed black hole in Kaluza–Klein space–time

The metric of the charged Kaluza–Klein (KK) black hole with squashed horizon is described by[40]

$$\begin{aligned} \mathrm{d}s^2 = &- f(r)\mathrm{d}t^2 + \frac{k^2(r)}{f(r)}\mathrm{d}r^2 + \frac{r^2}{4}k(r)\mathrm{d}\Omega^2 \\ &+ \frac{r^2}{4}(\mathrm{d}\psi + \cos\theta\mathrm{d}\,\phi), \end{aligned} \tag{1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi$, $0 < \theta < \pi$, $0 < \phi < 2\pi$, and $0 < \psi < 4\pi$. The functions $f(r)$ and $k(r)$ take the following forms:

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4},$$

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$$k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}. \tag{2}$$

There are three coordinate singularities, the outer event horizon r_+, the inner event horizon r_- and the spatial infinity r_∞, and they satisfy $r_- < r_+ < r_\infty$, $0 < r < r_\infty$. They are related to mass M and charge Q by

$$M = \frac{3\pi(r_+^2 r_\infty^2 + r_-^2 r_\infty^2 - 2r_+^2 r_-^2)}{8G\sqrt{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}},$$
$$Q = \frac{\sqrt{3}\pi r_+ r_-}{2G}. \tag{3}$$

The gauge potential is described by

$$A_t = \pm\frac{\sqrt{3}}{2}\left(\frac{r_+ r_-}{r^2} - \frac{r_+ r_-}{r_\infty^2}\right). \tag{4}$$

A new radial coordinate ρ can be introduced as[40]

$$\rho = \frac{r^2}{r_\infty^2 - r^2}\rho_0, \tag{5}$$

it yields the relation

$$r_\infty^2 = 4(\rho_+ + \rho_0)(\rho_- + \rho_0), \tag{6}$$

where

$$\rho_\pm = \frac{r_\pm^2}{r_\infty^2 - r_\pm^2}\rho_0. \tag{7}$$

Under this coordinate transformation, metric (1), gauge potential A_τ, black hole mass M and charge Q can be rewritten as

$$\mathrm{d}s^2 = -F(\rho)\mathrm{d}\tau^2 + \frac{K^2(\rho)}{F(\rho)}\mathrm{d}\rho^2 + \rho^2 K^2(\rho)\mathrm{d}\Omega^2 + \frac{r_\infty^2}{4K^2(\rho)}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)^2, \tag{8}$$

and

$$A_\tau = \pm\frac{\sqrt{3}}{2}\frac{\sqrt{\rho_+\rho_-}}{\rho},$$
$$M = \frac{3\pi r_\infty}{4G}(\rho_+ + \rho_-),$$
$$Q = \mp\frac{\sqrt{3}\pi r_\infty}{G}\sqrt{\rho_+\rho_-}, \tag{9}$$

where $\tau = 2\rho_0 t/r_\infty$, and the new functions $F(\rho)$ and $K(\rho)$ are given by

$$F = \left(1 - \frac{\rho_+}{\rho}\right)\left(1 - \frac{\rho_-}{\rho}\right), \quad K = 1 + \frac{\rho_0}{\rho}. \tag{10}$$

The determinant of the metric is

$$g = -\frac{\rho^4 r_\infty^2 K^4(\rho)\sin^2\theta}{4}, \tag{11}$$

and the non-zero inverse metrics can be expressed as

$$\boldsymbol{g}^{\tau\tau} = -\frac{1}{F(\rho)}, \quad \boldsymbol{g}^{\rho\rho} = \frac{F(\rho)}{K^2(\rho)}, \quad \boldsymbol{g}^{\theta\theta} = \frac{1}{\rho^2 K^2(\rho)},$$
$$\boldsymbol{g}^{\phi\phi} = \frac{1}{\rho^2 K^2(\rho)\sin^2\theta},$$
$$\boldsymbol{g}^{\psi\psi} = \frac{4K^2(\rho)}{r_\infty^2} + \frac{\cos^2\theta}{\rho^2 K^2(\rho)\sin^2\theta},$$
$$\boldsymbol{g}^{\phi\psi} = -\frac{\cos\theta}{\rho^2 K^2(\rho)\sin^2\theta}. \tag{12}$$

In cured spacetime, Dirac particle motion satisfies the following covariant Dirac equation:

$$\mathrm{i}\boldsymbol{\gamma}^\mu\left(\partial_\mu + \Omega_\mu + \frac{\mathrm{i}e}{\hbar}A_\mu\right)\Psi + \frac{m}{\hbar}\Psi = 0, \tag{13}$$

where m and e are the mass and the electric charge of the emitted particle, and

$$\Omega_\mu = \frac{\mathrm{i}}{2}\Gamma_\mu^{\alpha\beta}\Sigma_{\alpha\beta}, \quad \Sigma_{\alpha\beta} = \frac{\mathrm{i}}{4}\left[\boldsymbol{\gamma}^\alpha, \boldsymbol{\gamma}^\beta\right]. \tag{14}$$

The matrices satisfy

$$\{\boldsymbol{\gamma}^\mu, \boldsymbol{\gamma}^\nu\} = 2\boldsymbol{g}^{\mu\nu} \times 1. \tag{15}$$

In this section, we choose the following matrices $\boldsymbol{\gamma}^\mu$ form for the rotating KK black hole with a squashed horizon:

$$\boldsymbol{\gamma}^\tau = \sqrt{-\boldsymbol{g}^{\tau\tau}}\begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}^\rho = \sqrt{\boldsymbol{g}^{\rho\rho}}\begin{pmatrix} 0 & \boldsymbol{\sigma}^3 \\ \boldsymbol{\sigma}^3 & 0 \end{pmatrix},$$
$$\boldsymbol{\gamma}^\theta = \sqrt{\boldsymbol{g}^{\theta\theta}}\begin{pmatrix} 0 & \boldsymbol{\sigma}^1 \\ \boldsymbol{\sigma}^1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}^\phi = \sqrt{\boldsymbol{g}^{\phi\phi}}\begin{pmatrix} 0 & \boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^2 & 0 \end{pmatrix},$$
$$\boldsymbol{\gamma}^\psi = G\begin{pmatrix} -\boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} + F\begin{pmatrix} 0 & \boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^2 & 0 \end{pmatrix}, \tag{16}$$

where

$$F = -\frac{\boldsymbol{g}^{\phi\psi}}{\sqrt{\boldsymbol{g}^{\phi\phi}}}, \qquad G = \sqrt{\boldsymbol{g}^{\psi\psi} - \frac{(\boldsymbol{g}^{\phi\psi})^2}{\boldsymbol{g}^{\phi\phi}}}, \tag{17}$$

and $\boldsymbol{I}$ is a 2×2 identity matrix and gamma matrices $\boldsymbol{\gamma}^a (a = 0, 1, 2, 3, 4)$ obey the anti-commutation relations

$$\{\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^b\} = 2\boldsymbol{\eta}^{ab} \tag{18}$$

with $\boldsymbol{\eta}^{ab} = \mathrm{diag}(-1, 1, 1, 1, 1)$ being the flat (lorentz) metric tensor. The $\boldsymbol{\sigma}^i (i = 1, 2, 3)$ are Pauli matrices. Here, we will only analyse the spin-up case since the final result is the same as that in the spin-down case, as can be presented by using the methods described

below. We employ the ansatz for the spin-up spinor field $\boldsymbol{\Psi}$ as follows:

$$\begin{aligned}&\boldsymbol{\Psi}_{\uparrow}(\tau,\rho,\theta,\phi,\psi)\\&=\begin{pmatrix}A(\tau,\rho,\theta,\phi,\psi)\\0\\B(\tau,\rho,\theta,\phi,\psi)\\0\end{pmatrix}\times\exp\left[\frac{\mathrm{i}}{\hbar}I_{\uparrow}(\tau,\rho,\theta,\phi,\psi)\right].\end{aligned}\tag{19}$$

Thus, we have the following equations:

$$\begin{aligned}&B\Big(\sqrt{-g^{\tau\tau}}(\partial_\tau I_\uparrow+eA_\tau)+\sqrt{g^{\rho\rho}}\partial_\rho I_\uparrow\Big)\\&-A\Big(G\partial_\psi I_\uparrow+m\Big)=0,\end{aligned}\tag{20}$$

$$B\Big(\sqrt{g^{\theta\theta}}\partial_\theta I_\uparrow+\mathrm{i}\sqrt{g^{\phi\phi}}\partial_\phi I_\uparrow+\mathrm{i}F\partial_\psi I_\uparrow\Big)=0,\tag{21}$$

$$\begin{aligned}&-A\Big(\sqrt{-g^{\tau\tau}}(\partial_\tau I_\uparrow+eA_\tau)-\sqrt{g^{\rho\rho}}\partial_\rho I_\uparrow\Big)\\&+B\Big(G\partial_\psi I_\uparrow-m\Big)=0,\end{aligned}\tag{22}$$

$$A\Big(\sqrt{g^{\theta\theta}}\partial_\theta I_\uparrow+\mathrm{i}\sqrt{g^{\phi\phi}}\partial_\phi I_\uparrow+\mathrm{i}F\partial_\psi I_\uparrow\Big)=0.\tag{23}$$

From the above equations, we can find that the action can be separated. Because there are three Killing vectors $(\partial/\partial_\tau)^\mu$, $(\partial/\partial_\phi)^\mu$ and $(\partial/\partial_\psi)^\mu$ in 5D rotating Kaluza–Klein black hole with squashed horizon, we employ the following ansatz:

$$I_\uparrow=-\omega\tau+J\phi+L\psi+R(\rho,\theta)+K,\tag{24}$$

where ω, J and L are the emitted Dirac particle energy and angular momenta corresponding to angles ϕ and ψ, and K is a complex constant. Inserting ansatz (24) into Eqs. (20)–(23) (where we only consider the positive frequency contributions without loss of generality), we have

$$\begin{aligned}&B\Big((-\omega+eA_\tau)\sqrt{-g^{\tau\tau}}+\sqrt{g^{\rho\rho}}\partial_\rho R(\rho,\theta)\Big)\\&-A\Big(GL+m\Big)=0,\end{aligned}\tag{25}$$

$$B\Big(\sqrt{g^{\theta\theta}}\partial_\theta R(\rho,\theta)+\mathrm{i}\sqrt{g^{\phi\phi}}J+\mathrm{i}FL\Big)=0,\tag{26}$$

$$\begin{aligned}&-A\Big((-\omega+eA_\tau)\sqrt{-g^{\tau\tau}}-\sqrt{g^{\rho\rho}}\partial_\rho R(\rho,\theta)\Big)\\&+B\Big(GL-m\Big)=0,\end{aligned}\tag{27}$$

$$A\Big(\sqrt{g^{\theta\theta}}\partial_\theta R(\rho,\theta)+\mathrm{i}\sqrt{g^{\phi\phi}}J+\mathrm{i}FL\Big)=0.\tag{28}$$

From Eqs. (25) and (27), we can easily see that the two equations have a non-trivial solution for A and B if, and only if, the determinant of the coefficient matrix vanishes. Then we have

$$\begin{aligned}&\partial_\rho R(\rho,\theta)\\&=\pm\sqrt{\frac{(\omega-eA_\tau)^2-(-g^{\tau\tau})^{-1}(L^2G^2-m^2)}{g^{\rho\rho}(-g^{\tau\tau})^{-1}}}\\&=\pm\frac{\rho^2}{(\rho-\rho_+)(\rho-\rho_-)}\sqrt{\frac{\rho+\rho_0}{\rho}}\\&\quad\times\sqrt{(\omega-eA_\tau)^2-(-g^{\tau\tau})^{-1}(L^2G^2-m^2)}.\end{aligned}\tag{29}$$

From Eqs. (26) and (28), we obtain

$$\partial_\theta R(\rho,\theta)=-\mathrm{i}\frac{J\sqrt{g^{\phi\phi}}+LF}{\sqrt{g^{\theta\theta}}}.\tag{30}$$

Since the leading contribution comes from the event horizon, for simplicity we may use the near-horizon approximation and expand the equations near the horizon as follows:

$$(-g^{\tau\tau})^{-1}=(-g^{\tau\tau})^{-1}_{,\rho}(\rho_+)(\rho-\rho_+),\tag{31}$$

where

$$(-g^{\tau\tau})^{-1}_{,\rho}(\rho_+)=\left.\frac{\partial(-g^{\tau\tau})^{-1}}{\partial\rho}\right|_{\rho=\rho_+}.\tag{32}$$

Thus, we can obtain

$$\partial_\theta R(\rho,\theta)=-\mathrm{i}\frac{J\sqrt{g^{\phi\phi}}(\rho_+,\theta)+LF(\rho_+,\theta)}{\sqrt{g^{\theta\theta}(\rho_+,\theta)}},\tag{33}$$

and

$$\begin{aligned}&\partial_\rho R(\rho,\theta)\\&=\pm\frac{\rho^2}{(\rho-\rho_+)(\rho-\rho_-)}\sqrt{\frac{\rho+\rho_0}{\rho}}\\&\quad\times\sqrt{(\omega-eA_\tau)^2-(-g^{\tau\tau})^{-1}_{,\rho}(\rho-\rho_+)(L^2G^2-m^2)}.\end{aligned}\tag{34}$$

Near the event horizon ρ_+, $\partial_\theta R(\rho,\theta)$ has no explicit ρ dependence and $\partial_\rho R(\rho,\theta)$ is independent of θ. Therefore, near the horizon, $R(\rho,\theta)$ can be separated into $R(\rho,\theta)=R(\rho)+\Theta(\theta)$, which is corresponding to $\partial_\theta R(\rho,\theta)=\partial_\theta\Theta(\theta)$ and $\partial_\rho R(\rho,\theta)=\partial_\rho R(\rho)$. From Eq. (24), we can find that $R(\rho,\theta)$ and K will make contributions to the imaginary parts of the action. Now, we will focus on calculating $R(\rho)$ and $\Theta(\theta)$. Since the contributions from $\Theta(\theta)$ are completely the same for both the outgoing and the ingoing solutions, the contributions of $\Theta(\theta)$ to the imaginary parts of the action are canceled out. Solving the radial function $R(\rho)$ yields

$$R_{\pm}(\rho)=\pm\int\frac{\rho^2}{(\rho-\rho_+)(\rho-\rho_-)}\sqrt{\frac{\rho+\rho_0}{\rho}}\times\sqrt{(\omega-eA_\tau)^2-(-g^{\tau\tau})^{-1}_{,\rho}(\rho-\rho_+)(L^2G^2-m^2)}\,\mathrm{d}\rho. \quad (35)$$

So we obtain the function $R(\rho)$

$$R_{\pm}(\rho)=\pm\pi\mathrm{i}\sqrt{\frac{\rho_++\rho_0}{\rho_+}}\frac{\rho_+^2}{\rho_+-\rho_-}(\omega-e\Phi_+), \quad (36)$$

where $\Phi_+=A_\tau(\rho_+)$. One solution $R_+(\rho)$ corresponds to the Dirac particle moving away from the outer event horizon (i.e., + outgoing), and the other solution corresponds to the Dirac particle moving towards the outer event horizon (i.e., − ingoing). Imaginary parts of the action can only come from the pole at the event horizon or from the imaginary parts of K. So when a particle crosses the event horizon each way, the outgoing and the ingoing probabilities are respectively given by

$$\begin{aligned}P_{\mathrm{out}}&=\exp\left(-\frac{2}{\hbar}\mathrm{Im}\,I_\uparrow\right)\\&=\exp\left[-\frac{2}{\hbar}(\mathrm{Im}R_++\mathrm{Im}K)\right],\\P_{\mathrm{in}}&=\exp\left(-\frac{2}{\hbar}\mathrm{Im}\,I_\uparrow\right)\\&=\exp\left[-\frac{2}{\hbar}(\mathrm{Im}R_-+\mathrm{Im}K)\right].\end{aligned} \quad (37)$$

To ensure that the probability is normalized so that any particle classically can completely enter the black hole, the probability for incoming particle crossing the horizon should be in unity. In our case, we order $\mathrm{Im}K=-\mathrm{Im}R_-$. Since $\mathrm{Im}R_+=-\mathrm{Im}R_-$, the tunneling probability of a Dirac particle crossing from inside to outside the event horizon becomes

$$\begin{aligned}\Gamma&=\exp[-4\mathrm{Im}R_+(\rho)]\\&=\exp\left[-\frac{4\pi\rho_+^2}{\rho_+-\rho_-}\sqrt{\frac{\rho_++\rho_0}{\rho_+}}\right](\omega-e\Phi_+).\end{aligned} \quad (38)$$

From the tunneling probability, the Hawking temperature of the black hole can be determined as

$$T=\frac{\rho_+-\rho_-}{4\pi\rho_+^2}\sqrt{\frac{\rho_+}{\rho_++\rho_0}}. \quad (39)$$

This result is exactly consistent with that in Refs. [40] and [41].

3. Fermion tunneling from a squashed black hole in the Gödel universe

The metric of the Schwarzschild–Gödel black hole with a squashed horizon is described by[42]

$$\begin{aligned}&\mathrm{d}s^2\\&=-k(r)\mathrm{d}t^2+\frac{\chi^2(r)}{V(r)}\mathrm{d}r^2+\frac{r^2}{4}\chi(r)\mathrm{d}\theta^2-2g(r)\mathrm{d}t\mathrm{d}\psi\\&+\left(h(r)\cos^2\theta+\frac{r^2}{4}\chi(r)\sin^2\theta\right)\mathrm{d}\phi^2+h(r)\mathrm{d}\psi^2\\&-2g(r)\cos\theta\mathrm{d}t\mathrm{d}\phi+2h(r)\cos\theta\mathrm{d}\phi\mathrm{d}\psi,\end{aligned} \quad (40)$$

where

$$\begin{aligned}&k(r)=1-\frac{2m}{r^2},\qquad g(r)=jr^2,\\&h(r)=-j^2r^2(r^2+2m),\\&\chi(r)=\frac{c^2+2cm(1-8j^2m)}{(r^2+c)^2},\\&V(r)=1-\frac{2m}{r^2}+\frac{16j^2m^2}{r^2}.\end{aligned} \quad (41)$$

Here $c=-r_0^2$ is a real constant and controls the squashing of the three-sphere. The black hole horizon is

$$r_{\mathrm{H}}^2=2m(1-8j^2m). \quad (42)$$

The inverse metrics are

$$\begin{aligned}&\boldsymbol{g}^{tt}=-\frac{h(r)}{k(r)h(r)+g^2(r)},\quad \boldsymbol{g}^{rr}=\frac{V(r)}{\chi^2(r)},\\&\boldsymbol{g}^{\theta\theta}=\frac{4}{r^2\chi(r)},\quad \boldsymbol{g}^{\phi\phi}=\frac{4}{r^2\chi(r)\sin^2\theta},\\&\boldsymbol{g}^{t\phi}=-\frac{g(r)}{k(r)h(r)+g^2(r)},\quad \boldsymbol{g}^{\phi\psi}=-\frac{4\cos\theta}{r^2\chi(r)\sin^2\theta},\\&\boldsymbol{g}^{\psi\psi}=\frac{4\cos^2\theta}{r^2\chi(r)\sin^2\theta}+\frac{k(r)}{k(r)h(r)+g^2(r)}.\end{aligned} \quad (43)$$

In this section, we choose the following matrices $\boldsymbol{\gamma}^\mu$ form for the Schwarzschild–Gödel black hole with squashed horizon:

$$\boldsymbol{\gamma}^t=\sqrt{-\boldsymbol{g}^{tt}}\begin{pmatrix}0&\boldsymbol{I}\\-\boldsymbol{I}&0\end{pmatrix},\quad \boldsymbol{\gamma}^r=\sqrt{\boldsymbol{g}^{rr}}\begin{pmatrix}0&\boldsymbol{\sigma}^3\\\boldsymbol{\sigma}^3&0\end{pmatrix},$$

$$\boldsymbol{\gamma}^\theta=\sqrt{\boldsymbol{g}^{\theta\theta}}\begin{pmatrix}0&\boldsymbol{\sigma}^1\\\boldsymbol{\sigma}^1&0\end{pmatrix},\quad \boldsymbol{\gamma}^\phi=\sqrt{\boldsymbol{g}^{\phi\phi}}\begin{pmatrix}0&\boldsymbol{\sigma}^2\\\boldsymbol{\sigma}^2&0\end{pmatrix},$$

$$\boldsymbol{\gamma}^{\psi} = H \begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix} + F \begin{pmatrix} 0 & \boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^2 & 0 \end{pmatrix} + G \begin{pmatrix} -\boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix}, \tag{44}$$

where

$$H = \sqrt{-\frac{(\boldsymbol{g}^{\tau\psi})^2}{\boldsymbol{g}^{\tau\tau}}}, \quad F = -\frac{\boldsymbol{g}^{\phi\psi}}{\sqrt{\boldsymbol{g}^{\phi\phi}}},$$
$$G = \sqrt{\boldsymbol{g}^{\psi\psi} - \frac{(\boldsymbol{g}^{\tau\psi})^2}{\boldsymbol{g}^{\tau\tau}} - \frac{(\boldsymbol{g}^{\phi\psi})^2}{\boldsymbol{g}^{\phi\phi}}}. \tag{45}$$

Taking the same method, we can obtain the function $R(r)$

$$R(r) = \pm\pi \mathrm{i} \frac{r_0^2\sqrt{2m(1-8j^2m)^3}}{2[r_0^2 - 2m(1-8j^2m)]}. \tag{46}$$

The tunneling probability for a Dirac particle crossing from inside to outside the event horizon becomes

$$\Gamma = \exp\left[\frac{-2\pi r_0^2\sqrt{2m(1-8j^2m)^3}}{r_0^2 - 2m(1-8j^2m)}\right]\omega. \tag{47}$$

From the tunneling probability, the Hawking temperature of the black hole can be determined as

$$T = \frac{r_0^2 - 2m(1-8j^2m)}{2\pi r_0^2\sqrt{2m(1-8j^2m)^3}}. \tag{48}$$

As a result, the Hawking temperature of the black hole is successfully recovered.

4. Conclusion

It should be pointed out that for spacetimes with different horizon topologies and different dimensions, choosing appropriate $\boldsymbol{\gamma}^{\mu}$ matrices for the general covariant Dirac equation is an important technique for the fermion tunneling method, or we can not correctly recover teh Hawking temperature. In this paper, we construct two sets of $\boldsymbol{\gamma}^{\mu}$ matrices for the squashed black holes in the Gödel universe and charged Kaluza–Klein space–time, and recover the Hawking temperature by calculating the tunneling probability. The result shows that both scalar particles and Dirac particles with spin tunnel cross the horizon at the same Hawking temperature.

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