A meshless method for the nonlinear generalized regularized long wave equation*

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This paper presents a meshless method for the nonlinear generalized regularized long wave (GRLW) equation based on the moving least-squares approximation. The nonlinear discrete scheme of the GRLW equation is obtained and is solved using the iteration method. A theorem on the convergence of the iterative process is presented and proved using theorems of the infinity norm. Compared with numerical methods based on mesh, the meshless method for the GRLW equation only requires the scattered nodes instead of meshing the domain of the problem. Some examples, such as the propagation of single soliton and the interaction of two solitary waves, are given to show the effectiveness of the meshless method.

Keywords: generalized regularized long wave equation, meshless method, moving least-squares approximation, convergence

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1. Introduction

The generalized regularized long wave (GRLW) equation

$$u_t + u_x + \alpha(u^p)_x - \beta u_{xxt} = 0, \tag{1}$$

where p is a positive integer, and α and β are positive constants, was proposed many years ago as a model for small-amplitude long waves on the surface of water by Peregrine.[1,2] The GRLW equation is an alternative to the Korteweg–de Vries (KdV) equation for describing nonlinear dispersive waves and can be used to describe phenomena with weak nonlinearity and dispersion waves, including nonlinear transverse waves in shallow water, ion-acoustic and magneto hydrodynamic waves in plasma and phonon packets in nonlinear crystals.

For $p = 2$, Eq. (1) reduces to a special case,

$$u_t + u_x + \varepsilon u u_x - \beta u_{xxt} = 0, \tag{2}$$

which is called the regularized long wave (RLW) equation, where $\varepsilon = 2\alpha$. Equation (2) has been studied extensively in the past both analytically and numerically by means of, for example, the Adomian decomposition method, the Fourier method, the finite difference method, teh finite element method, the variational iteration method, the meshless method and the cubic B-spline collocation procedure,[3−8] etc.

For $p \geq 3$, however, presumably because of its higher nonlinearity, the GRLW equation has been seldom studied before. Some previous theoretical studies on the GRLW equation were carried out. In Ref. [9], authors studied the stability of its solutions and found that it depends on the velocity. Hamdi and Enright derived some exact solitary wave solutions for it.[10] However, in many cases, it is very difficult to find the analytical solutions. The numerical method for the approximate solution of the GRLW equation is very important. Soliman obtained the solution for a short GRLW equation based on the variational iteration method.[11] Zhang presented a finite difference method for a GRLW equation Cauchy problem.[12] Ramos studied solitary wave interactions based on the separation of the temporal and spatial derivatives.[13] Kaya implemented the Adomian decomposition method to obtain the numerical solution of the GRLW equation.[14]

In recent years, meshless methods have attracted a lot of attention from researchers and have gained considerable importance for the numerical solutions to various types of engineering problems which are not suited to conventional numerical methods, such as the finite element method and the boundary element method.[15] Based on different approximation functions, many kinds of meshless methods have been

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proposed, such as the smooth particle hydrodynamics method,[16] the radial basis function method,[17,18] the element-free Galerkin (EFG) method,[19−22] the meshless local Petrov–Galerkin method,[23,24] the finite point method,[25,26] the meshless local boundary integral equation method,[27] the reproducing kernel particle method,[28] the boundary element-free method,[29−31] the complex variable meshless method,[32−36] and the meshless point collocation method.[37,38]

Now a few nonlinear problems are solved with meshless methods. It is very important to develop the meshless method for nonlinear problems. To the best of our knowledge, the meshless method based on moving least-squares (MLS) approximation has not been used for the GRLW equation. The aim of this study is to obtain the numerical solution of the GRLW equation by the meshless method.

This paper is organized as follows. In Section 2, we introduce the method of MLS approximation. In Section 3, the governing equation with initial and boundary conditions are given, the meshless method based on the MLS approximation is presented and the discrete scheme of the problem is obtained. In Section 4, a theorem on the convergence of the iterative process is presented and proved using the theorems of the infinity norm. In Section 5, we give several examples which are solved by using the meshless method in this study and the numerical results of these examples are in good agreement with the exact solutions. Section 6 ends this paper with a brief conclusion.

2. The MLS approximation

Let $u(\boldsymbol{x})$ be the function of the field variable defined in the domain Ω. The approximation of the function $u(\boldsymbol{x})$ is

$$u^h(\boldsymbol{x}) = \sum_{i=1}^{m} p_i(\boldsymbol{x}) a_i(\boldsymbol{x}) \equiv \boldsymbol{P}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{a}(\boldsymbol{x}), \tag{3}$$

where $p_i(\boldsymbol{x})$ is a monomial basis function, $a_i(\boldsymbol{x})$ is the coefficient of the basis, m is the number of terms in the basis and

$$\boldsymbol{P}^{\mathrm{T}}(\boldsymbol{x}) = (p_1(\boldsymbol{x}), p_2(\boldsymbol{x}), \ldots, p_m(\boldsymbol{x})), \tag{4}$$

$$\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{x}) = (a_1(\boldsymbol{x}), a_2(\boldsymbol{x}), \ldots, a_m(\boldsymbol{x})). \tag{5}$$

In order to obtain the expression of $u^h(\boldsymbol{x})$, Lancaster and Salkauskas[39] defined a local approximation,

$$u^h(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \sum_{i=1}^{m} p_i(\bar{\boldsymbol{x}}) a_i(\boldsymbol{x}) \equiv \boldsymbol{P}^{\mathrm{T}}(\bar{\boldsymbol{x}}) \boldsymbol{a}(\boldsymbol{x}), \tag{6}$$

where $\bar{\boldsymbol{x}}$ is the point in the local approximation of $\boldsymbol{x}$.

To precisely obtain the local approximation of the function $u(\boldsymbol{x})$, we must minimize the difference between the local approximation $u^h(\boldsymbol{x}, \bar{\boldsymbol{x}})$ and the function $u(\boldsymbol{x})$ by a weighted least-squares method.

Define the weighted discrete L_2 norm as

$$\begin{aligned} J &= \sum_{I=1}^{N} w(\boldsymbol{x} - \boldsymbol{x}_I)[u^h(\boldsymbol{x}, \boldsymbol{x}_I) - u_I]^2 \\ &= \sum_{I=1}^{N} w(\boldsymbol{x} - \boldsymbol{x}_I) \left[\sum_{i=1}^{m} p_i(\boldsymbol{x}_I) a_i(\boldsymbol{x}) - u_I \right]^2, \end{aligned} \tag{7}$$

where $w(\boldsymbol{x} - \boldsymbol{x}_I)$ is a weight function with a domain of influence, $\boldsymbol{x}_I$ $(I = 1, 2, \ldots, N)$ are the nodes with domains of influence which cover the point $\boldsymbol{x}$, and u_I refers to the nodal parameter of $u(\boldsymbol{x})$ at $\boldsymbol{x}_I$.

From $\partial J / \partial a_i = 0$ $(i = 1, 2, \ldots, m)$, we can obtain

$$\boldsymbol{A}(\boldsymbol{x}) \boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x}) \boldsymbol{U}, \tag{8}$$

where

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{I=1}^{N} w(\boldsymbol{x} - \boldsymbol{x}_I) \boldsymbol{P}(\boldsymbol{x}_I) \boldsymbol{P}^{\mathrm{T}}(\boldsymbol{x}_I), \tag{9}$$

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{x}) = {} & (w(\boldsymbol{x} - \boldsymbol{x}_1) \boldsymbol{P}(\boldsymbol{x}_1), w(\boldsymbol{x} - \boldsymbol{x}_2) \\ & \times \boldsymbol{P}(\boldsymbol{x}_2), \ldots, w(\boldsymbol{x} - \boldsymbol{x}_N) \boldsymbol{P}(\boldsymbol{x}_N)), \end{aligned} \tag{10}$$

$$\boldsymbol{U}^{\mathrm{T}} = (u_1, u_2, \ldots, u_N). \tag{11}$$

Then the approximate function of $u(\boldsymbol{x})$ can be obtained as

$$\begin{aligned} u^h(\boldsymbol{x}) &= u^h(\boldsymbol{x}, \boldsymbol{x}) = \sum_{i=1}^{m} p_i(\boldsymbol{x}) a_i(\boldsymbol{x}) \\ &= \sum_{I=1}^{N} \phi_I(\boldsymbol{x}) u_I \equiv \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{U}, \end{aligned} \tag{12}$$

where $\boldsymbol{\Phi}$ is a matrix of shape function defined as

$$\begin{aligned} \boldsymbol{\Phi}^{\mathrm{T}} &= (\phi_1(\boldsymbol{x}), \phi_2(\boldsymbol{x}), \ldots, \phi_N(\boldsymbol{x})) \\ &= \boldsymbol{P}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{A}^{-1}(\boldsymbol{x}) \boldsymbol{B}(\boldsymbol{x}). \end{aligned} \tag{13}$$

The derivatives of the unknown $u(\boldsymbol{x})$ are approximated by

$$\frac{\partial^k}{\partial \boldsymbol{x}^k} u^h(\boldsymbol{x}) = \sum_{I=1}^{N} \left(\frac{\partial^k}{\partial \boldsymbol{x}^k} \phi_I(\boldsymbol{x}) \right) u_I. \tag{14}$$

The cubic spline weight function used in the present analysis can be written as a function of normalized radius r as follows:

$$w(\boldsymbol{x}-\boldsymbol{x}_I)\equiv w(r)=\begin{cases}2/3-4r^2+4r^3, & (r\le 1/2),\\ 4/3-4r+4r^2-4r^3/3, & (1/2<r\le 1),\\ 0, & (r>1),\end{cases} \tag{15}$$

where

$$r=d_I/d_{\rm m},\quad d_I=||\boldsymbol{x}-\boldsymbol{x}_I||,\quad d_{\rm m}=d_{\max}\cdot c_I, \tag{16}$$

and $d_{\max}$ is a scaling parameter, the distance c_I are chosen so that the moment matrix $\boldsymbol{A}(\boldsymbol{x})$ is not singular.

3. The meshless method for the GRLW equation

The GRLW equation in Cartesian coordinates can be written as

$$u_t+u_x+\alpha(u^p)_x-\beta u_{xxt}=0, \tag{17}$$

where $p\ge 2$ is a positive integer, α and β are positive constants.

The boundary and initial conditions are

$$u(a,t)=\varphi_1(t),\quad u(b,t)=\varphi_2(t), \tag{18}$$

$$u(x,0)=\psi(x). \tag{19}$$

Now the meshless method based on the MLS approximation can be implemented to obtain the discrete equations of the problem. We discretize the time derivative by using the usual finite difference method. From Eq. (17), we have

$$\frac{u^{(m+1)}-u^{(m)}}{\Delta t}+\frac{u_x^{(m+1)}+u_x^{(m)}}{2} +\alpha\frac{(u^p)_x^{(m+1)}+(u^p)_x^{(m)}}{2}-\beta\frac{u_{xx}^{(m+1)}-u_{xx}^{(m)}}{\Delta t}=0, \tag{20}$$

where

$$u^{(m)}=u(x,t_m),\quad u_x^{(m)}=\frac{\partial}{\partial x}u(x,t_m),$$

$$u_{xx}^{(m)}=\frac{\partial^2}{\partial x^2}u(x,t_m).$$

Equation (20) can be expressed as

$$L(u^{(m+1)})+\alpha p\Delta t(u^{(m+1)})^{p-1}u_x^{(m+1)}=Z(u^{(m)}), \tag{21}$$

where

$$\begin{aligned}L(u^{(m+1)}) &= 2u^{(m+1)}+\Delta t u_x^{(m+1)}-2\beta u_{xx}^{(m+1)},\\ Z(u^{(m)}) &= 2u^{(m)}-\Delta t u_x^{(m)}-2\beta u_{xx}^{(m)}\\ &\quad -\alpha p\Delta t(u^{(m)})^{p-1}u_x^{(m)}.\end{aligned}$$

Let $\{a=x_1<\cdots<x_N=b\}$ be the partition in the interval $[a,b]$. Depending on the expressions of the approximation functions Eqs. (12) and (14), we let

$$\begin{aligned}(u_I^h)^{(m)} &= u^h(x_I,t_m)\\ &= \sum_{k=1}^{N}\phi_k(x_I)u_k(t_m)\equiv \boldsymbol{\Phi}_I^{\rm T}\boldsymbol{U}^{(m)}, \qquad (22)\\ (u_I^h)_x^{(m)} &= \frac{\partial}{\partial x}u^h(x_I,t_m)\\ &= \sum_{k=1}^{N}\frac{\partial}{\partial x}\phi_k(x_I)u_k(t_m)\equiv (\boldsymbol{\Phi}_I)_x^{\rm T}\boldsymbol{U}^{(m)}, \qquad (23)\\ (u_I^h)_{xx}^{(m)} &= \frac{\partial^2}{\partial x^2}u^h(x_I,t_m)\\ &= \sum_{k=1}^{N}\frac{\partial^2}{\partial x^2}\phi_k(x_I)u_k(t_m)\equiv (\boldsymbol{\Phi}_I)_{xx}^{\rm T}\boldsymbol{U}^{(m)}, \qquad (24)\end{aligned}$$

where $\phi_k(x_I)$ is determined by Eq. (13), $u_k(t_m)$ refers to the nodal parameter of $u(x,t)$ at the node x_k and the time t_m.

From Eqs. (21) and (18), we can obtain

$$L(u_I^{(m+1)})+\alpha p\Delta t(u^{(m+1)})^{p-1}u_x^{(m+1)}|_{x=x_I}=Z(u_I^{(m)}),\quad (I=2,3,\ldots,N-1), \tag{25}$$

$$u_1^{(m+1)}=\varphi_1(t_{m+1}),\quad u_N^{(m+1)}=\varphi_2(t_{m+1}). \tag{26}$$

Substituting Eqs. (22)–(24) into Eqs. (25) and (26) yields

$$L^h((u_I^h)^{(m+1)})+\alpha p\Delta t((u_I^h)^{(m+1)})^{p-1}(u_I^h)_x^{(m+1)}=Z^h((u_I^h)^{(m)}),\quad (I=2,3,\ldots,N-1), \tag{27}$$

$$u^h(x_1,t_{m+1})=\varphi_1(t_{m+1}),u^h(x_N,t_{m+1})=\varphi_2(t_{m+1}), \tag{28}$$

where

$$L^h((u_I^h)^{(m+1)})=2(u_I^h)^{(m+1)}+\Delta t(u_I^h)_x^{(m+1)}-2\beta(u_I^h)_{xx}^{(m+1)}, \tag{29}$$

$$Z^h((u_I^h)^{(m)})=2(u_I^h)^{(m)}-\Delta t(u_I^h)_x^{(m)}-2\beta(u_I^h)_{xx}^{(m)}-\alpha p\Delta t((u_I^h)^{(m)})^{p-1}(u_I^h)_x^{(m)}, \tag{30}$$

where the values of $(u_I^h)^{(m)}$, $(u_I^h)_x^{(m)}$ and $(u_I^h)_{xx}^{(m)}$ are determined by Eqs. (22)–(24), respectively.

Equations (27) and (28) can be expressed as follows:

$$\begin{aligned} &\boldsymbol{K}\boldsymbol{U}^{(m+1)} + \boldsymbol{H}(\boldsymbol{U}^{(m+1)}) \\ &= \boldsymbol{F}\boldsymbol{U}^{(m)} - \boldsymbol{H}(\boldsymbol{U}^{(m)}) + \boldsymbol{G}, \end{aligned} \tag{31}$$

which is the discrete scheme of the GRLW equation. Here $\boldsymbol{K}$ is a $N \times N$ matrix, whose first and last rows are, respectively, $\boldsymbol{\Phi}_1^{\mathrm{T}}$ and $\boldsymbol{\Phi}_N^{\mathrm{T}}$, and the ith $(2 \le i \le N-1)$ row is $2\boldsymbol{\Phi}_I^{\mathrm{T}} + \Delta t(\boldsymbol{\Phi}_I)_x^{\mathrm{T}} - 2\beta(\boldsymbol{\Phi}_I)_{xx}^{\mathrm{T}}$; $\boldsymbol{H}(\boldsymbol{U}^{(m+1)})$ and $\boldsymbol{H}(\boldsymbol{U}^{(m)})$ are similar to each other, and $\boldsymbol{H}(\boldsymbol{U}^{(m+1)})$ is a $N \times 1$ matrix, whose first and last rows are zero, the ith $(2 \le i \le N-1)$ row is $\alpha p \Delta t(\boldsymbol{\Phi}_I^{\mathrm{T}}\boldsymbol{U}^{(m+1)})^{p-1}(\boldsymbol{\Phi}_I)_x^{\mathrm{T}}\boldsymbol{U}^{(m+1)}$; $\boldsymbol{F}$ is a $N \times N$ matrix, whose first and last rows are zero, and the ith $(2 \le i \le N-1)$ row is $2\boldsymbol{\Phi}_I^{\mathrm{T}} - \Delta t(\boldsymbol{\Phi}_I)_x^{\mathrm{T}} - 2\beta(\boldsymbol{\Phi}_I)_{xx}^{\mathrm{T}}$; $\boldsymbol{G}$ is a $N \times 1$ matrix, whose first and last rows are, respectively, $\varphi_1(t_{m+1})$ and $\varphi_2(t_{m+1})$, and the others are zero.

When the value of $\boldsymbol{U}^{(m)}$ is given, because there are nonlinear terms in the expression of $\boldsymbol{H}(\boldsymbol{U}^{(m+1)})$, Eq. (31) is a nonlinear system for unknown $\boldsymbol{U}^{(m+1)}$ and cannot be solved directly. In order to obtain the value of $\boldsymbol{U}^{(m+1)}$, we will solve Eq. (31) with the method of iteration.

The iterative steps to obtain the solution are as follows:

Step 1 Let $\boldsymbol{U}^{(m+1)} = \boldsymbol{U}^{(m)}$ (here, we suppose that the value of $\boldsymbol{U}^{(m)}$ has been known from the initial condition when $m = 0$). Substituting it into the expression of $\boldsymbol{H}(\boldsymbol{U}^{(m+1)})$, we can compute the value of $\boldsymbol{H}(\boldsymbol{U}^{(m+1)})$. Then Eq. (31) reduces to a system of linear algebraic equations, from which we can easily obtain the solution of $\boldsymbol{U}^{(m+1)}$. Now the solution is denoted by the expression of $\boldsymbol{U}_1^{(m+1)}$.

Step 2 Let $\boldsymbol{U}^{(m+1)} = \boldsymbol{U}_1^{(m+1)}$. We also substitute it into the expression of $\boldsymbol{H}(\boldsymbol{U}^{(m+1)})$, and Eq. (31) becomes a linear system, whose solution can be similarly obtained and denoted by $\boldsymbol{U}_2^{(m+1)}$.

Step 3 Repeating the above process, until the successive solutions $\boldsymbol{U}_k^{(m+1)}$ and $\boldsymbol{U}_{k+1}^{(m+1)}$ converge to a prescribed number of significant figures, we stop the iteration. For example, we can use the following condition for stopping the iterations at each time level,

$$||\boldsymbol{U}_{k+1}^{(m+1)} - \boldsymbol{U}_k^{(m+1)}|| < \varepsilon, \tag{32}$$

where ε is a given constant.

Step 4 When the condition (32) is satisfied, we let $\boldsymbol{U}^{(m+1)} = \boldsymbol{U}_{k+1}^{(m+1)}$, which is regarded as the final solution of Eq. (31).

Now the algebraic equations can be solved to obtain the solution at new time levels from Eq. (31). Before we proceed, we shall discuss the convergence of the above iterative process.

4. Convergence of the iterative process

Let $\boldsymbol{S}_I = 2\boldsymbol{\Phi}_I - 2\beta(\boldsymbol{\Phi}_I)_{xx}$, $I = 1, 2, \ldots, N$. We can prove that the iterative process is convergent and has the following theorem.

Theorem If the $N \times N$ matrix $K_0 = (\boldsymbol{\Phi}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_{N-1}, \boldsymbol{\Phi}_N)$ is invertible, then there is a constant M such that if the time step $\Delta t < 1/M$, the iterative process is convergent.

Proof Let $\varepsilon_k = \boldsymbol{U}_{k+1}^{(m+1)} - \boldsymbol{U}_k^{(m+1)}$, where $\boldsymbol{U}_k^{(m+1)}$ is the solution of the iterative process at the kth step. Now we will prove that when the time step Δt is sufficiently small, there exists $||\varepsilon_k||_\infty < ||\varepsilon_{k-1}||_\infty$.

From Eq. (31), we have

$$\boldsymbol{K}\varepsilon_k = \boldsymbol{H}(\boldsymbol{U}_{k-1}^{(m+1)}) - \boldsymbol{H}(\boldsymbol{U}_k^{(m+1)}),$$

i.e.,

$$\varepsilon_k = \boldsymbol{K}^{-1}[\boldsymbol{H}(\boldsymbol{U}_{k-1}^{(m+1)}) - \boldsymbol{H}(\boldsymbol{U}_k^{(m+1)})]. \tag{33}$$

Taking the infinity norm from Eq. (33), we have

$$||\varepsilon_k||_\infty \le ||\boldsymbol{K}^{-1}||_\infty ||\boldsymbol{H}(\boldsymbol{U}_{k-1}^{(m+1)}) - \boldsymbol{H}(\boldsymbol{U}_k^{(m+1)})||_\infty.$$

Depending on the mean value theorem, we find that there exists $\xi = \theta\boldsymbol{U}_k^{(m+1)} + (1-\theta)\boldsymbol{U}_{k-1}^{(m+1)}$ $(0 \le \theta \le 1)$ such that

$$||\varepsilon_k||_\infty \le ||\boldsymbol{K}^{-1}||_\infty ||\boldsymbol{J}(\xi)||_\infty ||\varepsilon_{k-1}||_\infty, \tag{34}$$

where $\boldsymbol{J}(\xi)$ is Jacobian matrix computed from the matrix $\boldsymbol{H}(\boldsymbol{U}_k^{(m)})$ and whose first and last rows are zero, the ith $(2 \le i \le N-1)$ row is

$$\begin{aligned} &\alpha p \Delta t[(\boldsymbol{\Phi}_I^{\mathrm{T}}\xi)^{p-1}(\boldsymbol{\Phi}_I)_x^{\mathrm{T}} \\ &+(p-1)(\boldsymbol{\Phi}_I^{\mathrm{T}}\xi)^{p-2}((\boldsymbol{\Phi}_I)_x^{\mathrm{T}}\xi)\boldsymbol{\Phi}_I^{\mathrm{T}}]. \end{aligned}$$

On account of the boundness of the functions $(u_I^h)^{(m)}$, $(u_I^h)_x^{(m)}$, $\boldsymbol{\Phi}_I^{\mathrm{T}}$ and $(\boldsymbol{\Phi}_I)_x^{\mathrm{T}}$, there exists a constant M_1 such that

$$||\boldsymbol{J}(\xi)||_\infty \le M_1 \Delta t. \tag{35}$$

According to the definition of the matrix $\boldsymbol{K}$, we can obtain

$$\lim_{\Delta t \to 0} \boldsymbol{K}^{\mathrm{T}} = \boldsymbol{K}_0 = (\boldsymbol{\Phi}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_{N-1}, \boldsymbol{\Phi}_N).$$

If $\boldsymbol{K}_0$ is invertible, there exists the infinity norm $||\boldsymbol{K}_0^{-1}||_\infty$. Since the infinity norm is continuous on the elements of the matrix, $||\boldsymbol{K}^{-1}||_\infty$ is certainly bounded as $\Delta t \to 0$, i.e., when the time step Δt is sufficiently small, there exists a non-zero constant M_2 such that

$$||\boldsymbol{K}^{-1}||_\infty < M_2. \tag{36}$$

From Eqs. (34)–(36), we can know that if time step satisfies $\Delta t < 1/M$, and $M = M_2 M_1$, then the norm also satisfies

$$\begin{aligned} ||\varepsilon_k||_\infty &\leq ||K^{-1}||_\infty M_1 \Delta t ||\varepsilon_{k-1}||_\infty \\ &< M_2 M_1 \Delta t ||\varepsilon_{k-1}||_\infty < ||\varepsilon_{k-1}||_\infty. \end{aligned}$$

Therefore, the iterative process is bound to convergent. Then the proof of the above theorem is finished.

5. Numerical examples

In this section, we will present some numerical examples to show the accuracy of the method proposed in this paper. The accuracy of the method is measured by using differences between the analytical and numerical results at the node points by way of calculation of the discrete L_2 and L_∞ error norms

$$L_2 = \sqrt{h \sum_{I=1}^{N} (u_I - u_I^h)^2}, \tag{37}$$

$$L_\infty = \max_{1 \leq I \leq N} \{|u_I - u_I^h|\}, \tag{38}$$

where u_I and u_I^h are the exact and numerical solutions at $x = x_I$, respectively.

Firstly, we consider one solitary wave motion; secondly we study the interaction of two solitary waves. In our examples, a regular arrangement of nodes is used. The weight function is chosen to be the cubic spline weight function and the basis functions are chosen as a complete polynomial of order 3.

5.1. Single soliton

Example 1 We choose the parameter $p = 2$, i.e., the GRLW equation has the form

$$u_t + u_x + 2\alpha u u_x - \beta u_{xxt} = 0, \tag{39}$$

which is called the RLW equation and has the solitary wave solution as follows:[1]

$$u(x,t) = 3c\,\mathrm{sech}^2[k(\boldsymbol{x} - \boldsymbol{x}_0 - vt)], \tag{40}$$

which represents the motion of a single solitary wave with amplitude $3c$, where $v = (1 + 2\alpha c)$ is the wave velocity and

$$k = \frac{1}{2}\sqrt{\frac{2\alpha c}{\beta(1 + 2\alpha c)}}.$$

The initial and boundary conditions are computed from the exact solution Eq. (40).

We use the method presented in this paper to solve Eq. (39). The computation is performed with $[a, b] = [-40, 60]$, $\alpha = 1/2$, $\beta = 1$, $c = 0.1$, $x_0 = 0$, $\Delta t = 0.2$, and a regular arrangement of nodes with space step $h = 0.4$. The simulation is run to time $t = 20$ and the numerical results with the meshless method in this paper are shown in Table 1. In Fig. 1, we plot the motion of the solitary waves from $t = 0$ to $t = 20$ at intervals of $t = 4$, which shows that a single soliton with amplitude $3c$ is moving from left to right at a constant speed. From the numerical results, we can find that they are in good agreement with the exact solutions in this case.

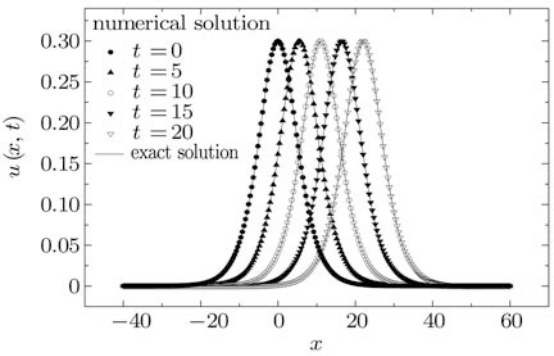


Fig. 1. The motion of the solitary waves from $t = 0$ to $t = 20$ at intervals of $t = 4$ for Example 1.

Table 1. The values of the error norms at different times for Example 1.

t	2	4	6	8	10	12	14	16	18	20
$L_2 \times 10^3$	0.070	0.139	0.205	0.268	0.329	0.391	0.447	0.493	0.530	0.579
$L_\infty \times 10^3$	0.026	0.052	0.075	0.093	0.106	0.114	0.118	0.119	0.119	0.129

Example 2 In this example, we consider the case with $p \geq 3$. The GRLW equation has the solitary wave solution

$$u(x,t) = a\{\mathrm{sech}^2[k(x + x_0 - ct)]\}^{1/(p-1)}, \quad (41)$$

where $k = \frac{p-1}{2\beta}\sqrt{(c-1)/c}$, $a = [(p+1)(c-1)/(2\alpha)]^{1/(p-1)}$.

We will give the numerical solution with the initial and boundary conditions extracted from the exact solution Eq. (41). The computation is carried out with $[a,b] = [-60,\ 60]$, $\alpha = 1$, $\beta = 1$, $c = 1.01$, $x_0 = 10$, $\Delta t = 0.1$, space step $h = 0.48$. In Table 2, we show the discrete L_2 and L_∞ error norms with different values of t and p. The numerical and exact solutions are also given in Fig. 2 at time $t = 20$.

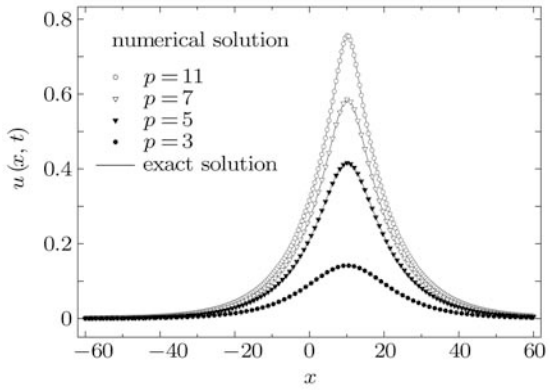


Fig. 2. The motion of the solitary waves at $t = 20$ with different values of p for Example 2.

Table 2. The values of the error norms at different values of t and p for Example 2.

	t	2	4	6	8	10
$p = 3$	$L_2 \times 10^3$	0.07	0.13	0.17	0.19	0.19
	$L_\infty \times 10^3$	0.02	0.03	0.04	0.04	0.05
$p = 5$	$L_2 \times 10^3$	0.16	0.30	0.40	0.46	0.50
	$L_\infty \times 10^3$	0.04	0.07	0.09	0.09	0.10
$p = 7$	$L_2 \times 10^3$	0.32	0.61	0.84	1.02	1.17
	$L_\infty \times 10^3$	0.12	0.21	0.27	0.32	0.37
$p = 9$	$L_2 \times 10^3$	0.63	1.18	1.61	1.95	2.24
	$L_\infty \times 10^3$	0.29	0.48	0.61	0.70	0.78
$p = 11$	$L_2 \times 10^3$	1.09	1.97	2.64	3.18	3.65
	$L_\infty \times 10^3$	0.55	0.85	1.07	1.23	1.36

From the numerical results, it is evident that the present results are in excellent agreement with the analytical solutions. Although the higher nonlinearity at $p = 3, 5, \ldots, 11$, the L_2 and L_∞ error norms for these simulations are satisfactorily small at all times, where $L_\infty = 1.36 \times 10^{-3}$ and $L_2 = 3.65 \times 10^{-3}$ at $t = 10$ and $p = 11$.

5.2. Interaction of two solitary waves

In this section, we study the interaction of two solitary waves with different amplitudes and traveling in the same direction. We consider the GRLW equation with an initial condition given by the linear sum of two well-separated solitary waves of various amplitudes.

Example 3 In this example, the initial condition extracted from Eq. (40) is

$$u(x,0) = \sum_{j=1}^{2} 3c_j \,\mathrm{sech}^2[k_j(x + x_{0j} - v_j t)],$$

where x_{0j} are arbitrary constants, $v_j = (1 + 2\alpha c_j)$, $k_j = \frac{1}{2}\sqrt{2\alpha c_j/\beta(1+2\alpha c_j)}$.

The computation is performed with $[a,b] = [-40,60]$, $\alpha = 1/2$, $\beta = 1$, $c_1 = 0.1$, $c_2 = 0.05$, $x_{01} = 0$, $x_{02} = 25$, $\Delta t = 0.1$, and $h = 0.5$. The simulation is run to time $t = 10$ and our results are shown in Table 3 and Fig. 3. It is evident that the results agree well with the analytical solutions, where $L_\infty = 1.65 \times 10^{-3}$ and $L_2 = 4.48 \times 10^{-3}$ at $t = 10$.

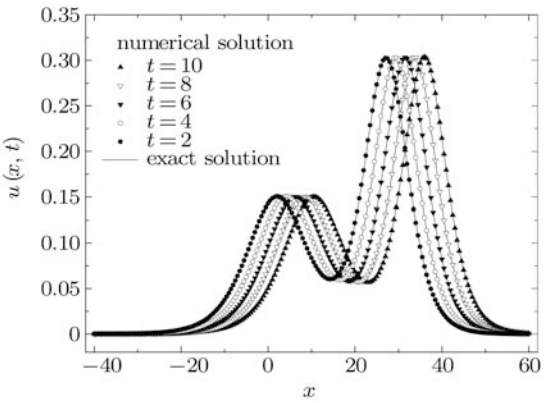


Fig. 3. The motion of two solitary waves from to $t = 10$ at intervals of 2 for Example 3.

Table 3. The values of the error norms at different times for Example 3.

t	1	2	3	4	5	6	7	8	9	10
$L_2 \times 10^3$	0.48	0.96	1.43	1.88	2.33	2.77	3.20	3.63	4.05	4.48
$L_\infty \times 10^3$	0.18	0.37	0.55	0.72	0.89	1.06	1.21	1.33	1.52	1.65

Example 4 We study the interaction of two GRLW solitary waves in the case of $p \geq 3$. We consider the GRLW equation with the following initial condition:

$$u(x,0) = \sum_{j=1}^{2} a_j \{\mathrm{sech}^2[k_j(x + x_{0j} - c_j t)]\}^{1/(p-1)}, \quad (42)$$

where $k_j = \frac{p-1}{2\beta}\sqrt{(c_j-1)/c_j}$, $a_j = [(p+1)(c_j - 1)/(2\alpha)]^{1/(p-1)}$, x_{0j} are arbitrary constants.

In our computation, we choose the parameters $[a,b] = [-150,\ 100]$, $\alpha = 1$, $\beta = 1$, $c_1 = 1.005$, $c_2 = 1.01$, $x_{01} = 40$, $x_{02} = -30$, $\Delta t = 0.05$, $h = 0.625$. In Table 4, we show the discrete L_2 and L_∞error norms. And we plot the numerical and exact solutions in Fig. 4. The L_2 and L_∞ error norms for these simulations are satisfactorily small in the case of high nonlinearity, where $L_\infty = 3.17 \times 10^{-3}$ and $L_2 = 8.94 \times 10^{-3}$ at $t = 5$ and $p = 9$.

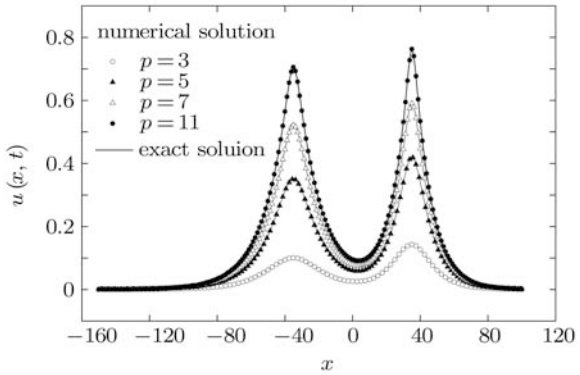


Fig. 4. The motion of two solitary waves at $t = 5$ with different values of p for Example 4.

Table 4. The values of the error norms at different values of t and p for Example 4.

	t	1	2	3	4	5
$p=3$	$L_2 \times 10^3$	0.33	0.51	0.70	0.86	1.00
	$L_\infty \times 10^3$	0.05	0.11	0.17	0.21	0.23
$p=5$	$L_2 \times 10^3$	0.87	1.38	1.91	2.37	2.81
	$L_\infty \times 10^3$	0.12	0.27	0.41	0.52	0.56
$p=7$	$L_2 \times 10^3$	1.37	2.35	3.35	4.28	5.19
	$L_\infty \times 10^3$	0.35	0.66	0.99	1.27	1.55
$p=9$	$L_2 \times 10^3$	2.09	3.84	5.59	7.29	8.94
	$L_\infty \times 10^3$	0.78	1.46	2.08	2.64	3.17

6. Conclusions

Presumably because of its high nonlinearity, the numerical methods for the GRLW equation have been studied by a few authors before. In this paper, the meshless method based on the MLS approximation is presented to deal with the numerical solution of the GRLW equation. The nonlinear discrete scheme of the GRLW equation is obtained and solved with the method of iteration. Under certain conditions, the iterative process is proved to be convergent. In order to show the accuracy and effectiveness of the meshless method, we present several examples. The numerical results are in good agreement with the exact solutions. The meshless method in this paper has important advantages over the traditional mesh-dependent techniques, such as finite element method and boundary element method, and can be considered to be one of the possible ways for solving other nonlinear partial differential equations.

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