Evolutionary games in a generalized Moran process with arbitrary selection strength and mutation*

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By using a generalized fitness-dependent Moran process, an evolutionary model for symmetric 2×2 games in a well-mixed population with a finite size is investigated. In the model, the individuals' payoff accumulating from games is mapped into fitness using an exponent function. Both selection strength β and mutation rate ε are considered. The process is an ergodic birth-death process. Based on the limit distribution of the process, we give the analysis results for which strategy will be favoured when ε is small enough. The results depend on not only the payoff matrix of the game, but also on the population size. Especially, we prove that natural selection favours the strategy which is risk-dominant when the population size is large enough. For arbitrary β and ε values, the 'Hawk–Dove' game and the 'Coordinate' game are used to illustrate our model. We give the evolutionary stable strategy (ESS) of the games and compare the results with those of the replicator dynamics in the infinite population. The results are determined by simulation experiments.

Keywords: evolutionary games, fitness-dependent Moran process, birth–death process, evolutionary stable strategy

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1. Introduction

Traditional game theory imposes severe restrictions on the information processing capacity and the rationality degree of the players. Moreover, even the strongest assumptions on rationality cannot lead us to predict the opponent equilibrium-strategy choice if the game has multiple Nash equilibriums. In contrast, evolutionary game theory is within the framework of bounded rationality. It is based on the population of individuals and introduces a learning process from which equilibrium selection is the limit outcome of the dynamic process.

The fundamental notion of evolutionary game theory was introduced by Maynard and Price in 1973.[1] Later, the famous replicator dynamics were proposed by Taylor and Jonker[2] and further studied in Refs.[3] and [4], in which differential equations were used to describe the strategy ratio change in the population and a powerful framework was provided for studying the deterministic dynamics in an infinitely large size population. These dynamics have been used in extensive studies, ranging from evolutionary biology, sociology to economics.[5−9]

In realistic systems, the population size is finite and the deterministic selection process is disturbed by stochastic effects and random drift, so stochastic evolutionary game dynamics in a finite size population needs to be considered. Assuming a finite but constant size population, Kandori *et al.*[10] built a dynamic model using a Markov chain in discrete time and used this model to analyse the long-run equilibrium selection of symmetric 2×2 games. Later, Amir and Berninghaus[11] extended the model proposed by Kandori *et al.* to continuous time condition.

The Moran process is a classical model of population genetics[12] and has only recently been transferred to game theory.[13,14] In their models, individuals of the population play the symmetric games with each other and accumulate payoff, which is mapped into fitness using a linear function. The fitness-dependent Moran process consists of three steps: (i) selection, i.e., an individual is randomly selected for reproduction with a probability proportional to its fitness; (ii) reproduction, i.e., the selected individual produces one offspring; (iii) replacement, i.e., the offspring replaces a randomly selected individual. This process can be seen as a study process of bounded rational players based on Darwin's natural selection.

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Within the framework of Moran process, Taylor *et al.*[13] presented a fixation probability method to evaluate which strategy would be favourable for symmetric 2×2 games without mutation. Nowak *et al.*[14] used this method to compare the 'Always defect' and 'Tit for Tat' strategies in the finitely repetitive prisoner dilemma game. Later, Fudenberg *et al.*[15] extended the classical Moran process by combining the frequency-dependent selection with the mutation. They gave a complete analysis of the long-run behaviour with a small mutation rate and strong selection strength in symmetric 2×2 games. Traulsen *et al.*[16] discussed the relationship between the replicator dynamics in an infinite size population and the Moran process in a large but finite size population. In Ref. [17], another payoff to fitness function was introduced which makes the analytical results possible in any selection strength without mutation. Recent papers that were based on these models and dealt with the fitness-dependent Moran process include Refs. [18]–[23].

In the present paper, we continue their work on symmetric 2×2 games in a well-mixed population with a finite size. We set up a dynamic evolutionary model using a more generalized fitness-dependent Moran process. In the model, both selection strength and mutation are considered. The Moran process is a birth-death process. Based on the limit distribution of the process, we give the analysis results under some conditions and illustrate our model for arbitrary parameters by simulation experiments.

The rest of the present paper is organized as follows. The model and the analytical results are described in Section 2. The simulation results and qualitative analysis are given in Section 3. Finally, some conclusions are drawn from the present study in the last section.

2. Model and analysis results when the mutation rate is small

Consider a finite population with size N playing symmetric 2×2 games. The strategy set of the game is $\{A, B\}$ and the payoff matrix is

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array},$$

which means an A player will obtain a when playing against another A or b when playing against B. Choosing strategy B results in obtaining either c (against A) or d (against B).

Assuming that the population is well mixed. Based on the payoff matrix, when the number of individuals choosing A is i, the expected payoffs of choosing strategy A and B are, respectively,

Choosing A:

$$\pi_A^i = \frac{a(i-1) + b(N-i)}{N-1},\ i \in \{1, 2, \cdots, N\}, \quad (1)$$

Choosing B:

$$\pi_B^i = \frac{ci + d(N-1-i)}{N-1},\ i \in \{0, 1, \cdots N-1\}. \quad (2)$$

Individual fitness f is a function of its payoff π from the games. The linear function

$$f = 1 - w + w\pi, \quad (3)$$

as used in Ref. [14] requires an upper bound of the parameter w to ensure $f > 0$. Instead, the nonlinear function

$$f = \exp(\beta\pi), \quad (4)$$

in Ref. [17] allows arbitrary values as long as $\beta > 0$, so we use the exponential function in the present paper. The parameter β that denotes selection strength can be seen as the degree to which the payoffs contribute to fitness.

At each time step, one individual is chosen to reproduce an offspring. The offspring uses the same strategy as that of its mother, i.e., with probability $1 - \varepsilon$, and uses a different strategy with probability ε (ε is mutation rate, $0 < \varepsilon < 1/2$). The probability for one individual to be chosen to reproduce is proportional to its fitness. Then, let the offspring replace a randomly chosen individual in the population, so the populatiob number is fixed. This is a study process reflecting 'survival of the fittest' and also considers both stochastic effects and random drift.

Let T denote the set of game phase. Introduce stochastic process $z(t)$ that denotes the number of the individuals choosing strategy A at time $t (t \in T)$. Let $S = \{0, 1, \ldots, N\}$ be the state space of the system, and $P(t) = \{p_{ij}(t)\}_{i,j \in S}$ be the t-step probability transition matrix. The process is homogeneous, thus

$$p_{ij}(t) = p\{z(s+t) = j | z(s) = i\}, \forall s \in T. \quad (5)$$

According to the above evolutionary rules, when the state $z(t) = i$, through one time step, system states

can only become $i+1$, $i-1$ or stay unchanged. Let $\lambda_i(\mu_i)$ denote such a probability of the number of individuals choosing A that it increases (decreases) by one when the system is in state i. Then

$$\lambda_i = \frac{i \cdot f_A^i \cdot (1-\varepsilon) + (N-i) \cdot f_B^i \cdot \varepsilon}{i f_A^i + (N-i) f_B^i} \cdot \frac{N-i}{N},$$

$$\mu_i = \frac{i \cdot f_A^i \cdot \varepsilon + (N-i) \cdot f_B^i \cdot (1-\varepsilon)}{i f_A^i + (N-i) f_B^i} \cdot \frac{i}{N},$$

with $f_A^i = \exp(\beta \pi_A^i)$, $f_B^i = \exp(\beta \pi_B^i)$, $(1 \le i \le N-1)$,

$$\lambda_0 = \varepsilon, \quad \lambda_N = 0, \quad \mu_0 = 0, \quad \mu_N = \varepsilon. \tag{6}$$

So $z(t)$ is a discrete time Markov chain with state space $S = \{0, 1, \ldots, N\}$. Obviously, this Markov chain is a birth–death process. The one-step Markov transition probability matrix can be written as

$$P(1) = \begin{bmatrix} p_{00} & p_{01} & 0 & \cdots & 0 & 0 & 0 \\ p_{10} & p_{11} & p_{12} & \cdots & 0 & 0 & 0 \\ 0 & p_{21} & p_{22} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & p_{N-1,N-2} & p_{N-1,N-1} & p_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & p_{N,N-1} & p_{N,N} \end{bmatrix},$$

with

$$p_{ij} = \begin{cases} \lambda_i, & j = i+1, \\ 1 - (\lambda_i + \mu_i), & j = i, \\ \mu_i, & j = i-1, \\ 0, & \text{otherwise}, \end{cases} \qquad 0 \le i,\ j \le N. \tag{7}$$

By using the Chapman–Kolmogorov equation, we can obtain the n-step Markov transition probability matrix

$$P(n) = P(1)^n. \tag{8}$$

This is also the Markov transition probability matrix of n-time-repeat evolutionary games.

As $\varepsilon > 0$, the Markov chain is ergodic. Based on the results in the birth–death process, $\forall i \in S$, $\lim\limits_{t\to\infty} p_{ij}(t)$ exist and all of them are equal. Let

$$\lim_{t\to\infty} p_{ij}(t) = v_j^\varepsilon, \tag{9}$$

then $v^\varepsilon = (v_0^\varepsilon, v_1^\varepsilon, \cdots, v_N^\varepsilon)$ will be the limit distribution of the process when the mutation rate is ε, where

$$v_j^\varepsilon = \frac{\xi_j}{\sum_{k=0}^N \xi_k}, \qquad j \in S,$$

$$\xi_0 = 1, \quad \xi_k = \frac{\lambda_0 \lambda_1 \ldots \lambda_{k-1}}{\mu_1 \mu_2 \ldots \mu_k}, \quad (1 \le k \le N). \tag{10}$$

We first analyse the limit case when $\varepsilon \to 0^+$. Let

$$v = (v_0, v_1, \ldots, v_N) = \lim_{\varepsilon\to 0^+} (v_0^\varepsilon, v_1^\varepsilon, \ldots, v_N^\varepsilon). \tag{11}$$

When $1 \le k \le N-1$, then

$$\lim_{\varepsilon\to 0^+} \xi_k = \lim_{\varepsilon\to 0^+} \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} = 0.$$

So,

$$v_j = \lim_{\varepsilon\to 0^+} v_j^\varepsilon = \lim_{\varepsilon\to 0^+} \frac{\xi_j}{\sum_{k=0}^N \xi_k} = 0, \quad (1 \le j \le N-1).$$

$$\lim_{\varepsilon\to 0^+} \xi_N = \lim_{\varepsilon\to 0^+} \frac{\lambda_0 \lambda_1 \cdots \lambda_{N-1}}{\mu_1 \mu_2 \cdots \mu_N} = \prod_{i=1}^{N-1} \frac{\exp(\beta \pi_A^i)}{\exp(\beta \pi_B^i)} = \exp\left(\beta \sum_{i=1}^{N-1} (\pi_A^i - \pi_B^i)\right),$$

with

$$\sum_{i=1}^{N-1} (\pi_A^i - \pi_B^i) = \sum_{i=1}^{N-1} \left(\frac{a-b-c+d}{N-1} i + \frac{-a+bN-dN+d}{N-1}\right) = \frac{1}{2}(a+b-c-d)N - a + d, \tag{12}$$

So,

$$v_0 = \lim_{\varepsilon\to 0^+} v_0^\varepsilon = \lim_{\varepsilon\to 0^+} \frac{\xi_0}{\sum_{k=0}^N \xi_k} = \frac{1}{1 + \exp\left(\beta\left[\frac{1}{2}(a+b-c-d)N - a + d\right]\right)},$$

$$v_N = \lim_{\varepsilon\to 0^+} v_N^\varepsilon = \lim_{\varepsilon\to 0^+} \frac{\xi_N}{\sum_{k=0}^N \xi_k} = \frac{\exp\left(\beta\left[\frac{1}{2}(a+b-c-d)N - a + d\right]\right)}{1 + \exp\left(\beta\left[\frac{1}{2}(a+b-c-d)N - a + d\right]\right)}.$$

When $\frac{1}{2}(a+b-c-d)N - a + d > 0$, $v_N > v_0$, evolutionary selection favours A;

When $\frac{1}{2}(a+b-c-d)N - a + d < 0$, $v_N < v_0$, evolutionary selection favours B;

When $N \gg 1$ (N is large enough), $\frac{1}{2}(a+b-c-d)N-a+d>0$ degenerates into $a+b>c+d$, and in this case, evolutionary selection favours the strategy which is risk-dominant.

3. Simulation results

In order to know more about the ESS in our model when the selection strength and mutation rate are arbitrary values, we present two illustrative examples for symmetric 2×2 games.

In the first example, we consider the Hawk–Dove game. The strategy set is {Hawk(H), Dove(D)} and the payoff matrix is given as

$$\begin{array}{c} \\ H \\ D \end{array} \begin{array}{c} \begin{array}{cc} H & D \end{array} \\ \begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix} \end{array}$$

with $a=-5,\ b=2,\ c=0,\ d=1$. Let the population size $N=10$. Based on the replicator equations in the infinite size population, $\frac{b-d}{b+c-a-d}=\frac{1}{6}$, the mixed strategy $(\frac{1}{6}H,\ \frac{5}{6}D)$ is the only ESS in this game. We give the simulation results for our model in the following.

Figures 1 and 2 show the results after $n\to\infty$ repeated 'Hawk–Dove' games, when the system reaches the stationary state, the relationship of the expected population frequency of choosing H (denote it f_H), the selection strength β and the mutation rate ε. When β is fixed and is not too small ($\beta \geq \frac{1}{\max\limits_i\left(\left|\pi_A^i-\pi_B^i\right|\right)}=0.20$), ε has a dominant influence on the system stationary state. That is, when $\varepsilon\to 0^+$, there are few individuals ($<5\%$) choosing strategy H , which validates the conclusions in Section 2 (when $\varepsilon\to 0^+$, $\frac{1}{2}(a+b-c-d)N-a+d=-14\ll 0$). As ε increases, f_H also increases, and there is a state ($\varepsilon\approx 0.02$) in which f_H is $\frac{1}{6}$, the same as the result of the ESS in an infinite population. When ε reaches 0.5, f_H also reaches 0.5, which means that individuals choose their strategies almost randomly. When ε is fixed, $\beta\to 0^+$, individuals choose their strategies randomly; as β increases, f_H decreases; when $\beta>0.4$, the selection strength β has little influence on the system stationary state. Thus, either a too low selection strength or a too large mutation rate can cause the system to get into a state of chaos.

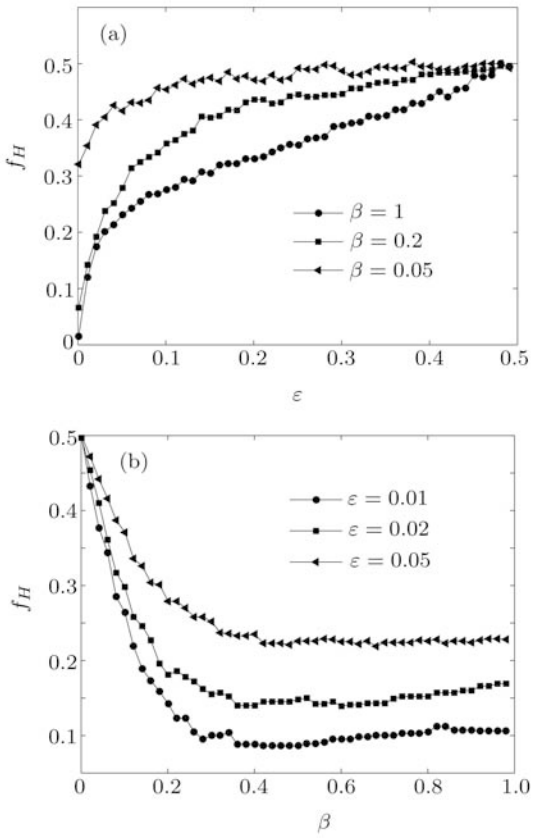


Fig. 1. (a) Characteristics of ε-frequency in the 'Hawk–Dove' game when $\beta=1, 0.2, 0.05$. The horizontal axis is ε, the vertical axis is f_H; (b) Characteristics of β-frequency in the 'Hawk–Dove' game when $\varepsilon=0.01, 0.02, 0.05$. The horizontal axis is β, the vertical axis is f_H.

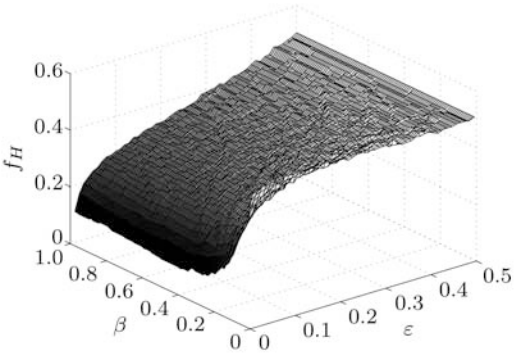


Fig. 2. Characteristics of (ε,β)-frequency in the 'Hawk–Dove' game. When mutation rate ε is small enough and the selection strength β reaches a certain level, f_H is close to 0, which implies that the evolutionary model favours the risk-dominant strategy. There is also a state ($\varepsilon\approx 0.02$, β is not too small) in which the ESS is the same as the replicator dynamics in an infinite population. When ε is too large or β is too small, players choose their strategy almost randomly.

Figure 3 shows the evolutionary process in the 'Hawk–Dove' game for different ε and β values. When ε is fixed, the greater the β, the lower the iteration

number required until the system reaches the stationary state. When β is fixed, the greater the ε, the lower iteration number required until the system reaches the stationary state. As shown in Fig. 3, when $\varepsilon = 0.02$ and $\beta = 1$, the iteration needs no more than 300 times until the system reaches the stationary state under which f_H is approximately 17%.

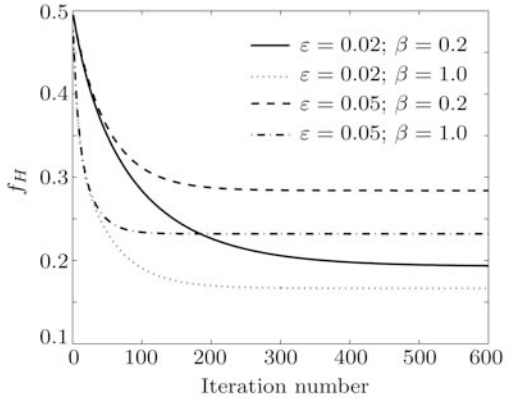


Fig. 3. Evolutionary processes in the 'Hawk–Dove' game ($\varepsilon = 0.02, 0.05;\ \beta = 0.2, 1$); The horizontal axis shows the iteration number, the vertical axis shows f_H. When the curve reaches horizontal, we have the ESS of the game and also the corresponding time required for the system to reach the equilibrium state.

In the second example, we consider the 'Coordinate' game. The strategy set is $\{A, B\}$ and the payoff matrix is given as

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 5 & 3 \\ 0 & 6 \end{bmatrix} \end{array}$$

with $a = 5$, $b = 3$, $c = 0$, and $d = 6$. Let population size $N = 10$. Based on the replicator equations in the infinite size population, the states of 'all the individuals choosing A' and 'all the individuals choosing B' are of two ESSs in this game. We give the simulation results for our model below.

Figures 4 and 5 show the results after $n \to \infty$ repeat 'Coordinate' games when the system reaches the stationary state, the relationship of the expected population frequency of choosing strategy A (denoted as f_A), the selection strength β and the mutant rate ε.

When β is fixed and is not too small ($\beta \geq \dfrac{1}{\max\limits_i \left(\left| \pi_A^i - \pi_B^i \right| \right)} = 0.18$), ε has a dominant influence on system stationary state. That is, when $\varepsilon \to 0^+$, most of the individuals ($> 85\%$) choose strategy A,

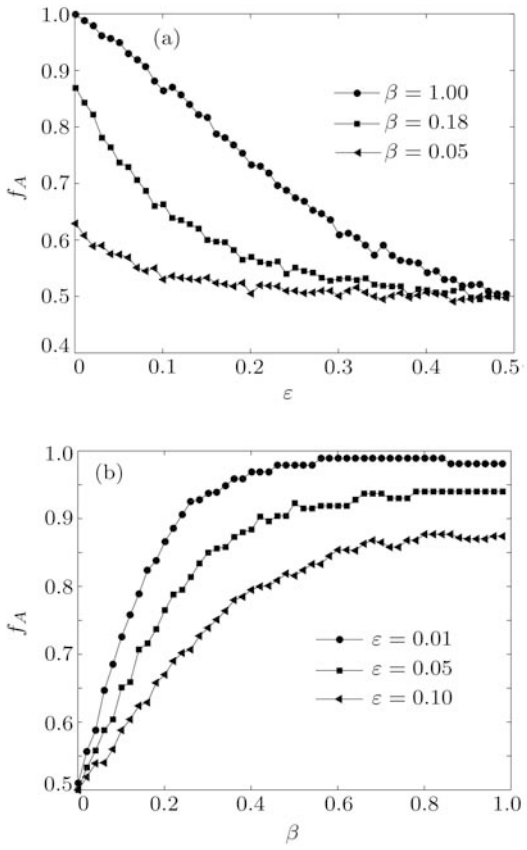


Fig. 4. (a) Characteristics of ε-frequency in the 'Coordinate' game ($\beta = 1.00, 0.18, 0.05$). The horizontal axis is ε, the vertical axis is f_A; (b) Characteristics of β-frequency in the 'Coordinate' game ($\varepsilon = 0.01, 0.05, 0.10$). The horizontal axis is β, the vertical axis is f_A.

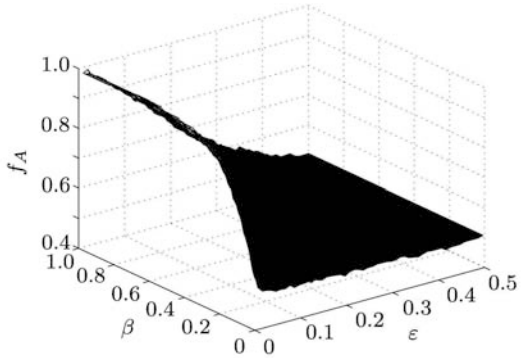


Fig. 5. Characteristics of (ε, β)-frequency in the 'Coordinate' game. When mutation rate ε is small enough and the selection strength β reaches a certain level, f_H is close to 1, validating that the evolutionary model favours the risk-dominant strategy. When ε is too large or β is too small, players choose their strategy almost randomly.

which validates the conclusions in Section 2 (when $\varepsilon \to 0^+$, $\dfrac{1}{2}(a + b - c - d)N - a + d = 11 \gg 0$). As ε increases, f_A decreases. When ε reaches 0.5, f_A also reaches 0.5, which means that individuals choose their strategies almost randomly.

When ε is fixed, $\beta \to 0^+$, individuals choose their strategies randomly; as β increases, f_A increases; when $\beta > 0.5$, the selection strength β has little influence on the system stationary state.

Figure 6 shows the evolutionary processes in the 'Coordinate' game for different ε and β values. When ε is fixed, the greater the β, the higher iteration number required until the system reaches the stationary state. When β is fixed, the greater the ε, the lower the iteration number that is required until the system reaches the stationary state. As shown in the figure, when $\varepsilon = 0.02$ and $\beta = 1$, the iteration needs more than 12000 times until the system reaches the stationary state under which the expected population frequency of choosing A is nearly 97%.

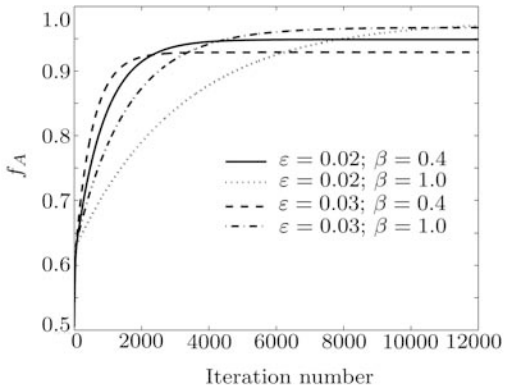


Fig. 6. Evolutionary processes in the 'Coordinate' game ($\varepsilon = 0.02, 0.03$; $\beta = 0.4, 1$). The horizontal axis shows the iteration number, the vertical axis shows f_A. When the curve reaches the horizontal level, the ESS of the game and also the corresponding time required for the system to reach the equilibrium state are obtained.

4. Conclusions

We have studied an evolutionary model of symmetric 2×2 games in a well-mixed population with a finite size. In the model, two parameters β and ε, to describe selection strength and mutation rate respectively, have been introduced. When ε is small enough, we give the analysis results for which strategy will be favoured. The results depend on not only payoff matrix, but also on population size. We prove that natural selection favours the strategy which is risk-dominant when the population size is large enough. For arbitrary β and ε values, the 'Hawk–Dove' game and the 'Coordinate' game are used to illustrate our model. In traditional game theory, the 'Hawk–Dove' game has only one mixed strategy Nash equilibrium and it is of ESS in the infinitely well-mixed population without mutation. Experimental results show that the same ESS can be achieved when ε and β are under some values in our model, but when ε is too large or β is too small, the system will be chaotic. Unlike the 'Hawk–Dove' game, the 'Coordinate' game has two pure-strategy Nash equilibriums and both of them are of ESS, but in our model, only the risk-dominant one can be achieved when ε is small enough and β is larger than a certain value; when ε is too large or β is too small, the system will be chaotic. The model we have investigated is applicable only for symmetric 2×2 games and the well-mixed population. How to extend it to non-symmetric 2×2 games or to a heterogeneous mixed population is the subject of our future study.

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