Mei symmetries and Mei conserved quantities for higher-order nonholonomic constraint systems*

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This paper presents the Mei symmetries and new types of non-Noether conserved quantities for a higher-order nonholonomic constraint mechanical system. On the basis of the form invariance of differential equations of motion for dynamical functions under general infinitesimal transformation, the determining equations, the constraint restriction equations and the additional restriction equations of Mei symmetries of the system are constructed. The criterions of Mei symmetries, weak Mei symmetries and strong Mei symmetries of the system are given. New types of conserved quantities, i.e. the Mei symmetrical conserved quantities, the weak Mei symmetrical conserved quantities and the strong Mei symmetrical conserved quantities of a higher-order nonholonomic system, are obtained. Then, a deduction of the first-order nonholonomic system is discussed. Finally, two examples are given to illustrate the application of the method and then the results.

Keywords: higher-order nonholonomic system, Mei symmetry, Mei conserved quantity

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1. Introduction

In 1894, the German scholar Hertz[1] first separated constraint systems into holonomic and nonholonomic systems, so as to open up a new field of the mechanics of nonholonomic constraint systems. With the development of science and technology, study of the mechanics of nonholonomic constraint systems plays an important theoretical and practical role,[2−30] and has become a popular subject in classical and quantum field theory,[4,11,12,16,20−22] relativistic systems,[8,9,13] mechanics,[2−7,14,15,17−20,23] mathematics,[10,27] spacecraft attitude dynamics,[24,25] multibody systems,[24−26] robot control theory,[26,28] machine engineering,[29,30] etc.

The principle of symmetries is a higher level of law in physics, and the conserved quantities of the dynamical systems can better reveal the profound physical laws. The conserved quantities of a physical system are closely related to its symmetries. In modern times, there are two main methods used for obtaining conserved quantities: Noether symmetries and Lie symmetries. In 1918, the German scientist Noether first discovered that conserved quantities correspond to some symmetries and conserved quantities can be found through the symmetries of the system, the theory of Noether symmetries was then established.[31] In 1979, Lutzky[32,33] applied Lie theory to the differential equations of motion for a mechanical system and studied the Lie symmetries and conserved quantities of a dynamical system. Over the past 50 years, researchers have developed the methods of Noether symmetries and Lie symmetries, and obtained many important results in mechanics, relativistic systems, rotational relativistic systems, quantum field theory, atomic physics and molecular physics.[1,4,6,16,20−22,34−56] In 2000, Mei[57] gave a new symmetry, i.e. form invariance, now known as Mei symmetry by academics,[58,59] which means that when the dynamical functions are replaced by the transformed functions under the infinitesimal transformations of group, the forms of differential equations of motion stay invariant. It is different from Noether symmetries and Lie symmetries, which can also lead to conserved quantities under certain conditions. Over the past 10 years, study of Mei symmetry has become a hot topic and includes a series of results.[58−75]

For a higher-order nonholonomic system, Zhang,[11] Luo[20,39] and Mei[40], respectively, studied the Noether symmetrical method and the Lie symmetrical method. However, their studies were confined to giving the Noether conserved quantities. In this paper, we further study the Mei symmetries of the higher-order nonholonomic system and obtain new types of non-Noether conserved quantities.

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First, on the basis of the form invariance of differential equations of motion for dynamical functions under general infinitesimal transformation, the determining equations, the constraint restriction equations and the additional restriction equations of Mei symmetries of the system are constructed. Second, the criterions of Mei symmetries, weak Mei symmetries and strong Mei symmetries of the system are given. Then, new types of conserved quantities (i.e. Mei symmetrical conserved quantities, weak Mei symmetrical conserved quantities and strong Mei symmetrical conserved quantities) of a higher-order nonholonomic system are obtained. Furthermore, a deduction of the first-order nonholonomic system is discussed. Two examples are given to illustrate the application of the method and the results.

2. Equations of motion of higher-order nonholonomic systems

Let us consider a mechanical system whose configuration is described by the n generalized coordinates q_s $(s = 1, 2, \ldots, n)$. The motion of the system is subjected to the g ideal bilateral m-order nonlinear nonholonomic constraints

$$f_\rho(t, q_s, \dot{q}_s, \ldots, \overset{(m)}{q_s}) = 0,$$
$$(\rho = 1, 2, \ldots, g; \quad s = 1, 2, \ldots, n; \quad m = 1, 2, \ldots). \quad (1)$$

The virtual displacements of the system satisfy Chetaev's condition[6]

$$\sum_{s=1}^{n} \frac{\partial f_\rho}{\partial \overset{(m)}{q_s}} \delta \overset{(m)}{q_s} = 0,$$
$$\delta t = \delta q_s = \delta \dot{q}_s = , \ldots, = \delta \overset{(m-1)}{q_s} = 0, \delta \overset{(m)}{q_s} \neq 0. \quad (2)$$

Suppose that the Lagrangian is

$$L = L(t, q_s, \dot{q}_s), \quad (s = 1, 2, \ldots, n), \quad (3)$$

using the Lagrange method of multipliers, we can obtain Lagrange equations with multiplier for a higher-order nonholonomic system

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \sum_{\rho=1}^{g} \lambda_\rho \frac{\partial f_\rho}{\partial \overset{(m)}{q_s}},$$
$$(s = 1, 2, \ldots, n; \quad m = 1, 2, \ldots). \quad (4)$$

Before integrating the differential equations (4), using Eqs. (1) and (4), we can express multipliers λ_ρ as the functions of $t, q_s, \dot{q}_s, \ldots, \overset{(m)}{q_s}$. So, Eq. (4) can be written in the form of the corresponding holonomic system

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \Lambda_s, \quad (5)$$

where

$$\Lambda_s = \sum_{\rho=1}^{g} \lambda_\rho \frac{\partial f_\rho}{\partial \overset{(m)}{q_s}}, \quad (s = 1, 2, \ldots, n; \quad m = 1, 2, \ldots).$$

3. Criterions and determining equations of Mei symmetries for the higher-order nonholonomic systems

Choose the infinitesimal transformations of group of t and q_s as follows:

$$t^* = t + \Delta t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s, \quad (6)$$

their expanding forms are

$$t^* = t + \varepsilon \xi_0(t, q_s, \dot{q}_s, \ldots, \overset{(m)}{q_s}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, q_s, \dot{q}_s, \ldots, \overset{(m)}{q_s}), \quad (7)$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are the generating functions of the infinitesimal transformations. And the infinitesimal generator vectors are

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s},$$
$$X^{(1)} = X^{(0)} + \dot{\eta}_s \frac{\partial}{\partial \dot{q}_s}, \quad \dot{\eta}_s = \dot{\xi}_s - \dot{q}_s \dot{\xi}_0,$$
$$X^{(2)} = X^{(1)} + \ddot{\eta}_s \frac{\partial}{\partial \ddot{q}_s}, \quad \ddot{\eta}_s = (\dot{\eta}_s)^{\cdot} - \ddot{q}_s \dot{\xi}_0,$$
$$\vdots$$
$$X^{(m)} = X^{(m-1)} + \overset{(m)}{\eta_s} \frac{\partial}{\partial \overset{(m)}{q_s}},$$
$$\overset{(m)}{\eta_s} = (\overset{(m-1)}{\eta_s})^{\cdot} - \overset{(m)}{q_s}\, \dot{\xi}_0. \quad (8)$$

Under the infinitesimal transformation Eq. (7), the Lagrangian L and the generalized constraint reactions Λ_s become

$$L^* = L(t, q_s, \dot{q}_s) + \varepsilon X^{(1)}(L) + o(\varepsilon^2),$$
$$\Lambda_s^* = \Lambda_s(t, q_s, \dot{q}_s, \ldots, \overset{(m)}{q_s}) + \varepsilon X^{(m)}(\Lambda_s) + o(\varepsilon^2), \quad (9)$$

and f_ρ become

$$f_\rho^* = f_\rho(t, q_s, \dot{q}_s, \ldots, \overset{(m)}{q_s}) + \varepsilon X^{(m)}(f_\rho) + o(\varepsilon^2). \quad (10)$$

Mei symmetry is the form invariance of differential equations of motion for dynamical functions under infinitesimal transformation, so Eq. (5) and the constraints Eq. (1) become, respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \dot{q}_s}-\frac{\partial L^*}{\partial q_s}=\Lambda_s^*,$$
$$(s=1,2,\ldots,n;\ \ m=1,2,\ldots), \tag{11}$$

and

$$f_\rho^*(t^*,q_s^*,\dot{q}_s^*,\overset{(2)}{q}{}_s^*,\ldots,\overset{(m)}{q}{}_s^*)=0,$$
$$(\rho=1,2,\ldots,g;\ \ s=1,2,\ldots,n;\ \ m=1,2,\ldots). \tag{12}$$

Substituting Eq. (9) into Eq. (11), neglecting the terms of ε^2 and the higher order terms, and using Eq. (5), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}-\frac{\partial X^{(1)}(L)}{\partial q_s}=X^{(m)}(\Lambda_s),$$
$$(s=1,2,\ldots,n;\ \ m=1,2,\ldots). \tag{13}$$

We call Eq. (13) the Mei symmetrical determining equations of the higher-order nonholonomic system.

Substituting Eq. (10) into (12), we can obtain

$$X^{(m)}(f_\rho)=0,\quad (\rho=1,2,\ldots,g;\quad m=1,2,\ldots). \tag{14}$$

We call Eq. (14) the constraint restriction equations of the higher-order nonholonomic system.

Considering the Chetaev condition Eq. (2) and

$$\delta\overset{(m)}{q}_s=\Delta\overset{(m)}{q}_s-\overset{(m+1)}{q}_s\Delta t=\varepsilon(\overset{(m)}{\eta}_s-\overset{(m+1)}{q}_s\xi_0), \tag{15}$$

we have

$$\sum_{s=1}^{n}\frac{\partial f_\rho}{\partial\overset{(m)}{q}_s}(\overset{(m)}{\eta}_s-\overset{(m+1)}{q}_s\xi_0)=0,$$
$$(\rho=1,2,\ldots,g;\quad m=1,2,\ldots). \tag{16}$$

We call Eq. (16) the additional restriction equations of the higher-order nonholonomic system.

Criterion 1 For the higher-order nonholonomic system Eqs. (1) and (4), if the generating functions ξ_0 and ξ_s of dynamical functions under general infinitesimal transformation Eq. (7) satisfy the determining equation (13), then the invariance is the Mei symmetry of the corresponding holonomic system Eq. (5).

Criterion 2 For the higher-order nonholonomic system Eqs. (1) and (4), if the generating functions ξ_0 and ξ_s of dynamical functions under general infinitesimal transformation Eq. (7) satisfy the determining equation (13) and the constraint restriction equation (14), then the invariance is the weak Mei symmetry of the higher-order nonholonomic system.

Criterion 3 For the higher-order nonholonomic system Eqs. (1) and (4), if the generating functions ξ_0 and ξ_s of dynamical functions under general infinitesimal transformation Eq. (7) satisfy the determining equation (13), the constraint restriction equation (14) and the additional restriction equation (16), then the invariance is the strong Mei symmetry of the higher-order nonholonomic system.

4. Mei conserved quantitities for higher-order nonholonomic systems

For the higher-order nonholonomic system, the Mei symmetries can directly lead to new types of non-Noether conserved quantities, i.e. Mei conserved quantities. The following theorems give the conditions for the existence of conserved quantities directly induced by Mei symmetries for a higher-order nonholonomic system.

Theorem 1 For the holonomic system Eq. (5) corresponding to the higher-order nonholonomic system Eqs. (1) and (4), if the generating functions ξ_0 and ξ_s of dynamical functions under general infinitesimal transformation Eq. (7) satisfy the determining equation (13), and there is a gauge function $G_{\mathrm{F}}=G_{\mathrm{F}}(t,q_s,\dot{q}_s,\ldots,\overset{(m)}{q}_s)$ satisfies the structure equations

$$X^{(1)}[X^{(1)}(L)]+X^{(1)}(L)\dot{\xi}_0$$
$$+X^{(m)}(\Lambda_s)(\xi_s-\dot{q}_s\xi_0)+\dot{G}_{\mathrm{F}}=0, \tag{17}$$

then the corresponding holonomic system Eq. (5) possesses the following Mei symmetrical conserved quantities:

$$I_{\mathrm{F}}=X^{(1)}(L)\xi_0+\frac{\partial X^{(1)}(L)}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G_{\mathrm{F}}$$
$$=\text{const.} \tag{18}$$

Proof We have

$$\frac{\mathrm{d}I_{\mathrm{F}}}{\mathrm{d}t}=\left(\frac{\partial X^{(1)}(L)}{\partial t}+\frac{\partial X^{(1)}(L)}{\partial q_s}\dot{q}_s+\frac{\partial X^{(1)}(L)}{\partial\dot{q}_s}\ddot{q}_s\right)\xi_0$$
$$+X^{(1)}(L)\dot{\xi}_0+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X^{(1)}(L)}{\partial\dot{q}_s}\right)(\xi_s-\dot{q}_s\xi_0)$$
$$+\frac{\partial X^{(1)}(L)}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0-\ddot{q}_s\xi_0)$$
$$-X^{(1)}(L)\dot{\xi}_0-\frac{\partial X^{(1)}(L)}{\partial t}\xi_0-\frac{\partial X^{(1)}(L)}{\partial q_s}\xi_s$$

$$
\begin{aligned}
&-\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)-X^{(m)}(\Lambda_s)(\xi_s-\dot{q}_s\xi_0)\\
&=\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}-\frac{\partial X^{(1)}(L)}{\partial q_s}\right.\\
&\left.-X^{(m)}(\Lambda_s)\right](\xi_s-\dot{q}_s\xi_0). \qquad (19)
\end{aligned}
$$

By the determining equation (13) of the Mei symmetries, we know that Eq. (19) is zero.

Theorem 2 For the higher order nonholonomic system Eqs. (1) and (4), if the generating functions ξ_0 and ξ_s of dynamical functions under general infinitesimal transformation Eq. (7) satisfy the determining equation (13) and the constraint restriction equation (14), and there is a gauge function $G_{\mathrm{F}}=G_{\mathrm{F}}(t,q_s,\dot{q}_s,\ldots,q_s^{(m)})$ satisfying structure equation (17), then the system possesses the weak Mei symmetrical conserved quantities Eq. (18).

Theorem 3 For the higher-order nonholonomic system Eqs. (1) and (4), if the generating functions ξ_0 and ξ_s of dynamical functions under general infinitesimal transformation Eq. (7) satisfy determining equation (13), the constraint restriction equation (14) and the additional restriction equation (16), and there is a gauge function $G_{\mathrm{F}}=G_{\mathrm{F}}(t,q_s,\dot{q}_s,\ldots,\overset{(m)}{q_s})$ which satisfies structure equation (17), then the system possesses the strong Mei symmetrical conserved quantities Eq. (18).

When $m=1$, the higher-order nonholonomic system becomes a 1-order nonholonomic system.

Deduction For the nonholonomic system Eqs. (1) and (4) when $m=1$, if the generating functions ξ_0 and ξ_s of dynamical functions under general infinitesimal transformation Eq. (7) satisfy the Mei symmetrical determining equation (13), and there is a gauge function $G_{\mathrm{F}}=G_{\mathrm{F}}(t,q_s,\dot{q}_s)$ which satisfying the structure equations

$$
\begin{aligned}
&X^{(1)}[X^{(1)}(L)]+X^{(1)}(L)\dot{\xi}_0+X^{(1)}(\Lambda_s)(\xi_s-\dot{q}_s\xi_0)\\
&+\dot{G}_{\mathrm{F}}=0, \qquad (20)
\end{aligned}
$$

then the corresponding holonomic system Eq. (5) possesses the following Mei symmetrical conserved quantities

$$
\begin{aligned}
I_{\mathrm{F}}&=X^{(1)}(L)\xi_0+\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G_{\mathrm{F}}\\
&=\text{const.} \qquad (21)
\end{aligned}
$$

In addition, if the generating functions ξ_0 and ξ_s of the dynamical functions under general infinitesimal transformation Eq. (7) satisfy the determining equation (13) and the constraint restriction equation (14), and there is a gauge function $G_{\mathrm{F}}=G_{\mathrm{F}}(t,q_s,\dot{q}_s)$ which satisfies the structure equation (20), then the system possesses weak Mei symmetrical conserved quantities Eq. (21). Furthermore, if the generating functions ξ_0 and ξ_s of dynamical functions under general infinitesimal transformation Eq. (7) satisfy the determining equation (13), the constraint restriction equation (14) and the additional restriction equation (16), and there is a gauge function $G_{\mathrm{F}}=G_{\mathrm{F}}(t,q_s,\dot{q}_s)$ which satisfy the structure equations (20), then the system possesses strong Mei symmetrical conserved quantities Eq. (21).

5. Examples

Example 1 We study directly leading conserved quantities by the Mei symmetries of the higher-order nonholonomic system, of which the Lagrangian and the higher-order nonholonomic constraint are respectively[41]

$$L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2)-q_2, \qquad (22)$$

$$f^{(2)}=\ddot{q}_2-t\ddot{q}_1=0. \qquad (23)$$

The equations of motion of the system have the form

$$\ddot{q}_1=-\lambda t,\quad \ddot{q}_2-1=\lambda. \qquad (24)$$

From Eqs. (23) and (24), we obtain

$$\lambda=-\frac{1}{1+t^2}, \qquad (25)$$

then

$$\Lambda_1=\frac{t}{1+t^2},\quad \Lambda_2=-\frac{1}{1+t^2}. \qquad (26)$$

Through some calculations, we have

$$
\begin{aligned}
X^{(1)}(L)=&-\xi_2+\dot{q}_1(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0)+\dot{q}_2(\dot{\xi}_2-\dot{q}_2\dot{\xi}_0), \qquad (27a)\\
X^{(2)}(f)=&-\ddot{q}_1\xi_0-t(\ddot{\xi}_1-2\ddot{q}_1\dot{\xi}_0-\dot{q}_1\ddot{\xi}_0)\\
&+(\ddot{\xi}_2-2\ddot{q}_2\dot{\xi}_0-\dot{q}_2\ddot{\xi}_0), \qquad (27b)\\
X^{(2)}(\Lambda_1)=&\frac{\partial}{\partial t}\left(\frac{t}{1+t^2}\right)\xi_0,\\
X^{(2)}(\Lambda_2)=&\frac{\partial}{\partial t}(-\frac{1}{1+t^2})\xi_0. \qquad (27c)
\end{aligned}
$$

Choose the generating functions as

$$\xi_0=0,\quad \xi_1=\dot{q}_2-t\dot{q}_1+q_1+t,\quad \xi_2=1, \qquad (28)$$

then we have

$$\dot{\xi}_1=1,\quad X^{(1)}(L)=\dot{q}_1-1,$$

$$X^{(1)}[X^{(1)}(L)] = \dot{\xi}_1 = 1, \tag{29a}$$

$$X^{(2)}(f^{(2)}) = 0, \quad X^{(2)}(\Lambda_s) = 0, \quad (s = 1, 2). \tag{29b}$$

Substituting Eqs. (28) and (29) into the structure equation (17) yields

$$G_{\rm F} = -t, \tag{30}$$

then, we can obtain weak Mei symmetrical conserved quantity

$$I_{\rm F} = \dot{q}_2 - t\dot{q}_1 + q_1 = \text{const.} \tag{31}$$

Example 2 We study directly leading conserved quantities by the Mei symmetries of the higher-order nonholonomic system, of which the Lagrangian and the higher-order nonholonomic constraint of a system are

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2), \tag{32}$$

$$f^{(4)} = \overset{(3)}{q}_1 \overset{(4)}{q}_1 + \overset{(3)}{q}_2 \overset{(4)}{q}_2 - t\overset{(3)}{q}_3 \overset{(4)}{q}_3 = 0. \tag{33}$$

The equations of motion of the system have the form

$$m\ddot{q}_1 = \lambda \overset{(3)}{q}_1, \quad m\ddot{q}_2 = \lambda \overset{(3)}{q}_2, \quad m\ddot{q}_3 = -t\lambda \overset{(3)}{q}_3. \tag{34}$$

From Eqs. (33) and (34), we obtain

$$\lambda = \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3} \mathrm{d}t\right). \tag{35}$$

Then

$$\begin{aligned}
\Lambda_1 &= \overset{(3)}{q}_1 \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3 \, \mathrm{d}t}\right), \\
\Lambda_2 &= \overset{(3)}{q}_2 \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3 \quad dt}\right), \\
\Lambda_3 &= -t\overset{(3)}{q}_3 \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3 \, \mathrm{d}t}\right).
\end{aligned} \tag{36}$$

Through some calculations, we have

$$X^{(1)}(L) = m\dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + m\dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) + m\dot{q}_3(\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0), \tag{37a}$$

$$\begin{aligned}
X^{(4)}(f^{(4)}) = {} & \overset{(3)}{q}_3 \overset{(4)}{q}_3 \xi_0 + \overset{(4)}{q}_s(\overset{(3)}{\xi}_s - 3\overset{(3)}{q}_s \dot{\xi}_0 - 3\ddot{q}_s\ddot{\xi}_0 - \dot{q}_s \overset{(3)}{\xi}_0 - t\overset{(4)}{q}_3(\overset{(3)}{\xi}_3 - 3\overset{(3)}{q}_3 \dot{\xi}_0 \\
& - 3\ddot{q}_3\ddot{\xi}_0 - \dot{q}_3 \overset{(3)}{\xi}_0) + \overset{(3)}{q}_s(\overset{(4)}{\xi}_s - 4\overset{(4)}{q}_s \dot{\xi}_s - 6\overset{(3)}{q}_s \ddot{\xi}_0 - 4\ddot{q}_s \overset{(3)}{\xi}_0 - \dot{q}_s \overset{(4)}{\xi}_0) \\
& - t\overset{(3)}{q}_3(\overset{(4)}{\xi}_3 - 4\overset{(4)}{q}_3 \dot{\xi}_3 - 6\overset{(3)}{q}_3 \ddot{\xi}_0 - 4\ddot{q}_3 \overset{(3)}{\xi}_0 - \dot{q}_3 \overset{(4)}{\xi}_0), \quad (s = 1, 2),
\end{aligned} \tag{37b}$$

$$\begin{aligned}
X^{(4)}(\Lambda_s) = {} & \frac{\partial}{\partial t}\left(\overset{(3)}{q}_s \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3 \, \mathrm{d}t}\right)\right)\xi_0 \\
& + \frac{\partial}{\partial \ddot{q}_s}\left(\overset{(3)}{q}_s \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3 \, \mathrm{d}t}\right)\right)(\ddot{\xi}_s - 2\ddot{q}_s\dot{\xi}_0 - \dot{q}_s\ddot{\xi}_0) \\
& + \frac{\partial}{\partial \overset{(3)}{q}_s}\left(\overset{(3)}{q}_s \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3}\mathrm{d}t\right)\right) \\
& \times (\overset{(3)}{\xi}_s - 3\overset{(3)}{q}_s \dot{\xi}_0 - 3\ddot{q}_s\ddot{\xi}_0 - \dot{q}_s \overset{(3)}{\xi}_0), \quad (s = 1, 2),
\end{aligned} \tag{37c}$$

$$\begin{aligned}
X^{(4)}(\Lambda_3) = {} & -\frac{\partial}{\partial t}\left(t\overset{(3)}{q}_3 \exp\int\left(\frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3}\mathrm{d}t\right)\right)\xi_0 \\
& - \frac{\partial}{\partial \ddot{q}_3}\left(t\overset{(3)}{q}_3 \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3}\mathrm{d}t\right)\right)(\ddot{\xi}_3 - 2\ddot{q}_3\dot{\xi}_0 - \dot{q}_3\ddot{\xi}_0) \\
& - \frac{\partial}{\partial \overset{(3)}{q}_3}\left(t\overset{(3)}{q}_3 \exp\left(\int \frac{t(\overset{(3)}{q}_1)^2 + t(\overset{(3)}{q}_2)^2 + t(\overset{(3)}{q}_3)^2 - \ddot{q}_3 \overset{(3)}{q}_3}{t\ddot{q}_1 \overset{(3)}{q}_1 + t\ddot{q}_2 \overset{(3)}{q}_2 + t\ddot{q}_3 \overset{(3)}{q}_3}\mathrm{d}t\right) \times (\overset{(3)}{\xi}_3 - 3\overset{(3)}{q}_3 \dot{\xi}_0 - 3\ddot{q}_3\ddot{\xi}_0 - \dot{q}_3 \overset{(3)}{\xi}_0).\right.
\end{aligned} \tag{37d}$$

Choose the generating functions as

$$\xi_0 = 0, \quad \xi_1 = (\overset{(3)}{q}_1)^2 + (\overset{(3)}{q}_2)^2 - t(\overset{(3)}{q}_3)^2 + \int (\overset{(3)}{q}_3)^2 \mathrm{d}t + t, \quad \xi_2 = \xi_3 = 1, \tag{38}$$

then we have

$$\dot{\xi}_1 = 1, \quad X^{(1)}(L) = m\dot{q}_1,$$
$$X^{(1)}[X^{(1)}(L)] = \dot{\xi}_1 = 1, \tag{39a}$$
$$X^{(4)}(f^{(4)}) = 0, \quad X^{(4)}(\Lambda_s) = 0, \quad (s = 1, 2, 3). \tag{39b}$$

Substituting Eqs. (38) and (39) into the structure equation (17) yields

$$G_{\rm F} = -t, \tag{40}$$

then, we can obtain weak Mei symmetrical conserved quantity

$$I_{\rm F} = m[(\overset{(3)}{q_1})^2 + (\overset{(3)}{q_2})^2 - t(\overset{(3)}{q_3})^2 + \int (\overset{(3)}{q_2})^2 {\rm d}t] + (m-1)t = \text{const.} \tag{41}$$

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