Conformal invariance and a kind of Hojman conserved quantity of the Nambu system*

Li Yan(李 燕), Fang Jian-Hui(方建会)†, and Zhang Ke-Jun(张克军)

College of Physics Science and Technology, China University of Petroleum, Dongying 257061, *China*

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Conformal invariance and a kind of Hojman conserved quantity of the Nambu system under infinitesimal transformations are studied. The definition and the determining equation of conformal invariance of the system are presented. The necessary and sufficient condition under which the conformal invariance of the system would have Lie symmetry under infinitesimal transformations is derived. Then, the condition of existence and a kind of Hojman conserved quantity are obtained. Finally, an example is given to illustrate the application of the results.

Keywords: conformal invariance, Nambu system, Hojman conserved quantity

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1. Introduction

Research on the symmetries and conserved quantities of a mechanical system possesses important theoretical and practical significance, and plays an important role in modern mathematics, mechanics and physics. The modern methods of finding conserved quantities by using symmetries in analytical dynamics are mainly: Noether symmetry,[1] Lie symmetry,[2] and Mei symmetry.[3,4] The conserved quantities induced by these three symmetries are mainly Noether, Hojman and Mei conserved quantities. Mei has conducted thorough research on the above symmetries and the conserved quantities of constrained mechanical systems in Ref. [5]. In 1997, Galiullin *et al.*[6] studied the conformal invariance of a Birkhoff system under special infinitesimal transformations. In 2001, Robert and Matthew[7] investigated the conformal Hamiltonian systems using geometric methods, discussed the geometric structure of conformal invariance and pointed out the difference of conformal invariance from the arbitrary symmetry of a dynamical system. In Ref. [8] the conformal invariance of a first-order differential equation was studied. At present, research on the conformal invariance and conserved quantities of mechanical systems has already made some important achievements.[9−15]

In 1973, Nambu proposed a possible generalization of ordinary Hamilton mechanics, which is now called Nambu mechanics.[16] Thereafter, Mukunda and Sudarshan[17] studied the relation between the Nambu system and the Hamilton system. Ogawa and Sagae[18] constructed a fundamental theorem based on the invariance of geometry modality and gave the first application—Euler equation. Now, the Nambu mechanics system is widely used in particle physics, cosmography, string theory, hydrodynamics, mathematical physics, computational mathematics, *etc.* Recently, Zhang *et al.*[19] discussed the Lie symmetry and the inverse problem of Nambu mechanical systems. In Ref. [20] the Lie symmetry and Hojman conserved quantity of the Nambu system were studied.

In the present paper, the conformal invariance of the Nambu system is studied, the expression of a conformal factor is deduced and a kind of Hojman conserved quantity is obtained.

2. Differential equations of the Nambu system

We suppose that the motion in n-dimensions phase space is determined by n generalized coordinates q_i $(i = 1, \ldots, n)$. For an arbitrary function $A_i(\boldsymbol{q})$, we define the Nambu–Poisson bracket as

$$\{A_1(\boldsymbol{q}), A_2(\boldsymbol{q}), \ldots, A_n(\boldsymbol{q})\} = \frac{\partial(A_1, A_2, \ldots, A_n)}{\partial(q_1, q_2, \ldots, q_n)}$$

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†Corresponding author. E-mail: fangjh@upc.edu.cn

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$$= \begin{pmatrix} \dfrac{\partial A_1}{\partial q_1} & \dfrac{\partial A_1}{\partial q_2} & \cdots & \dfrac{\partial A_1}{\partial q_n} \\ \dfrac{\partial A_2}{\partial q_1} & \dfrac{\partial A_2}{\partial q_2} & \cdots & \dfrac{\partial A_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial A_n}{\partial q_1} & \dfrac{\partial A_n}{\partial q_2} & \cdots & \dfrac{\partial A_n}{\partial q_n} \end{pmatrix}. \quad (1)$$

By introducing $n-1$ Nambu–Hamilton functions $H_k(\boldsymbol{q})(k=1,2,\ldots,n-1)$, the equation of motion of the Nambu system can be written as[21]

$$\dot{q}_i = \{q_i, H_1(q), \ldots, H_{n-1}(q)\}. \quad (2)$$

For an arbitrary function $F(q_1,\ldots,q_n)$ independent of time, we have

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial(F, H_1, \ldots, H_{n-1})}{\partial(q_1, q_2, \ldots, q_n)} = \{F, H_1, \ldots, H_{n-1}\}.$$

3. Conformal invariance of the Nambu system

We introduce the infinitesimal transformations of the time and spatial coordinates as

$$t^* = t + \Delta t, \quad q_i^*(t^*) = q_i(t) + \Delta q_i. \quad (3)$$

or their expansion formulae

$$t^* = t + \varepsilon\tau(t,\boldsymbol{q}), \quad q_i^*(t^*) = q_i(t) + \varepsilon\xi_i(t,\boldsymbol{q}), \quad (4)$$

where ε is an infinitesimal parameter, τ and ξ_i are the infinitesimal generators.

Thus Eq. (2) can be expanded as

$$\dot{q}_i = g_i(t,\boldsymbol{q}), \quad (i=1,2,\ldots,n). \quad (5)$$

Introduce an infinitesimal transformation vector of generator

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_i\frac{\partial}{\partial q_i}, \quad (6)$$

and its first extension

$$X^{(1)} = X^{(0)} + (\dot{\xi}_i - \dot{q}_i\dot{\tau})\frac{\partial}{\partial \dot{q}_i} \quad (7)$$

and let

$$F_i = \dot{q}_i - g_i(t,\boldsymbol{q}), \quad (8)$$

then we will discuss the conformal invariance of the Nambu system Eq. (5), that is, look for the mapping ensemble of independent or dependent variables corresponding to the conformal invariance of Eq. (8).

Definition In the infinitesimal transformations of infinitesimal generators $\tau(t,\boldsymbol{q})$ and $\xi_i(t,\boldsymbol{q})$, if there is a matrix $\boldsymbol{l}_i^j$ satisfying the following condition:

$$X^{(1)}F_i = \boldsymbol{l}_i^j F_j, \quad (9)$$

this kind of invariance is called conformal invariance of the Nambu system Eq. (5) under infinitesimal transformations Eq. (4). Equation (9) is a determining equation of conformal invariance of the Nambu system Eq. (5), and $\boldsymbol{l}_i^j$ is called a conformal factor.

In order to obtain the conformal factor of conformal invariance of the differential equations of the Nambu system, we calculate the difference value of the following expression:

$$X^{(1)}F_i - X^{(1)}F_i\,|_{F_i=0}\,. \quad (10)$$

Because

$$\begin{aligned} & X^{(1)}F_i \\ &= X^{(1)}(\dot{q}_i - g_i) \\ &= \dot{\xi}_i - \dot{q}_i\dot{\tau} - X^{(0)}(g_i) \\ &= \frac{\partial\xi_i}{\partial t} + \frac{\partial\xi_i}{\partial q_j}\dot{q}_j - \dot{q}_i\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial q_j}\dot{q}_j\right) - X^{(0)}(g_i), \end{aligned} \quad (11)$$

and

$$\begin{aligned} X^{(1)}F_i\,|_{F_i=0} \;=\; & \frac{\partial\xi_i}{\partial t} + \frac{\partial\xi_i}{\partial q_j}g_j - g_i\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial q_j}g_j\right) \\ & - X^{(0)}(g_i), \end{aligned} \quad (12)$$

we have

$$\begin{aligned} & X^{(1)}F_i - X^{(1)}F_i\,|_{F_i=0} \\ &= \frac{\partial\xi_i}{\partial q_j}(\dot{q}_j - g_j) - \frac{\partial\tau}{\partial t}(\dot{q}_i - g_i) - \frac{\partial\tau}{\partial q_j}(\dot{q}_i\dot{q}_j - g_ig_j) \\ &= \left(\frac{\partial\xi_i}{\partial q_j} - \delta_i^j\frac{\partial\tau}{\partial t}\right)F_j - \frac{\partial\tau}{\partial q_j}(\dot{q}_i\dot{q}_j - g_ig_j). \end{aligned} \quad (13)$$

Due to $F_i = \dot{q}_i - g_i$, we have

$$\begin{aligned} \dot{q}_i\dot{q}_j \;&=\; (F_i + g_i)(F_j + g_j) \\ &= F_iF_j + F_ig_j + F_jg_i + g_ig_j. \end{aligned} \quad (14)$$

Substituting Eq. (14) into Eq. (13), we obtain

$$\begin{aligned} & X^{(1)}F_i - X^{(1)}F_i\,|_{F_i=0} \\ &= \left(\frac{\partial\xi_i}{\partial q_j} - \delta_i^j\frac{\partial\tau}{\partial t}\right)F_j \\ & \quad - \frac{\partial\tau}{\partial q_j}(F_iF_j + F_ig_j + F_jg_i). \end{aligned} \quad (15)$$

Ignoring the higher order terms of F_i, we have

$$\begin{aligned} & X^{(1)}F_i - X^{(1)}F_i\,|_{F_i=0} \\ &= \left\{\frac{\partial\xi_i}{\partial q_j} - \delta_i^j\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial q_j}g_j\right) - \frac{\partial\tau}{\partial q_j}g_i\right\}F_j \\ &= \beta_i^jF_j, \end{aligned} \quad (16)$$

where

$$\beta_i^j = \frac{\partial \xi_i}{\partial q_j} - \delta_i^j \left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q_j} g_j \right) - \frac{\partial \tau}{\partial q_j} g_i. \tag{17}$$

If the infinitesimal transformations have the Lie symmetry for the differential equations, we have

$$\boldsymbol{l}_i^j F_j - \beta_i^j F_j = X^{(1)} F_i \,|_{F_i=0} \,, \tag{18}$$

that is,

$$(\boldsymbol{l}_i^j - \beta_i^j) F_j = 0, \tag{19}$$

Namely $\boldsymbol{l}_i^j = \beta_i^j$. And then we have the following proposition:

Proposition 1 For the Nambu system, the necessary and sufficient conditions under which its conformal invariance is Lie symmetric simultaneously by the action of the infinitesimal transformations are the infinitesimal generators fulfilling the following expressions:

$$\boldsymbol{l}_i^j = \frac{\partial \xi_i}{\partial q_j} - \delta_i^j \left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q_j} g_j \right) - \frac{\partial \tau}{\partial q_j} g_i, \quad (i, j = 1, 2, \ldots, n). \tag{20}$$

4. Conserved quantity of the Nambu system

Under certain conditions, conformal invariance of the Nambu system can lead to conserved quantities.

Proposition 2 For the Nambu system Eq. (5), if the infinitesimal generators τ and ξ_i satisfy the conformal factor Eq. (20) and there is a gauge function $\mu(t, \boldsymbol{q})$ satisfying the following structure equation:

$$\frac{\partial g_i}{\partial q_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{21}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + g_i \frac{\partial}{\partial q_i}, \tag{22}$$

then conformal invariance can lead to a kind of Hojman conserved quantity

$$I_H = \frac{\partial [\mu(\xi_i - \tau g_i)]}{\mu \partial q_i} = \text{const.} \tag{23}$$

Proof Taking the derivative of I_H with respect to t, we have

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} I_H &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \left[(\xi_i - \tau g_i) \frac{\partial \ln \mu}{\partial q_i} + \frac{\partial (\xi_i - \tau g_i)}{\partial q_i} \right] \\ &= \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_i - g_i \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \tau - \tau \frac{\bar{\mathrm{d}}}{\mathrm{d}t} g_i \right) \frac{\partial \ln \mu}{\partial q_i} + (\xi_i - \tau g_i) \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\partial \ln \mu}{\partial q_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\partial}{\partial q_i} (\xi_i - \tau g_i). \end{aligned} \tag{24}$$

The infinitesimal generators τ and ξ_i satisfy the determining equation of Lie symmetry, i.e.,

$$X^{(1)} F_i \,|_{F_i=0} = 0. \tag{25}$$

Equation (25) can also be expressed as

$$\frac{\bar{\mathrm{d}} \xi_i}{\mathrm{d}t} - g_i \frac{\bar{\mathrm{d}} \tau}{\mathrm{d}t} = \tau \frac{\partial g_i}{\partial t} + \xi_j \frac{\partial g_i}{\partial q_j}. \tag{26}$$

Subtracting Eq. (26) into Eq. (24), we have

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} I_H &= \left(\tau \frac{\partial g_i}{\partial t} + \xi_j \frac{\partial g_i}{\partial q_j} - \tau \frac{\partial g_i}{\partial t} - \tau \frac{\partial g_i}{\partial q_j} g_j \right) \frac{\partial \ln \mu}{\partial q_i} + (\xi_i - \tau g_i) \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\partial \ln \mu}{\partial q_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\partial}{\partial q_i} (\xi_i - \tau g_i) \\ &= \left(\tau \frac{\partial g_i}{\partial t} + \xi_j \frac{\partial g_i}{\partial q_j} - \tau \frac{\partial g_i}{\partial t} - \tau \frac{\partial g_i}{\partial q_j} g_j \right) \frac{\partial \ln \mu}{\partial q_i} + (\xi_i - \tau g_i) \left[\frac{\partial}{\partial q_i} \frac{\bar{d}}{dt} \ln \mu - \frac{\partial \ln \mu}{\partial q_j} \frac{\partial g_j}{\partial q_i} \right] \\ &\quad + \frac{\partial}{\partial q_i} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} (\xi_i - \tau g_i) - \frac{\partial (\xi_i - \tau g_i)}{\partial q_j} \frac{\partial g_j}{\partial q_i} \\ &= (\xi_j - \tau g_j) \frac{\partial g_i}{\partial q_j} \frac{\partial \ln \mu}{\partial q_i} + (\xi_i - \tau g_i) \left[\frac{\partial}{\partial q_i} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu - \frac{\partial \ln \mu}{\partial q_j} \frac{\partial g_j}{\partial q_i} \right] \\ &\quad + \frac{\partial}{\partial q_i} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} (\xi_i - \tau g_i) - \frac{\partial (\xi_i - \tau g_i)}{\partial q_j} \frac{\partial g_j}{\partial q_i} \\ &= (\xi_i - \tau g_i) \frac{\partial}{\partial q_i} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu + \frac{\partial}{\partial q_i} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} (\xi_i - \tau g_i) - \frac{\partial (\xi_i - \tau g_i)}{\partial q_j} \frac{\partial g_j}{\partial q_i}. \end{aligned} \tag{27}$$

Substituting Eqs. (21) and (26) into Eq. (27), we can obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_H = -(\xi_i - \tau g_i)\frac{\partial^2 g_j}{\partial q_i \partial q_j} + \frac{\partial}{\partial q_i}\left[(\xi_j - \tau g_j)\frac{\partial g_i}{\partial q_j}\right] - \frac{\partial(\xi_i - \tau g_i)}{\partial q_j}\frac{\partial g_j}{\partial q_i} = 0.$$

So I_H is the conserved quantity of Nambu system.

5. Example

We study a Nambu system with three dimensions with two Nambu-Hamilton quantities, i.e.,

$$H_1 = \frac{1}{2}(q_1^2 + q_2^2) \quad \text{and} \quad H_2 = \frac{1}{2}(q_1^2 - q_2^2 + q_3). \tag{28}$$

Their differential equations of motion are

$$\begin{aligned} \dot{q}_1 &= \frac{1}{2}q_2, \\ \dot{q}_2 &= -\frac{1}{2}q_1, \\ \dot{q}_3 &= -2q_1 q_2. \end{aligned} \tag{29}$$

Let

$$F = \begin{pmatrix} F_1 = \dot{q}_1 - (1/2)q_2 \\ F_2 = \dot{q}_2 + (1/2)q_1 \\ F_3 = \dot{q}_3 + 2q_1 q_2 \end{pmatrix} \tag{30}$$

and choose the infinitesimal generators as

$$\tau = 1, \quad \xi_1 = \xi_2 = 0, \quad \text{and} \quad \xi_3 = \ln(2q_1^2 + q_3), \tag{31}$$

then we will have

$$X^{(1)} = \frac{\partial}{\partial t} + \xi_3\frac{\partial}{\partial q_3} + \dot{\xi}_3\frac{\partial}{\partial \dot{q}_3},$$

and

$$\begin{aligned} X^{(1)}F &= X^{(1)}\begin{pmatrix} F_1 = \dot{q}_1 - (1/2)q_2 \\ F_2 = \dot{q}_2 + (1/2)q_1 \\ F_3 = \dot{q}_3 + 2q_1 q_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \dfrac{\dot{q}_3 + 2q_1 q_2}{2q_1^2 + q_3} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{2q_1^2 + q_3} \end{pmatrix} \\ &\quad \times \begin{pmatrix} F_1 = \dot{q}_1 - (1/2)q_2 \\ F_2 = \dot{q}_2 + (1/2)q_1 \\ F_3 = \dot{q}_3 + 2q_1 q_2 \end{pmatrix}. \end{aligned} \tag{32}$$

We obtain conformal factors as

$$\boldsymbol{\iota}_i^j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{2q_1^2 + q_3} \end{pmatrix}. \tag{33}$$

Utilizing Proposition 1, we can also obtain

$$\boldsymbol{\iota}_i^j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{2q_1^2 + q_3} \end{pmatrix}. \tag{34}$$

Choosing $\mu = \exp(q_1^2 - q_2^2)$ and substituting it into Eq. (23), we can obtain a new type of Hojman conserved quantity as

$$I_H = \frac{1}{2q_1^2 + q_3} = \text{const.} \tag{35}$$

6. Conclusion

In this paper we have studied the conformal invariance and a kind of Hojman conserved quantity under infinitesimal transformations. The conformal factor of conformal invariance is obtained by using the Lie symmetry, which is the necessary and sufficient condition under which the conformal invariance can be Lie symmetry. The condition of existence and the form of a kind of Hojman conserved quantity are obtained. This conserved quantity is different from the Hojman conserved quantity in Ref. [20]. The Nambu system is more general than the Hamilton system, which is inapplicable to the study of space with an odd dimension, so the result given in this paper has more general significance. The present result can be generalized to space with any dimension.

References

[1] Noether A E 1918 *Nachr. Akad. Math.* **2** 235

[2] Lutzky M 1979 *J. Phys.* A: *Math. Gen.* **12** 973

[3] Mei F X 2000 *Beijing Inst. Technol.* **9** 120

[4] Mei F X 2001 *Chin. Phys.* **10** 177

[5] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[6] Galiullin A S, Gafarov G G, Malaishka R P and Khwan A M 1997 *Analytical Dynamics of Helmholtz, Birkhoff and Nambu Systems* (Moscow: UFN) p183 (in Russian)

[7] Robert M L and Matthew P 2001 *J. Geom. Phys.* **39** 276

[8] He G and Mei F X 2008 *Chin. Phys.* B **17** 2764

[9] Cai J L 2008 *Chin. Phys. Lett.* **25** 1523

[10] Liu C, Mei F X and Guo Y X 2008 *Acta Phys. Sin.* **57** 6704 (in Chinese)

[11] Fu J L, Wang X J and Xie F P 2008 *Chin. Phys. Lett.* **25** 2413

[12] Cai J L, Luo S K and Mei F X 2008 *Chin. Phys.* B **17** 3170

[13] Liu C, Liu S X, Mei F X and Guo Y X 2009 *Chin. Phys.* B **18** 856

[14] Chen X W, Zhao Y H and Liu C 2009 *Acta Phys. Sin.* **58** 5150 (in Chinese)

[15] Cai J L 2010 *Int. J. Theor. Phys.* **49** 201

[16] Nambu Y 1973 *Phys. Rev.* D **7** 2405

[17] Mukunda N and Sudarshan E C G 1976 *Phys. Rev.* D **13** 2846

[18] Ogawa T and Sagae T 2000 *Int. J. Theor. Phys.* **39** 2875

[19] Zhang K, Wang C and Zhou L B 2008 *Acta Phys. Sin.* **57** 6718 (in Chinese)

[20] Lin P, Fang J H and Pang T 2008 *Chin. Phys.* B **17** 4361

[21] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) p38 (in Chinese)