A new model analysis of the third harmonic voltage in inductive measurement for critical current density of superconducting films*

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The critical current density J_c is one of the most important parameters of high temperature superconducting films in superconducting applications, such as superconducting filter and superconducting Josephson devices. This paper presents a new model to describe inhomogeneous current distribution throughout the thickness of superconducting films applying magnetic field by solving the differential equation derived from Maxwell equation and the second London equation. Using this model, it accurately calculates the inductive third-harmonic voltage when the film applying magnetic field with the inductive measurement for J_c. The theoretic curve is consistent with the experimental results about measuring superconducting film, especially when the third-harmonic voltage just exceeds zero. The J_c value of superconducting films determined by the inductive method is also compared with results measured by four-probe transport method. The agreements between inductive method and transport method are very good.

Keywords: inhomogeneous current, third-harmonic voltage, critical current density, magnetic field

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1. Introduction

Superconducting films of large area with high, uniformly distributed critical current density J_c have been developed for electric-power applications.[1,2] For this reason the critical current density that represents the film quality needs to be measured in samples of different shapes nondestructively. The inductive measurement method using the third-harmonic voltage V_3 is one of the most reliable methods for non-destructive measurement of J_c.[3−7] The appearance of the harmonic component in the induced voltage originates from a nonlinear magnetization current due to flux pinning which can be described by the Bean critical state model.[3] Years ago Claassen *et al.*[4] proposed a nondestructive and contactless method to measure J_c in superconducting films. In this method, an AC magnetic field is generated by a sinusoidal driving current $I_0 \cos \omega t$ flowing in a coil placed near the upper surface of a superconducting film, and the amplitude of the third-harmonic voltage V_3 induced in the coil is simultaneously measured. The J_c is calculated by defining a threshold criterion I_{th} when a third harmonic voltage V_3 appears suddenly and gets to 0.1 mV.

Later, Mawatari *et al.*[5,6] derived the theoretical relationship between I_0 and V_3 by using the Bean critical state model with an assumption that the magnetic field is parallel to the surface of the film. They presented an expression of surface current K_s when films go beyond the Bean critical state. By developing this theoretical model they provide the scientific basis of the inductive method for measuring the critical density J_c in large-area superconducting films. However, the magnetic field is not parallel to the surface when the magnetic field penetrates into the film, especially around the central area of the coil where the magnetic field is strong. The inhomogeneity of magnetic field has the result that the measured m, which is the factor originating from the fitting of the data near $I_0 - I_{th}$ with $V_3 \propto (I_0 - I_{th})^m$, is larger than the theoretical m in Refs. [5] and [6]. Therefore their assumption is not very accurate to calculate the distribution of sheet current when the magnetic shielding is not effective.[5]

For yielding in an exact way the relationship between V_3 and I_0 near $I_0 - I_{th}$ in the critical state, we present a new model in this paper. Firstly, we

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analyse the distribution of current density inside superconducting films applying DC magnetic field, and obtain the inhomogeneous distribution of J_s in the critical state by using the magnetic screen. Then the expression of J_s for the superconducting film applying AC magnetic field is deduced. Finally we calculate the relationship between the third harmonic voltage and driving current. To confirm the accuracy of our method, we have measured Tl-2212 films under different frequencies. The theoretic curve (V_3–I_0) agrees well with the experimental result. Comparing with the actual J_c-value from four-probe transport method, we estimate the error of J_c based on our model as 5% that will be explained by Table 1 of Section 4 in detail.

2. The current density of superconducting film applying magnetic field

A superconducting film applying magnetic field will undergo the Meissner state, and the critical state as increasing the magnetic field. In the Meissner state, the magnetic field inside the film is mainly distributed within the penetration depth λ_L and the current density value in the whole film is lower than J_c. In the critical state, the magnetic field is still shielded above the film,[8] but the current density in some areas of the film has arrived at J_c.

In this section, the distribution of the current density inside the film applied in magnetic field is analysed thoroughly.

2.1. Distribution of current for the film applying DC magnetic field

As shown in Fig. 1, a small coil applying current I_0 is mounted above the film. Assuming the coil is a continuous conductor and the current flowing in it is uniform, we obtain the current density in the coil as follows:

$$J_{\text{coil}} = \frac{I_0 \cdot P}{(R_2 - R_1)(Z_2 - Z_1)}$$
$$R_1 < r < R_2, \quad Z_1 < z < Z_2. \tag{1}$$

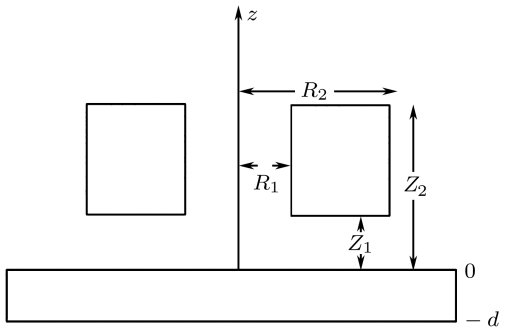


Fig. 1. The sketch of the cross section of coil in cylindrical coordinates, the small coil with the number of turns $P = 528$, the inner radius R_1 of 1 mm, the outer radius R_2 of 2.5 mm and the height of 1.0 mm, is mounted on a superconducting film. The film-coil distance Z_1 is 0.25 mm. The thickness d of the film is 700 nm.

According to Maxwell equation $\boldsymbol{\nabla} \times \boldsymbol{B} = -\mu_0(J_s + J_{\text{coil}})$ and the second London equation $\boldsymbol{\nabla} \times \boldsymbol{J}_s = \boldsymbol{B}/\lambda_L^2$, we obtain the second order differential equation about magnetic field in the Meissner state as

$$\nabla^2 \boldsymbol{B} = \boldsymbol{B}/\lambda_L^2 - \mu_0 \nabla \times \boldsymbol{J}_{\text{coil}}, \tag{2}$$

with the boundary conditions:

$$B_{Z1} = B_{Z2}, \quad \partial B_{Z1}/\partial z - \partial B_{Z2}/\partial z = \mu_0 I_0. \tag{3}$$

Solving the differential equation, we can calculate the parallel components and normal components of B inside the film ($-d < z < 0$) as:[7]

$$B_z(z,\rho) = \mu_0 J_{\text{coil}} \int_0^\infty \left[\int_{R_1}^{R_2} \mathrm{J}_1(\eta r) r \,\mathrm{d}r \cdot \int_{Z_1}^{Z_2} \mathrm{e}^{-\delta\eta} \,\mathrm{d}\delta \right] G(\eta, z, d, \lambda_L) \mathrm{J}_0(\rho\eta) \eta \,\mathrm{d}\eta, \tag{4}$$

$$B_\rho(z,\rho) = -\mu_0 J_{\text{coil}} \int_0^\infty \left[\int_{R_1}^{R_2} \mathrm{J}_1(\eta r) r \,\mathrm{d}r \cdot \int_{Z_1}^{Z_2} \mathrm{e}^{-\delta\eta} \,\mathrm{d}\delta \right] \frac{\partial G(\eta, z, d, \lambda_L)}{\partial z} \mathrm{J}_1(\rho\eta) \eta \,\mathrm{d}\eta, \tag{5}$$

where J_0 and J_1 are the zero and first order Bessel functions respectively, η is the wave number in the ρ–θ plane, and G is a scaling function describing the distribution of the magnetic field B along z axis,

$$G(\eta, z, d, \lambda_L) = \frac{2\eta\lambda_L\sqrt{1+\eta^2\lambda_L^2}\cosh[((z+d)/\lambda_L)\sqrt{1+\eta^2\lambda_L^2}] + 2\eta^2\lambda_L^2 \sinh[((z+d)/\lambda_L)\sqrt{1+\eta^2\lambda_L^2}]}{(1+2\eta^2\lambda_L^2)\sinh[(d/\lambda_L)\sqrt{1+\eta^2\lambda_L^2})] + 2\eta\lambda_L\sqrt{1+\eta^2\lambda_L^2}\cosh[(d/\lambda_L)\sqrt{1+\eta^2\lambda_L^2})]}.$$

Comparing with the work in Ref. [7], we derive continuous expressions of the current distribution in the coil by using our analysis. The continuous expression is good for calculating the inhomogeneous J_s of the film in the critical state by using the magnetic screen. According to the London equation, the current density of superconducting films under the Meissner state can be written as

$$J_s = (-1/2)J_{coil}Q(\rho, z), \tag{6}$$

where

$$Q(\rho,z) = \int_0^\infty \frac{G(\eta,z,d,\lambda_L)}{\lambda_L^2}\int_{R_1}^{R_2} \mathrm{J}_1(\eta r) r\,\mathrm{d}r \times \int_{Z_1}^{Z_2} \mathrm{e}^{-\delta\eta}\,\mathrm{d}\delta\cdot \mathrm{J}_1(\eta\rho)\,\mathrm{d}\eta. \tag{7}$$

As shown in Fig. 2, the shield current density J_s of the film increases with the increase of driving current. When the driving current is big enough to induce J_s at an arbitrary point of the film arriving at the critical value J_c, the film will enter the critical state. We define the corresponding driving current as the threshold driving current I_{th}. Since the maximal value of J_s is at the points with coordinate $\rho = \rho_0$ and $z = 0$ (as shown in Fig. 2), I_{th} can be written as

$$J_c = \frac{I_{th}\cdot P}{2(R_2-R_1)(Z_2-Z_1)}Q(\rho_0, 0). \tag{8}$$

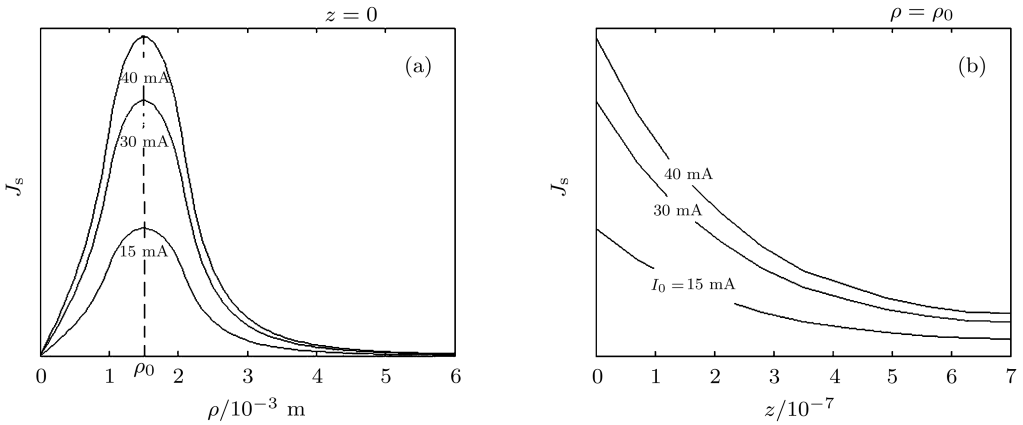


Fig. 2. The distribution of J_s in the Meissner state for different driving currents ($I_0 = 15, 30, 40$ mA): (a) at $z = 0$ plane, J_s value is maximal at $\rho = \rho_0$ in the horizontal direction; (b) $\rho = \rho_0$ camber, J_s value decreases along z axis.

A coefficient k can be used to link the threshold current I_{th} and the critical current density J_c and written as

$$k = \frac{P}{2(R_2-R_1)(Z_2-Z_1)}Q(\rho_0, 0), \tag{9}$$

where k is not affected by the material parameters of J_c and d, but is determined by the configuration of the coil.

As the increase of the applied current, the region where J_s equals J_c will expand both perpendicularly and horizontally. In a horizontal direction, the region spreads outward and inward with the axis of $\rho = \rho_0$. In vertical direction, the region spreads downwards from $z = 0$ plane. We define that the horizontal boundary of the region at $z = 0$ plane are ρ_1 and ρ_2 respectively. In the region of $\rho_1 < \rho < \rho_2$, the film can be considered as two parts along z axis: in the upper part, J_s value is equal to J_c, while in the lower part it is less than J_c and obeys the equation: $\nabla^2 J_s = J_s/\lambda_L^2$. We mark the dividing line as $z = -\gamma$ and the value of γ is related to the radius. So the edge condition for the lower part is $J_s(z = -\gamma(\rho)) = -J_c$. According to the differential equation and considering the continuity of J_s, we can write the expression of J_s along z axis for the region of $\rho_1 < \rho < \rho_2$ in the critical state as

$$J_s = \begin{cases} -J_c, & -\gamma < z < 0, \\ \dfrac{-I_{th}\cdot P}{2(R_2-R_1)(Z_2-Z_1)}Q(\rho, z+\gamma), & -d < z < -\gamma. \end{cases} \tag{10}$$

In the critical state, the magnetic field is shielded above the film and the magnetic field at the back of the film is considered to be zero.[3,8] Therefore we can obtain

$$H_- = K_s/2 \quad (\rho_1 < \rho < \rho_2), \tag{11}$$

where H_- is the magnetic field induced by the driving current at $z = -d$ and K_s is the sheet current density of the film inside $[\rho_1, \rho_2]$. Expanding Eq. (11), we can calculate and depict the depth γ for each radius ρ through the equation:

$$\frac{2I_0 \int_0^{\infty} \int_{R_1}^{R_2} \mathrm{J}_1(\eta r) r \,\mathrm{d}r \int_{Z_1}^{Z_2} \mathrm{e}^{-\delta\eta} \,\mathrm{d}\delta \cdot \mathrm{J}_1(\eta\rho)\eta \,\mathrm{d}\eta}{I_{\mathrm{th}}} = Q(\rho_0, 0) \cdot \gamma + \int_{-d}^{-\gamma} Q(\rho, z+\gamma) \,\mathrm{d}z. \tag{12}$$

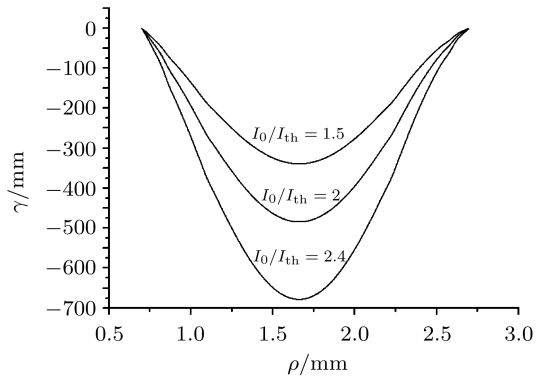


Fig. 3. The radius dependence of $\gamma(\rho)$ for different driving currents.

According to Eq. (12), we depict the distribution of current of the film under the critical state in Fig. 4.

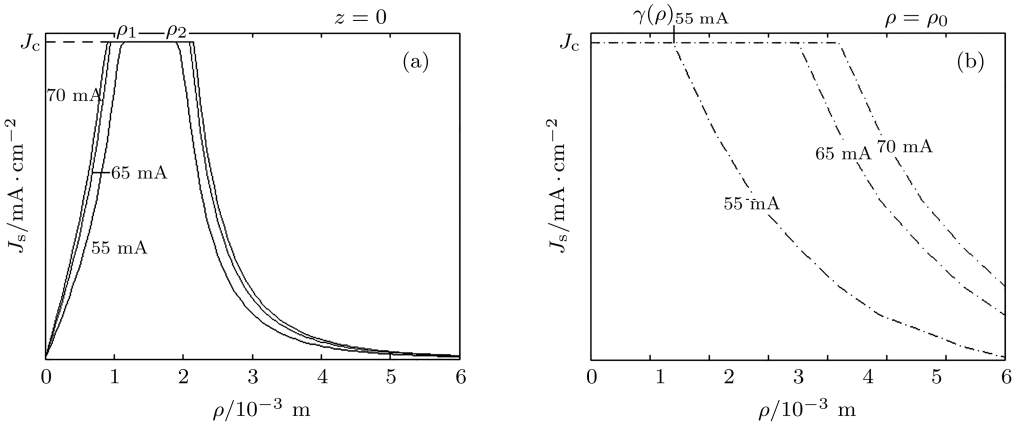


Fig. 4. The distribution of J_s in the critical state for different driving currents ($I_0 = 55, 65, 70$ mA): (a) at $z = 0$ plane; (b) $\rho = \rho_0$ camber.

When the driving current increases continuously, the film will be gradually penetrated by the magnetic field to the backside and goes beyond the critical state. Finally, the current density in the whole film will equal J_c.

2.2. The current for the film in the critical state during applying sinusoidal current

When a sinusoidal current $I = I_0 \cos \omega t$ is applied in the driving coil, the current of the film in the Meissner state will follow the sinusoidal change of the applied current because of Faraday's law. However, when the amplitude I_0 is larger than I_{th}, due to the flux pining, there will exist a region where the value of shield current density keeps at J_c. As mentioned above, the boundary of the region can be marked as ρ_1, ρ_2, $\gamma(\rho)$, which are proportional to the amplitude I_0 of the sinusoidal current and independent of time. Since J_s in this region is no longer a sinusoidal curve, it will contain harmonic components and induce third-harmonic voltage in the coil. When applied magnetic field changes, an electromotive force with opposite direction of J_s will be induced at the upper surface of the film and make the current at the surface in the reverse direction.[3] Thus two opposite direction currents will flow in the region of the critical state as shown in Figs. 5(a) and 5(b). So the current density induced by the AC magnetic field in the critical state is given as follows:[9]

for $0 < \omega t < \pi$,

$$\begin{cases} J_s = J_c, & 0 > z > -\gamma_1(\rho), \\ J_s = -J_c, & -\gamma_1(\rho) > z > -\gamma(\rho), \end{cases} \tag{13a}$$

for $\pi < \omega t < 2\pi$

$$\begin{cases} J_{\rm s} = -J_{\rm c}, & 0 > z > -\gamma_2(\rho), \\ J_{\rm s} = J_{\rm c}, & -\gamma_2(\rho) > z > -\gamma(\rho), \end{cases} \tag{13b}$$

where $\gamma_1(\rho) = \gamma(\rho)(1 - \cos\omega t)/2$, $\gamma_2(\rho) = \gamma(\rho)(1 + \cos\omega t)/2$, and the depth $\gamma(\rho)$ is given by Eq. (12).

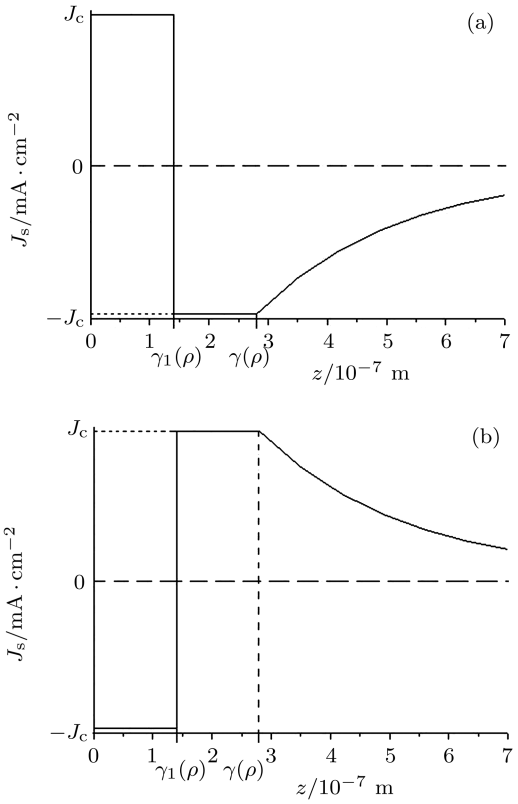


Fig. 5. The $J_{\rm s}(\rho, z, t)$ dependence of z axis in the critical state during applying AC driving current $I_0 \cos\omega t$ ($I_0 > I_{\rm th}$), for $\rho = \rho_0$: (a) $\omega t = \pi/2$; (b) $\omega t = 3\pi/2$.

3. The relationship between the third-harmonic voltage V_3 and driving current

After obtaining the expression of $J_{\rm s}(t)$, we can derive the magnetic flux in the coil $\Phi_{\rm coil}$ and then obtain the relationship between the third-harmonic voltage and driving current. The vector potential arising from $J_{\rm s}(t)$ is given by:

$$\begin{aligned} & A_{\rm s}(r, \delta, t) \\ = & \frac{\mu_0}{4\pi} \int_0^{\infty} \int_0^{2\pi} \rho {\rm d}\rho {\rm d}\theta \\ & \times \int_{-\gamma}^{0} \frac{J_{\rm s}(\rho, z, t)}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta + (z - \delta)^2}} {\rm d}z. \end{aligned}$$

The $A_{\rm s}$ is expressed as the multipole expansion $A_{\rm s} = A_{\rm s1} + A_{\rm s2} + \cdots$, where

$$\begin{aligned} & A_{\rm s1}(r, \delta, t) \\ = & \frac{\mu_0}{4\pi} \int_0^{\infty} \int_0^{2\pi} \rho {\rm d}\rho {\rm d}\theta \\ & \times \int_{-\gamma}^{0} \frac{J_{\rm s}(\rho, z, t)}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta + \delta^2}} {\rm d}z, \\ & A_{\rm s2}(r, \delta, t) \\ = & \frac{\mu_0}{4\pi} \int_0^{\infty} \int_0^{2\pi} \rho {\rm d}\rho {\rm d}\theta \\ & \times \int_{-\gamma}^{0} \frac{J_{\rm s}(\rho, z, t) z}{\sqrt{(\rho^2 + r^2 - 2\rho r\cos\theta + \delta^2)^3}} {\rm d}z. \end{aligned}$$

The whole flux in the coil can be approximately written as

$$\begin{aligned} & \Phi_{\rm coil} \\ \approx & L_{\rm c} I_0 \cos\omega t + \frac{2\pi P}{(R_2 - R_1)(Z_2 - Z_1)} \int_{R_1}^{R_2} r {\rm d}r \\ & \times \int_{Z_1}^{Z_2} {\rm d}\delta (A_{\rm s1} + A_{\rm s2}), \end{aligned} \tag{14}$$

where the first term on the right-hand side arises from the driven current, and $L_{\rm c}$ is the self-inductance of the coil. According to Faraday's law, the induced voltage in the coil is give by

$$V(t) = R_{\rm c} I_0 \cos\omega t - {\rm d}\Phi_{\rm coil}/{\rm d}t. \tag{15}$$

When the film is in the critical state, $A_{\rm s1}$ can be written as

$$\begin{aligned} & A_{\rm s1}(r, \delta, t) \\ = & \frac{\mu_0}{4\pi} \int_0^{\infty} \int_0^{2\pi} \rho {\rm d}\rho {\rm d}\theta \\ & \times \frac{K_{\rm s}(\rho, t)}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta + \delta^2}}. \end{aligned}$$

Since $K_{\rm s}(\rho, t) = -J_{\rm c}\gamma(\rho)\cos\omega t$, $A_{\rm s1}$ will not contribute to the nonlinear response in the coil. Only $A_{\rm s2}$ will cause the third-harmonic voltage V_3. So we can obtain the following expression:

$$V_3(t) = \frac{\mu_0 P}{2(R_2 - R_1)(Z_2 - Z_1)} \int_{R_1}^{R_2} r {\rm d}r \int_{Z_1}^{Z_2} {\rm d}\delta \int_{\rho_1}^{\rho_2} \int_0^{2\pi} \rho {\rm d}\rho {\rm d}\theta \frac{{\rm d}(\int_{-\gamma}^{0} J_{\rm s}(\rho, z, t) {\rm d}z)_{\rm 3rd}/{\rm d}t}{\sqrt{(\rho^2 + r^2 - 2\rho r\cos\theta + \delta^2)^3}}. \tag{16}$$

Substituting J_s into Eq. (16), we can obtain the amplitude of $V_3(t)$ as

$$V_3 = \frac{2\mu_0 P f J_c}{5(R_2-R_1)(Z_2-Z_1)} \int_{R_1}^{R_2} r\,\mathrm{d}r \int_{Z_1}^{Z_2} \mathrm{d}\delta \int_{\rho_1}^{\rho_2} \int_0^{2\pi} \rho\,\mathrm{d}\rho\,\mathrm{d}\theta \frac{\gamma^2(\rho)}{\sqrt{(\rho^2+r^2-2\rho r\cos\theta+\delta^2)^3}}. \quad (17)$$

4. Critical current density measurements and experimental error analysis

To verify our method, a $Tl_2Ba_2CaCu_2O_8$ (Tl-2212) superconducting film fabricated on 10×10 mm^2 CeO_2 buffered r-cut sapphire substrates[10,11] is measured. The film has the thickness of 700 nm and the penetration depth $\lambda_L = 250$ nm. We have measured the critical current density for four different positions of the same film by both the inductive method with the criterion $V_3 = 0.07$ mV[12] and four-probe transport method. Considering the mean of measuring transport J_c-value as the true critical current density and substituting this J_c into Eq. (17), we calculated the theoretical curves (shown in Fig. 6 by solid lines). The comparison of the theoretical curve and experimental result is also shown in Fig. 6.

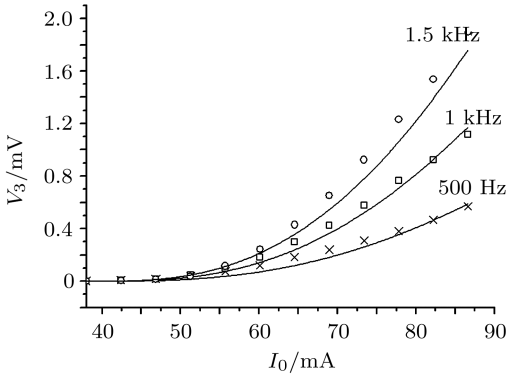


Fig. 6. The comparison of the theoretical curves and experimental results at three frequencies (500 Hz, 1 kHz, 1.5 kHz). The solid lines correspond to the theoretical curve calculated according to Eq. (17).

As shown in the figure, the theoretical curves agree well with the measuring results. Especially when the harmonic voltage happens suddenly (i.e. near $I_0 - I_{th}$), there is almost no scattering between the theoretical and experimental result. Originating from $V_3 \propto (I_0 - I_{th})^m$, we find that the fitting factor m of the theoretical curves are almost equal to those of experimental curves. It implies that the calculation of the distribution of the current in the film in the critical state is more accurate than that in Ref. [5] and the third-harmonic voltage evaluated by our model is reliable.

The measurement results and the relative error between transport and inductive method are shown in Table 1. In the inductive measurement, the linking coefficient k can be calculated by Eq. (9) to be 1.34×10^{12} m^{-2} according to the configuration of the coil described in Fig. 1, the experimental curve of 1 kHz is chosen to determine the I_{th} value with the criterion $V_3 = 0.07$ mV. As we considered that the mean value of transport J_c was the true value in the film, the error of inductive J_c is about 5%. It is goodish result for non-destruction measurement for J_c.

Table 1. Critical current density J_c measured by the inductive third-harmonic method and four-probe transport method.

position	transport J_c/(mA/cm^2) (at $E = 1 \times 10^{-3}$ V/m)	inductive J_c /(mA/cm^2)
1	3.50	3.53(3.4%)
2	3.373	3.21(5.97%)
3	3.438	3.57(4.57%)
4	3.345	3.54(3.69%)

5. Conclusion

In this paper, we consider the inhomogeneous distribution of the current density J_s through the superconducting films applying magnetic field. Utilizing the expression of J_s in the critical state for films applying AC magnetic field, the third-harmonic voltage in inductive measure for critical current density is calculated. The theoretical curve is highly consistent with the measuring results for the third-harmonic voltage of Tl-2212 film. Using our model, we estimate that the error of determining J_c is 5%.

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