Poisson theory and integration method for a dynamical system of relative motion*

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(Received 30 July 2010; revised manuscript received 21 August 2010)

This paper focuses on studying the Poisson theory and the integration method of dynamics of relative motion. Equations of a dynamical system of relative motion in phase space are given. Poisson theory of the system is established. The Jacobi last multiplier of the system is defined, and the relation between the Jacobi last multiplier and the first integrals of the system is studied. Our research shows that for a dynamical system of relative motion, whose configuration is determined by n generalized coordinates, the solution of the system can be found by using the Jacobi last multiplier if $(2n-1)$ first integrals of the system are known. At the end of the paper, an example is given to illustrate the application of the results.

Keywords: dynamics of relative motion, Poisson theory, method of integration, Jacobi last multiplier
PACS: 45.20.Jj, 02.30.Hq, 02.30.Jr **DOI:** 10.1088/1674-1056/20/2/024501

1. Introduction

The study on dynamics of complicated system has more important significance along with modern development of science and technology. The motion of the complicated system consists of the motion of the carrier body and the motion of the carried body relative to the carrier body. The motion of the complicated system can be described in the inertial coordinate system or in the moving coordinate system fixed on the carrier body. The dynamics in the moving coordinate system is called dynamics of relative motion. Whittaker[1] obtained the Lagrange equations of holonomic systems subjected to uniform rotation constraints. Lur'e[2] studied the dynamics of general relative motion of holonomic systems. Mei always takes the dynamics of relative motion as a special topic to summarize and research in his monographs.[3−8] In recent twenty years, many important results on study of the dynamics of relative motion have been obtained.[3−29] In this paper, we further study the integration theory of dynamics of relative motion for holonomic systems, and establish the Poisson theory of dynamics of relative motion and present a new integration method of dynamics of relative motion, i.e., the method of Jacobi last multiplier.

2. Equations of dynamics of relative motion

Suppose that the configuration of relative motion of a dynamical system is determined by n generalized coordinates q_s $(s=1,\ldots,n)$, and the equations of dynamics of relative motion can be written as follows:[5]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{\mathrm{r}}}{\partial \dot{q}_s}-\frac{\partial L_{\mathrm{r}}}{\partial q_s}=Q''_s+Q^{\dot{\omega}}_s+\Gamma_s,\quad (s=1,\ldots,n), \tag{1}$$

where

$$L_{\mathrm{r}}=T_{\mathrm{r}}-V-V^0-V^{\omega} \tag{2}$$

is the Lagrangian of relative motion, T_{r} is the kinetic energy of relative motion, V is the potential energy, V^0 is the potential energy of the uniform field, V^{ω} is the potential energy of the centrifugal forces; Q''_s are the non-potential generalized forces, $Q^{\dot{\omega}}_s$ are the generalized rotating inertial forces, and Γ_s are the generalized gyroscopic forces.[5]

Let

$$p_s=\frac{\partial L_{\mathrm{r}}}{\partial \dot{q}_s},\quad H=\left(p_s\dot{q}_s-L_{\mathrm{r}}\right)\big|_{(q,p,t)}\ , \tag{3}$$

then equations (1) can be written in the partial canonical form

$$\dot{q}_s=\frac{\partial H}{\partial p_s},\quad \dot{p}_s=-\frac{\partial H}{\partial q_s}+\tilde{Q}''_s+\tilde{Q}^{\dot{\omega}}_s+\tilde{\Gamma}_s,\quad (s=1,\ldots,n), \tag{4}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 10972151).
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where

$$\begin{aligned}
\tilde{Q}_s'' &= \tilde{Q}_s''(\boldsymbol{q},\boldsymbol{p},t) = Q_s''(\boldsymbol{q},\dot{\boldsymbol{q}}(\boldsymbol{q},\boldsymbol{p},t),t),\\
\tilde{Q}_s^{\dot{\omega}} &= \tilde{Q}_s^{\dot{\omega}}(\boldsymbol{q},\boldsymbol{p},t) = Q_s^{\dot{\omega}}(\boldsymbol{q},\dot{\boldsymbol{q}}(\boldsymbol{q},\boldsymbol{p},t),t),\\
\tilde{\Gamma}_s &= \tilde{\Gamma}_s(\boldsymbol{q},\boldsymbol{p},t) = \Gamma_s(\boldsymbol{q},\dot{\boldsymbol{q}}(\boldsymbol{q},\boldsymbol{p},t),t).
\end{aligned} \tag{5}$$

Suppose

$$\begin{aligned}
&\tilde{Q''}_s + \tilde{Q}_s^{\dot{\omega}} + \tilde{\Gamma}_s = -\Omega_{sk}\frac{\partial H}{\partial p_k},\\
&(\Omega_{sk}) = \begin{pmatrix} \Omega_{11} & \ddots & 0\\ 0 & & \Omega_{nn}\end{pmatrix};
\end{aligned} \tag{6}$$

and let

$$a^{\mu} = \begin{cases} q_{\mu}, & (\mu = 1,\ldots,n),\\ p_{\mu-n}, & (\mu = n+1,\ldots,2n),\end{cases} \tag{7}$$

then equations (4) can be written in the contravariant algebraic form

$$\dot{a}^{\mu} - S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = 0, \quad (\mu,\nu = 1,\ldots,2n), \tag{8}$$

where

$$\begin{aligned}
&S^{\mu\nu} = \omega^{\mu\nu} + T^{\mu\nu}, \quad (\omega^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & I_{n\times n}\\ -I_{n\times n} & 0_{n\times n}\end{pmatrix},\\
&(T^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & (-\Omega_{kk})_{n\times n}\end{pmatrix}.
\end{aligned} \tag{9}$$

3. Poisson theory of systems of dynamics of relative motion

Define a generalized Poisson bracket

$$[A,H] \overset{\text{def.}}{=} \frac{\partial A}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}. \tag{10}$$

Differentiating an arbitrary differentiable function $F = F(\boldsymbol{a},t)$ with respect to time t along the trajectory of dynamics of relative motion, we have

$$\frac{\bar{\mathrm{d}}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + [F,H], \tag{11}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\frac{\partial}{\partial a^{\mu}}. \tag{12}$$

From the formula (11), we obtain the follow proposition.

Proposition 1 For a dynamical system of relative motion under consideration, the necessary and sufficient condition under which $I(\boldsymbol{a},t) = \text{const.}$ is a first integral of the system is to satisfy the following generalized Poisson equation

$$\frac{\partial I}{\partial t} + [I,H] = 0. \tag{13}$$

Let $I = H$, then the condition (13) becomes

$$\begin{aligned}
0 &= \frac{\partial H}{\partial t} + [H,H]\\
&= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\\
&= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial a^{\mu}}T^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\\
&= \frac{\partial H}{\partial t} - \Omega_{ss}\left(\frac{\partial H}{\partial a^{n+s}}\right)^2\\
&= \frac{\partial H}{\partial t} + \left(\tilde{Q}_s'' + \tilde{Q}_s^{\dot{\omega}} + \tilde{\Gamma}_s\right)\dot{q}_s.
\end{aligned} \tag{14}$$

Therefore, we have the following proposition.

Proposition 2 For a dynamical system of relative motion under consideration, the necessary and sufficient condition under which the Hamiltonian H is a first integral of the system is to satisfy the formula (14).

From the formula (14), the Proposition 2 has the following deduction.

Proposition 3 For a dynamical system of relative motion under consideration, if t is not explicitly contained in H, and $\left(\tilde{Q}_s'' + \tilde{Q}_s^{\dot{\omega}} + \tilde{\Gamma}_s\right)$ are the generalized gyroscopic forces, then the Hamiltonian H is a first integral of the system.

Taking the partial derivative of Eq. (13) with respect to time t, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial I}{\partial a^{\mu}}\right)S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} + \frac{\partial I}{\partial a^{\mu}}\frac{\partial S^{\mu\nu}}{\partial t}\frac{\partial H}{\partial a^{\nu}}\\
&\quad + \frac{\partial I}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial a^{\nu}}\right) = 0.
\end{aligned} \tag{15}$$

If t is not explicitly contained in $S^{\mu\nu}$ and H, then the above formula becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial t}\right) + \left[\frac{\partial I}{\partial t},H\right] = 0. \tag{16}$$

Therefore, we have the following Proposition 4.

Proposition 4 For a dynamical system of relative motion under consideration, if $I(\boldsymbol{a},t) = \text{const.}$ is a first integral, and t is not explicitly contained in $S^{\mu\nu}$ and H, then $\partial I/\partial t, \partial^2 I/\partial t^2, \ldots$ are also first integrals of the system.

Similarly, we have the following Proposition 5.

Proposition 5 For a dynamical system of relative motion under consideration, if $I(\boldsymbol{a},t) = \text{const.}$ is

a first integral, and a^ρ is not explicitly contained in $S^{\mu\nu}$ and H, then $\partial I/\partial a^\rho, \partial^2 I/\partial {a^\rho}^2, \ldots$ are also first integrals of the system.

Propositions 1–5 constitute the Poisson theory of the dynamical system of relative motion under consideration.

4. Method of Jacobi last multiplier of systems dynamics of relative motion

Definition For a dynamical system of relative motion under consideration, if there exists some function $M = M(\boldsymbol{a}, t)$ satisfying the following condition

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial a^\mu}\left(MS^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right) = 0, \tag{17}$$

then function M is called the Jacobi last multiplier of the system.

Obviously we have

$$\frac{\partial}{\partial a^\mu}\left(\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right) = 0. \tag{18}$$

Therefore the condition (17) is equal to the following equation

$$\frac{\bar{\mathrm{d}}M}{\mathrm{d}t} + M\frac{\partial}{\partial a^\mu}\left(T^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right) = 0. \tag{19}$$

Equation (17) or (19) is called the determining equation of Jacobi last multiplier of the dynamical system of relative motion under consideration.

For the above definition, we have the following Proposition 6.

Proposition 6 For a dynamical system of relative motion under consideration, if the following condition is satisfied

$$\frac{\partial}{\partial a^\mu}\left(T^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right) = 0, \tag{20}$$

then $M = 1$ is the last multiplier of the system, and all the integrals of the system are its Jacobi last multipliers.

Proposition 7 For a dynamical system of relative motion under consideration, if $M = M(\boldsymbol{a}, t)$ and $N = N(\boldsymbol{a}, t)$ are two Jacobi last multipliers of the system, then its quotient M/N is a first integral of the system.

Proof According to Proposition 1 and using the determining equation of Jacobi last multiplier of the dynamical system of relative motion, we can easily obtain the result of this proposition.

Proposition 8 For a dynamical system of relative motion under consideration, if $(2n-1)$ first integrals of the system are known, i.e.

$$I_\sigma(\boldsymbol{a}, t) = C_\sigma, \quad (\sigma = 1, \ldots, 2n-1), \tag{21}$$

then the integral of the system is

$$\int \frac{M^*}{\Delta^*}\left[\mathrm{d}a^{2n} - \left(S^{2n,\nu}\frac{\partial H}{\partial a^\nu}\right)^* \mathrm{d}t\right] = C_{2n}, \tag{22}$$

where Δ denotes the Jacobian

$$\Delta = \frac{\partial(I_1, I_2, \ldots, I_{2n-1})}{\partial(a^1, a^2, \ldots, a^{2n-1})}. \tag{23}$$

Here $()^*$ denotes that a^σ are expressed as functions of a^{2n} and t, and C_μ $(\mu = 1, \ldots 2n)$ are the constants of integration.

Proof For the dynamical system of relative motion under consideration, similar to the deduction given by Whittaker in Ref. [1], we can easily obtain

$$\begin{aligned}\frac{\partial}{\partial a^\mu}\left(S^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right) = \Delta\Bigg\{&\frac{\partial}{\partial a^{2n}}\left[\left(S^{2n,\nu}\frac{\partial H}{\partial a^\nu}\right)^* \Big/ \Delta^*\right] \\ &+ \frac{\partial}{\partial t}\left(\frac{1}{\Delta^*}\right)\Bigg\}.\end{aligned} \tag{24}$$

Since M is a Jacobi last multiplier of the system, we have

$$\frac{1}{M}\frac{\bar{\mathrm{d}}M}{\mathrm{d}t} + \frac{\partial}{\partial a^\mu}\left(S^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right) = 0. \tag{25}$$

From Eqs. (24) and (25), we obtain

$$\begin{aligned}\frac{1}{\Delta M}\frac{\bar{\mathrm{d}}M}{\mathrm{d}t} &+ \frac{\partial}{\partial a^{2n}}\left[\left(S^{2n,\nu}\frac{\partial H}{\partial a^\nu}\right)^* \Big/ \Delta^*\right] \\ &+ \frac{\partial}{\partial t}\left(\frac{1}{\Delta^*}\right) = 0,\end{aligned} \tag{26}$$

or

$$\begin{aligned}&\frac{\partial}{\partial a^{2n}}\left[\left(S^{2n,\nu}\frac{\partial H}{\partial a^\nu}\right)^* M^* \Big/ \Delta^*\right] + \frac{\partial}{\partial t}\left(\frac{M^*}{\Delta^*}\right) \\ &= 0,\end{aligned} \tag{27}$$

which shows that the expression

$$\frac{M^*}{\Delta^*}\left[\mathrm{d}a^{2n} - \left(S^{2n,\nu}\frac{\partial H}{\partial a^\nu}\right)^* \mathrm{d}t\right]$$

is the perfect differential of some function of a^{2n} and t, and this establishes the proposition.

Proposition 8 presents a new method, i.e., the method of Jacobi last multiplier, to integrate the dynamical equations of relative motion. In general, the

method of Jacobi last multiplier has no restrictions such as those in the Hamilton–Jacobi method. The main difficulty of this method lies in: (i) a sufficient number of independent first integrals must be known in advance; (ii) the solution of the determining equation (19) should be found. However, as long as $(2n-1)$ first integrals are known and the Jacobi last multiplier M of the system might be found, one can obtain the solution of the dynamical system of relative motion.

5. Examples

To illustrate the above result, we give an example.

Consider a particle with mass m moving without friction in a plane P which goes through a fixed straight line Oz (the angle between the line Oz and a vertical line is a constant value α), the plane P at the same time rotating about the fixed line Oz with constant angular velocity ω.[7] Let us try to solve this problem by the method of Jacobi last multiplier.

Choose the coordinates z and r of the particle in the moving plane P as the generalized coordinates, which express the relative position of the particle.[7] The kinetic energy of relative motion is

$$T_{\rm r} = \frac{1}{2}m\left(\dot{r}^2+\dot{z}^2\right) = \frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2\right). \tag{28}$$

The potential energy of the centrifugal forces is

$$V^{\omega} = -\frac{1}{2}m\omega^2 q_2^2. \tag{29}$$

The non-potential generalized forces are

$$Q_1'' = -mg\cos\alpha, \quad Q_2'' = -mg\sin\alpha\sin\omega t. \tag{30}$$

The generalized momenta and the Hamiltonian are respectively

$$\begin{aligned} &p_1 = m\dot{q}_1, \quad p_2 = m\dot{q}_2,\\ &H = \frac{1}{2m}\left(p_1^2+p_2^2\right) - \frac{1}{2}m\omega^2 q_2^2. \end{aligned} \tag{31}$$

Let

$$a^1 = q_1, \quad a^2 = q_2, \quad a^3 = p_1, \quad a^4 = p_2, \tag{32}$$

then equations (8) yield

$$\dot{a}^{\mu} - S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = 0, \quad (\mu,\nu = 1,\ 2,\ 3,\ 4), \tag{33}$$

where

$$\begin{aligned} &H = \frac{1}{2m}\left[\left(a^3\right)^2+\left(a^4\right)^2\right] - \frac{1}{2}m\omega^2\left(a^2\right)^2,\\ &S^{\mu\nu} = \omega^{\mu\nu}+T^{\mu\nu}, \end{aligned}$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(T^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -\Omega_{11} & 0\\ 0 & 0 & 0 & -\Omega_{22} \end{pmatrix},$$

$$\begin{aligned} &\Omega_{11} = m^2 g\cos\alpha\left(a^3\right)^{-1},\\ &\Omega_{22} = m^2 g\sin\alpha\sin\omega t\left(a^4\right)^{-1}. \end{aligned} \tag{34}$$

The determining equation (19) of the Jacobi last multiplier gives

$$\begin{aligned} &\frac{\partial M}{\partial t} + \frac{\partial M}{\partial a^1}\frac{a^3}{m} + \frac{\partial M}{\partial a^2}\frac{a^4}{m} - \frac{\partial M}{\partial a^3}(mg\cos\alpha)\\ &\quad + \frac{\partial M}{\partial a^4}\left(m\omega^2 a^2 - mg\sin\alpha\sin\omega t\right) = 0. \end{aligned} \tag{35}$$

Equation (35) has the following solutions

$$M = 1, \tag{36}$$

$$M = a^3 + mgt\cos\alpha, \tag{37}$$

$$M = ma^1 - a^3 t - \frac{1}{2}mgt^2\cos\alpha, \tag{38}$$

$$\begin{aligned} M = {} & \frac{1}{2\omega}\exp(\omega t)\left[a^4 - m\omega a^2\right.\\ & \left. + \frac{mg}{2\omega}(\sin\omega t - \cos\omega t)\sin\alpha\right]. \end{aligned} \tag{39}$$

The formulae (37)–(39) are the first integrals of the system. Let

$$I_1 = a^3 + mgt\cos\alpha = C_1, \tag{40}$$

$$I_2 = ma^1 - a^3 t - \frac{1}{2}mgt^2\cos\alpha = C_2, \tag{41}$$

$$\begin{aligned} I_3 = {} & \frac{1}{2\omega}\exp(\omega t)\left[a^4 - m\omega a^2\right.\\ & \left. + \frac{mg}{2\omega}(\sin\omega t - \cos\omega t)\sin\alpha\right] = C_3. \end{aligned} \tag{42}$$

From Eqs. (40)–(42), we have

$$\begin{aligned} a^1 = {} & -\frac{1}{2}gt^2\cos\alpha + \frac{C_1}{m}t + \frac{C_2}{m},\\ a^2 = {} & \frac{1}{m\omega}a^4 + \frac{g}{2\omega}(\sin\omega t - \cos\omega t)\sin\alpha\\ & - \frac{2C_3}{m}\exp(-\omega t),\\ a^3 = {} & -mgt\cos\alpha + C_1. \end{aligned} \tag{43}$$

Performing some deriving work, we obtain

$$\Delta = \frac{\partial(I_1,I_2,I_3)}{\partial(a^1,a^2,a^3)} = -\frac{1}{2}m^2\exp(\omega t) = \Delta^*. \tag{44}$$

Expression (22) gives the following integral

$$\int \frac{1}{-\frac{1}{2}m^2 \exp(\omega t)} \left\{ \mathrm{d}a^4 - \left[\omega a^4 - \frac{1}{2} mg (\sin \omega t + \cos \omega t) \sin \alpha - 2\omega^2 C_3 \exp(-\omega t) \right] \mathrm{d}t \right\} = C_4. \tag{45}$$

After integration, we obtain

$$a^4 = -\frac{1}{2} m^2 C_4 \exp(\omega t) + \omega C_3 \exp(-\omega t) + \frac{1}{2\omega} mg \sin \alpha \cos \omega t. \tag{46}$$

Formulae (43) and (46) give the solution of the problem, and it contains four arbitrary constants.

References

[1] Whittaker E T 1952 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* 4th edn. (Cambridge: Cambridge University Press)

[2] Lur'e A I 1961 *Analytical Mechanics* (Moscow: GIFML) (in Russian)

[3] Mei F X 1985 *Foundations of Mechanics of Nonholonomic Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[4] Mei F X 1988 *Special Problems in Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[5] Mei F X, Liu D and Luo Y 1991 *Advanced Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[6] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)

[7] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[8] Mei F X and Wu H B 2009 *Dynamics of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press)

[9] Mei F X and Liu G L 1985 *Proceedings of ICNM* (Shanghai) p. 1262

[10] Liu G L, Qiao Y F, Zhang J F and Mei F X 1989 *Acta Mech. Sin.* **21** 742 (in Chinese)

[11] Mashhoon B 1988 *Phys. Lett.* A **126** 393

[12] Luo S K 1991 *Appl. Math. Mech.* **12** 927

[13] Luo S K 1993 *Appl. Math. Mech.* **14** 907

[14] Zhang Y and Mei F X 1998 *J. Beijing Institute of Technology* **7** 12

[15] Liu R W, Fu J L and Mei F X 1998 *J. Beijing Institute of Technology* **7** 221

[16] Luo S K, Chen X W and Fu J L 2001 *Mech. Res. Commun.* **28** 463

[17] Zakaria K 2003 *Physica* A **327** 221

[18] Chen X W, Wang M Q and Wang X M 2005 *Commun. Theor. Phys.* (Beijing, China) **43** 577

[19] Huang P T, Liu M P and Ma S J 2005 *Chin. Phys.* **14** 244

[20] Zhang X W 2006 *Acta Phys. Sin.* **55** 2669 (in Chinese)

[21] Zhao W, Liu B, Shi P M and Jiang J S 2006 *Acta Phys. Sin.* **55** 3852 (in Chinese)

[22] Qiao Y F, Zhao S H and Li R J 2007 *Commun. Theor. Phys.* (Beijing, China) **47** 217

[23] Hou Q B, Li Y C, Wang J and Xia L L 2007 *Commun. Theor. Phys.* (Beijing, China) **48** 795

[24] Shi P M, Liu B and Hou D X 2008 *Acta Phys. Sin.* **57** 1321 (in Chinese)

[25] Mei F X and Wu H B 2009 *Acta Phys. Sin.* **58** 5919 (in Chinese)

[26] Lynden-Bell D 2009 *Chaos in Astronomy: Conference* 2007 (Berlin, Heidelberg: Springer-Berlin) p. 319

[27] Chen X W, Zhao Y H and Li Y M 2009 *Chin. Phys.* B **18** 3139

[28] Zhang Y 2009 *Chin. Phys. Lett.* **26** 120305

[29] Zhang Y 2010 *Acta Armamentarii* **31** 58 (in Chinese)