Decoherence of photon-subtracted squeezed vacuum state in dissipative channel*

Xu Xue-Xiang(徐学翔)[a)b)], Yuan Hong-Chun(袁洪春)[b)†], and Fan Hong-Yi(范洪义)[b)]

[a)] *College of Physics and Communication Electronics, Jiangxi Normal University, Nanchang* 330022, *China*
[b)] *Department of Physics, Shanghai Jiao Tong University, Shanghai* 200030, *China*

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This paper investigates the decoherence of photo-subtracted squeezed vacuum state (PSSVS) in dissipative channel by describing its statistical properties with time evolution such as Wigner function, Husimi function, and tomogram. It first calculates the normalization factor of PSSVS related to Legendre polynomial. After deriving the normally ordered density operator of PSSVS in dissipative channel, one obtains the explicit analytical expressions of time evolution of PSSVS's statistical distribution function. It finds that these statistical distributions loss their non-Gaussian nature and become Gaussian at last in the dissipative environment as expected.

Keywords: photon-subtracted squeezed vacuum state, Wigner function, Husimi function, tomogram

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1. Introduction

Since 1991, when photon-added coherent states (PACS) was first presented by Agarwal and Tara,[1] subtracting photons from and/or adding photons to quantum states have been paid much attention. Because these states are of particular interest since they exhibit quite significant and unusual physical properties, studying them can not only deepen our understanding of the nature of quantum fields but also help us to realize these fields experimentally for possible future applications such as in weak-signal detection, quantum computing, and quantum coding.[2−10] For example, PACS exhibits some nonclassical properties such as negativity of Wigner function, higher-order squeezing and higher-order sub-Poissonian photon statistics, etc.[1,11,12] Photon-added entangled coherent state (PAECS)[13−15] can violate the Clauser, Horne, Shimony and Holt (CHSH) inequality and have the nonclassical quasiprobability distribution. Its amount of entanglement can be increased by repeatedly adding photon number.

In the present work, we shall focus our study on another nonclassical state, photo-subtracted squeezed vacuum state (PSSVS) and discuss its statistical properties with time evolution in dissipative channel. As is well known, dissipative quantum channels tend to deteriorate the degree of nonclassicality. Thus, it is necessary to investigate the decoherence properties in dissipative channels. Historically, for the single PSSVS, Biswas and Agarwal[16] investigated theoretically its nonclassical properties and decoherence in two different decoherent channels (amplitude decay and phase damping). In Ref. [17], statistical properties of PSSVS were investigated in a thermal environment.

This work is arranged as follows. In Section 2, we first calculate the normalization factor of PSSVS, which turns out to be a Legendre polynomial of the squeezing parameter, by using the principle of mathematical induction. In Section 3, based on our preceding results,[18] we directly derive the normally ordered density operator of PSSVS in dissipative channel, which is convenient for us to calculate the following statistical distribution function. We devote Sections 4 and 5 to deducing the explicit analytical expressions of time evolution of PSSVS's Wigner function (WF) and Husimi function (HF), respectively. In Section 6, The tomogram with time evolution of PSSVS is calculated via the intermediate coordinate-momentum representation. Through the limited discussion, it is found that these statistical distributions loss their non-Gaussian nature and become Gaussian at long times in the dissipative environment as expected. Conclusions are involved in the last section.

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†Corresponding author. E-mail: yuanhch@sjtu.edu.cn

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2. Normalization of PSSVS

Photon substraction is a useful way to conditionally manipulate a nonclassical state of the optical field. Recently, an SPSSV has also been experimentally prepared with a pulsed and continuous wave squeezed vacuum.[19,20] Theoretically, the PSSVS can be obtained by repeatedly operating the photon annihilation operator a on a squeezed vacuum state $S(r)\left|0\right\rangle$, i.e.,

$$\left|r,m\right\rangle \equiv N_{r,m}a^{m}S(r)\left|0\right\rangle , \tag{1}$$

where $\left|0\right\rangle$ is single mode vacuum, $S(r)$ is the single-mode squeezing operator $S(r)=\exp[r(a^{\dagger 2}-a^{2})/2]$ with r being the squeezing parameter, $[a,a^{\dagger}]=1$, and $N_{r,m}$ is the normalization factor to be determined.

In order to obtain the normalization factor, we let

$$\left|r\right\rangle _{m}\equiv a^{m}S(r)\left|0\right\rangle , \tag{2}$$

where

$$\left|r\right\rangle \equiv S(r)\left|0\right\rangle =\sqrt{\operatorname{sech}r}\exp\left(\frac{\tanh r}{2}a^{\dagger 2}\right)\left|0\right\rangle .$$

According to the principle of mathematical induction and $a^{\dagger}\left|r\right\rangle =a\coth r\left|r\right\rangle$, we have

$$\begin{aligned}{}_{1}\left\langle r\right|\left.r\right\rangle _{1} &= \coth^{2}r\operatorname{sech}r\left\langle 0\right|\exp\left(\frac{\tanh ra^{2}}{2}\right)a^{\dagger}a\\ &\quad\times\exp\left(\frac{\tanh ra^{\dagger 2}}{2}\right)\left|0\right\rangle -1\\ &= \mathrm{i}\sinh r\mathrm{P}_{1}(-\mathrm{i}\sinh r), \end{aligned}\tag{3}$$

where $\mathrm{P}_{1}(x)=x$ is the first-order Legendre polynomial, and

$$\begin{aligned}{}_{2}\left\langle r\right|\left.r\right\rangle _{2} &= \left\langle r\right|(aa^{\dagger}-1)(aa^{\dagger}-1)\left|r\right\rangle -{}_{1}\left\langle r\right|\left.r\right\rangle _{1}\\ &= 2(\mathrm{i}\sinh\lambda)^{2}\mathrm{P}_{2}(-\mathrm{i}\sinh\lambda), \end{aligned}\tag{4}$$

where $\mathrm{P}_{2}(x)=\frac{1}{2}(3x^{2}-1)$ is the second-order Legendre polynomial. Similarly, for the case $n\leq m$, we can easily obtain the following expression:

$$\begin{aligned}&{}_{n-1}\left\langle r\right|\left.r\right\rangle _{n-1}\\ &= (\mathrm{i}\sinh r)^{n-1}(n-1)!\mathrm{P}_{n-1}(-\mathrm{i}\sinh r), \end{aligned}\tag{5}$$

where $\mathrm{P}_{m}(x)$ is m-order Legendre polynomial whose definition is[21]

$$\mathrm{P}_{m}(x)=\sum_{l=0}^{[m/2]}\frac{(2m-2l)!\,(-1)^{l}\,x^{m-2l}}{2^{m}l!\,(m-l)!\,(m-2l)!}. \tag{6}$$

Therefore, we can also prove that

$$\begin{aligned}{}_{m}\left\langle r\right|\left.r\right\rangle _{m} &= \left\langle r\right|a^{\dagger m-1}(aa^{\dagger}-1)a^{m-1}\left|r\right\rangle\\ &= \coth^{2}r\;{}_{m}\left\langle r\right|\left.r\right\rangle _{m}\\ &\quad-(2m-1)\;{}_{m-1}\left\langle r\right|\left.r\right\rangle _{m-1}\\ &\quad-(m-1)^{2}\;{}_{m-2}\left\langle r\right|\left.r\right\rangle _{m-2}, \end{aligned}\tag{7}$$

i.e.,

$$\begin{aligned}{}_{m}\left\langle r\right|\left.r\right\rangle _{m} &= (-\sinh^{2}r)\{-(2m-1)\;{}_{m-1}\left\langle r\right|\left.r\right\rangle _{m-1}\\ &\quad-(m-1)^{2}\;{}_{m-2}\left\langle r\right|\left.r\right\rangle _{m-2}\}. \end{aligned}\tag{8}$$

Comparing Eq. (8) with the recurrence relation of Legendre polynomial

$$\begin{aligned}&(m+1)\mathrm{P}_{m+1}(x)-(2m+1)x\mathrm{P}_{m}(x)\\ &\quad+\;m\mathrm{P}_{m-1}(x)=0, \end{aligned}\tag{9}$$

we have

$${}_{m}\left\langle r\right|\left.r\right\rangle _{m}=m!(\mathrm{i}\sinh r)^{m}\mathrm{P}_{m}(-\mathrm{i}\sinh r), \tag{10}$$

which leads to compact expression for the normalization factor of PSSVS

$$N_{r,m}^{-2}=m!\,(\mathrm{i}\sinh r)^{m}\,\mathrm{P}_{m}\,(-\mathrm{i}\sinh r)\,. \tag{11}$$

This is a remarkable result. From Eq. (11) it is clearly seen that PSSVS's normalization factor is related to the m-order Legendre polynomial of the squeezing parameter r, where m is just the subtracted photon number.

3. Normally ordered form of PSSVS's density operator in dissipative channel

When the PSSVS evolves in the dissipative channel (or cavity at zero temperature), the evolution of the density matrix can be described by[22]

$$\frac{\mathrm{d}\rho\,(t)}{\mathrm{d}t}=\kappa\left[2a\rho\,(t)\,a^{\dagger}-a^{\dagger}a\rho\,(t)-\rho\,(t)\,a^{\dagger}a\right], \tag{12}$$

where $\left[a,a^{\dagger}\right]=1$ and κ is the dissipative coefficient. By virtue of the entangled state representation, we find that the density operator $\rho\,(t)$ for the dissipative channel satisfies the following relation:[18]

$$\rho\,(t)=\sum_{n=0}^{\infty}\frac{T^{n}}{n!}\,\mathrm{e}^{-\kappa ta^{\dagger}a}a^{n}\rho_{0}a^{\dagger n}\,\mathrm{e}^{-\kappa ta^{\dagger}a}, \tag{13}$$

where $T=1-\mathrm{e}^{-2\kappa t}$ and equation (13) is just the Kraus sum representation.

The initial density matrix ρ_{0} is the PSSVS in Eq. (13), i.e.,

$$\begin{aligned}\rho_{0} &= \left|r,m\right\rangle \left\langle r,m\right|\\ &\equiv N_{r,m}^{2}a^{m}S(r)\left|0\right\rangle \left\langle 0\right|S^{\dagger}(r)a^{\dagger m}. \end{aligned}\tag{14}$$

Substituting it into Eq. (13) yields

$$\rho_{r,m}\,(t)=\sum_{n=0}^{\infty}\frac{T^{n}}{n!}G_{r,m,n}\left(a,a^{\dagger};t\right), \tag{15}$$

where

$$G_{r,m,n}\left(a,a^{\dagger};t\right)=N_{r,m}^{2}\mathrm{e}^{-\kappa t a^{\dagger}a}a^{n}a^{m}S(r)\left|0\right\rangle\left\langle 0\right|S^{\dagger}(r)a^{\dagger m}a^{\dagger n}\mathrm{e}^{-\kappa t a^{\dagger}a}. \tag{16}$$

By noticing

$$S(r)\left|0\right\rangle=\operatorname{sech}^{1/2}r\exp\left(\frac{\tanh r}{2}a^{\dagger 2}\right)\left|0\right\rangle, \tag{17}$$

and using

$$\begin{aligned}&\mathrm{e}^{-\kappa t a^{\dagger}a}a\,\mathrm{e}^{\kappa t a^{\dagger}a}=a\,\mathrm{e}^{\kappa t},\\&\mathrm{e}^{-\kappa t a^{\dagger}a}a^{\dagger}\mathrm{e}^{\kappa t a^{\dagger}a}=a^{\dagger}\mathrm{e}^{-\kappa t},\end{aligned} \tag{18}$$

we can rewrite Eq. (16) as

$$\begin{aligned}G_{r,m,n}\left(a,a^{\dagger};t\right)=&\,N_{r,m}^{2}\operatorname{sech}r\,\mathrm{e}^{2(n+m)\kappa t}a^{n+m}\\&\times\mathrm{e}^{\mu^{2}a^{\dagger 2}}\left|0\right\rangle\left\langle 0\right|\mathrm{e}^{\mu^{2}a^{2}}a^{\dagger n+m},\end{aligned} \tag{19}$$

where $\mu\equiv\sqrt{\tanh r/2}\,\mathrm{e}^{-\kappa t}$.

On the other hand, using the Bogolyubov transformation

$$\mathrm{e}^{A}B\mathrm{e}^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots, \tag{20}$$

and the operator identity[23]

$$\begin{aligned}\left(xa+ya^{\dagger}\right)^{m}=&\left(-\mathrm{i}\sqrt{\frac{xy}{2}}\right)^{m}\\&:\mathrm{H}_{m}\left(\mathrm{i}\sqrt{\frac{x}{2y}}a+\mathrm{i}\sqrt{\frac{y}{2x}}a^{\dagger}\right):,\end{aligned} \tag{21}$$

where $:\ :$ denotes the normal ordering and $\mathrm{H}_{m}(x)$ is the Hermite polynomial[24]

$$\begin{aligned}\mathrm{H}_{m}(x)=&\sum_{l=0}^{[m/2]}\frac{(-1)^{l}\,m!}{l!(m-2l)!}(2x)^{m-2l}\\=&\frac{\partial^{m}}{\partial t^{m}}\mathrm{e}^{2tx-t^{2}}|_{t=0},\end{aligned} \tag{22}$$

we obtain the following two expressions:

$$\begin{aligned}a^{n+m}\mathrm{e}^{\mu^{2}a^{\dagger 2}}\left|0\right\rangle=&\,\mathrm{e}^{\mu^{2}a^{\dagger 2}}\left(a+2\mu^{2}a^{\dagger}\right)^{n+m}\left|0\right\rangle\\=&\,(-\mathrm{i}\,\mu)^{n+m}\,\mathrm{e}^{\mu^{2}a^{\dagger 2}}\\&\times\mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^{\dagger}\right)\left|0\right\rangle,\end{aligned} \tag{23}$$

and

$$\begin{aligned}\left\langle 0\right|\mathrm{e}^{\mu^{2}a^{2}}a^{\dagger n+m}=&\left\langle 0\right|\mathrm{H}_{n+m}\left(-\mathrm{i}\,\mu a\right)\\&\times\mathrm{e}^{\mu^{2}a^{2}}\left(\mathrm{i}\,\mu\right)^{n+m}.\end{aligned} \tag{24}$$

Substituting Eqs. (23) and (24) into Eq. (19) and considering the normal product form of vacuum projector $\left|0\right\rangle\left\langle 0\right|=:\,\exp(-a^{\dagger}a):$, we can simplify Eq. (19) as

$$\begin{aligned}G_{r,m,n}\left(a,a^{\dagger};t\right)=&\,N_{r,m}^{2}\operatorname{sech}r\left(\frac{\tanh r}{2}\right)^{n+m}\\&\times:\mathrm{e}^{\mu^{2}a^{\dagger 2}}\mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^{\dagger}\right)\mathrm{e}^{-a^{\dagger}a}\\&\times\mathrm{H}_{n+m}\left(-\mathrm{i}\,\mu a\right)\mathrm{e}^{\mu^{2}a^{2}}:.\end{aligned} \tag{25}$$

Therefore, we obtain the normal ordering form of $\rho_{r,m}\left(t\right)$ in Eq. (15), which is very convenient to calculate the following content. As the result of Eq. (15), when $\kappa t=0$, $T=0$, $\rho_{r,m}\left(0\right)=\rho_{0}$ as expected. When $\kappa t\to\infty$, $T=1$, we find $\rho_{r,m}\left(\infty\right)\to\left|0\right\rangle\left\langle 0\right|$, which implies that the system state reduces to a Gaussian state after a long time interaction in dissipative channel.

4. Wigner function of $\rho_{r,m}\left(t\right)$

It is well known that the WF evolution is usually a useful approach to monitor the decoherence of some quantum state. In the following section, using the normally ordered form of $\rho_{r,m}\left(t\right)$, we evaluate the WF with time evolution for PSSVS. For a single-mode system ρ, the Wigner function is given by[25]

$$W\left(\alpha,\alpha^{*}\right)=\mathrm{Tr}\left[\,\rho\Delta_{w}\left(\alpha,\alpha^{*}\right)\right], \tag{26}$$

where $\alpha=(q+\mathrm{i}\,p)/\sqrt{2}$ and the Wigner operator $\Delta_{w}\left(\alpha,\alpha^{*}\right)$ in the coherent representation $\left|z\right\rangle$ can be expressed as[26]

$$\Delta_{w}\left(\alpha,\alpha^{*}\right)=\frac{\mathrm{e}^{2|\alpha|^{2}}}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}^{2}z}{\pi}\left|z\right\rangle\left\langle -z\right|\mathrm{e}^{-2(z\alpha^{*}-z^{*}\alpha)}. \tag{27}$$

Knowing the normally ordered form of $\rho_{r,m}\left(t\right)$ for $\left|r,m\right\rangle$, we easily obtain

$$\begin{aligned}W\left(\alpha,\alpha^{*};t\right)=&\,\frac{\mathrm{e}^{2|\alpha|^{2}}}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}^{2}z}{\pi}\left\langle -z\right|\rho_{r,m}\left(t\right)\left|z\right\rangle\\&\times\mathrm{e}^{-2(z\alpha^{*}-z^{*}\alpha)}.\end{aligned} \tag{28}$$

Substituting Eqs. (15) into Eq. (28), we obtain the analytical expression of WF evolution of PASVS for any photon-added number m in the dissipative environment, which reads as

$$\begin{aligned}W\left(\alpha,\alpha^{*};t\right)=&\frac{N_{r,m}^{2}\operatorname{sech}r\,\mathrm{e}^{2|\alpha|^{2}}}{\pi}\sum_{n=0}^{\infty}\frac{T^{n}}{n!}\left(\frac{\tanh r}{2}\right)^{n+m}\\&\times\int_{-\infty}^{\infty}\frac{\mathrm{d}^{2}z}{\pi}\mathrm{e}^{-|z|^{2}-2\alpha^{*}z+2\alpha z^{*}+\mu^{2}z^{2}+\mu^{2}z^{*2}}\\&\times\mathrm{H}_{n+m}\left(-\mathrm{i}\,\mu z^{*}\right)\mathrm{H}_{n+m}\left(-\mathrm{i}\,\mu z\right).\end{aligned} \tag{29}$$

Noticing the generating function of single-mode Hermitian polynomials in Eq. (22) and using the following integral formula

$$\begin{aligned}&\int\frac{\mathrm{d}^{2}z}{\pi}\exp\left(\zeta\left|z\right|^{2}+\xi z+\eta z^{*}+fz^{2}+gz^{*2}\right)\\&=\frac{1}{\sqrt{\zeta^{2}-4fg}}\exp\left(\frac{-\zeta\xi\eta+\xi^{2}g+\eta^{2}f}{\zeta^{2}-4fg}\right),\end{aligned} \tag{30}$$

whose convergent condition is $\mathrm{Re}(\zeta \pm f \pm g) < 0$ and $\mathrm{Re}\left(\dfrac{\zeta^2 - 4fg}{\zeta \pm f \pm g}\right) < 0$, we can write Eq. (29) as

$$W(\alpha, \alpha^*; t) = \frac{N_{r,m}^2 \mathrm{sech}\, r}{\pi\sqrt{1-4\mu^4}} \sum_{n=0}^{\infty} \frac{T^n}{n!} \left(\frac{\tanh r}{2}\right)^{n+m} \mathrm{e}^{-2\mathcal{D}|\alpha|^2 - \mathcal{C}\left(\alpha^2 + \alpha^{*2}\right)} \times \frac{\partial^{2n+2m}}{\partial\lambda^{n+m}\partial\varepsilon^{n+m}} \exp\left(\mathcal{C}\lambda\varepsilon - \mathcal{B}\varepsilon^2 - \mathcal{B}\lambda^2 + \mathcal{A}\varepsilon + \mathcal{A}^*\lambda\right)|_{\lambda=\varepsilon=0}, \tag{31}$$

where we have set

$$\mathcal{A} = \frac{-4\,\mathrm{i}\,\mu\alpha + 8\,\mathrm{i}\,\mu^3\alpha^*}{1-4\mu^4}, \quad \mathcal{B} = \frac{1}{1-4\mu^4}, \quad \mathcal{C} = \frac{-4\mu^2}{1-4\mu^4}, \quad \mathcal{D} = \frac{1+4\mu^4}{1-4\mu^4}. \tag{32}$$

Further expanding the term $\exp(\mathcal{C}\lambda\varepsilon)$ to the form of the series and noticing the recurrence relation of Hermite polynomial

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l}\mathrm{H}_n(x) = \frac{2^l n!}{(n-l)!}\mathrm{H}_{n-l}(x), \tag{33}$$

we finally obtain the analytic expression of Wigner function for the PSSVS in the dissipative channel,

$$W(\alpha, \alpha^*; t) = \frac{N_{r,m}^2 \mathrm{sech}\, r\, \mathrm{e}^{-2\mathcal{D}|\alpha|^2 - \mathcal{C}\left(\alpha^2 + \alpha^{*2}\right)}}{\pi\sqrt{1-4\mu^4}} \times \sum_{n=0}^{\infty}\sum_{l=0}^{n+m} \frac{T^n \left(\frac{\tanh r}{2}\right)^{n+m} \mathcal{C}^l |\mathcal{B}|^{n+m-l} [(n+m)!]^2}{n!l! [(n+m-l)!]^2} \left|\mathrm{H}_{n+m-l}\left(\frac{\mathcal{A}^*}{2\sqrt{\mathcal{B}}}\right)\right|^2. \tag{34}$$

It is shown from Eq. (34) that the arbitrary PSSVS's WF evolution is related to Hermite polynomial.

In particular, when $\kappa t = 0$, $T = 0$, it leads to $\mathcal{B} = \cosh^2 r$, $\mathcal{C} = -\sinh 2r$, $\mathcal{D} = \cosh 2r$,

$$\mathcal{A} = \sqrt{\tanh r}\cosh^2 r[2p(1+\tanh r) + \mathrm{i}\,2q(\tanh r - 1)] \tag{35}$$

and $m = 0, 1$, equation (34) becomes, respectively,

$$W_{m=0}(\alpha, \alpha^*; 0) = \frac{1}{\pi}\mathrm{e}^{-\left(q^2\mathrm{e}^{-2r} + p^2\mathrm{e}^{2r}\right)}, \tag{36}$$

which is just the WF of squeezed vacuum state, a Gaussian distribution in phase space, and

$$W_{m=1}(\alpha, \alpha^*; 0) = \frac{1}{\pi}\left[2\left(q^2\mathrm{e}^{-2r} + p^2\mathrm{e}^{2r}\right) - 1\right] \times \mathrm{e}^{-\left(q^2\mathrm{e}^{-2r} + p^2\mathrm{e}^{2r}\right)}, \tag{37}$$

which shows that its WF becomes negative as $2\left(q^2\mathrm{e}^{-2r} + p^2\mathrm{e}^{2r}\right) < 1$. From Eqs. (36) and (37) we can see that when the added-photon number $m \geq 1$, the corresponding WF is a non-Gaussian distribution in phase space. Additionally, when $\kappa t \to \infty$, $T = 1$, then $\mathcal{A} \to 0, \mathcal{B} \to 1, \mathcal{C} \to 0, \mathcal{D} \to 1$, equation (34) is

$$W(\alpha, \alpha^*; \infty) = \frac{\mathrm{e}^{-q^2 - p^2}}{\pi}, \tag{38}$$

which indicated that the WF losses its non-Gaussian nature and becomes Gaussian at long times in the dissipative environment.

In fact, by compacting the summation in Eq. (31) and employing the similar procedure to obtain Eq. (34), one can obtain the Wigner function in another form as follows:

$$\begin{aligned} W(\alpha, \alpha^*, t) &= \frac{N_{r,m}^2\sqrt{\varkappa}}{\pi} \exp\left[-\varkappa\left[\left(4\mathrm{e}^{-2t\kappa}\sinh^2 r + 2\right)\right]|\alpha|^2 + \varkappa\,\mathrm{e}^{-2t\kappa}\sinh 2r\left(\alpha^2 + \alpha^{*2}\right)\right] \\ &\quad \times \frac{\partial^{2m}}{\partial\lambda^m\partial\varepsilon^m} \exp\left(c_1\lambda^2 + c_1\varepsilon^2 + c_2\lambda\varepsilon + c_3\lambda + c_3^*\varepsilon\right)|_{\lambda=\varepsilon=0} \\ &= \frac{m!\sqrt{\varkappa}\left(-\frac{\mathrm{i}\,\varkappa\cosh r}{2}\right)^m}{\pi \mathrm{P}_m(-\mathrm{i}\sinh r)} \exp\left[-2\varkappa\left(2\mathrm{e}^{-2t\kappa}\sinh^2 r + 1\right)|\alpha|^2 + \varkappa\mathrm{e}^{-2t\kappa}\sinh 2r\left(\alpha^2 + \alpha^{*2}\right)\right] \\ &\quad \times \sum_{k=0}^{m} \frac{\left[2\tanh r\left(1 - 2\mathrm{e}^{-2\kappa t}\right)\right]^k}{k![(m-k)!]^2} |\mathrm{H}_{m-k}(\mathcal{M})|^2 \end{aligned} \tag{39}$$

with

$$\varkappa = \left(4\mathrm{e}^{-2t\kappa}T\sinh^2 r + 1\right)^{-1}, \quad c_1 = \frac{\varkappa \sinh 2r}{4},$$
$$c_2 = \varkappa\left(1 - 2\,\mathrm{e}^{-2t\kappa}\right)\sinh^2 r, \quad c_3 = 2\,\mathrm{e}^{-t\kappa}\left(c_2\alpha + 2c_1\alpha^*\right),$$
$$\mathcal{M} = -\,\mathrm{i}\,\mathrm{e}^{-\kappa t}\sqrt{\varkappa}\left(\left(1 - 2\,\mathrm{e}^{-2t\kappa}\right)\sinh r\sqrt{2\tanh r}\alpha + \sqrt{\sinh 2r}\alpha^*\right). \tag{40}$$

Using Eq. (39), we show the variation of the Wigner function at different time scales in Fig. 1. It is easy to see how the negative region of the Wigner function gradually diminishes. Note that at the centre of the phase space, the Wigner function Eq. (39) can be expressed as

$$W(0,0,t) = \frac{m!\sqrt{\varkappa}\left(-\,\mathrm{i}\,\frac{\varkappa\cosh r}{2}\right)^m}{\pi \mathrm{P}_m\left(-\,\mathrm{i}\sinh r\right)} \sum_{k=0}^{m} \frac{\left[2\tanh r\left(1 - 2\,\mathrm{e}^{-2\kappa t}\right)\right]^k}{k!\left[(m-k)!\right]^2}\left|\mathrm{H}_{m-k}(0)\right|^2. \tag{41}$$

From Eq. (41), one can verify that when the following condition $\left(1 - 2\mathrm{e}^{-2\kappa t}\right) > 0$ is satisfied, i.e.

$$\kappa t > \kappa t_c = \frac{1}{2}\ln 2, \tag{42}$$

the Wigner function has no chance to show the negative region (see Fig. 2). This is in agreement with the conclusion in Ref. [16].

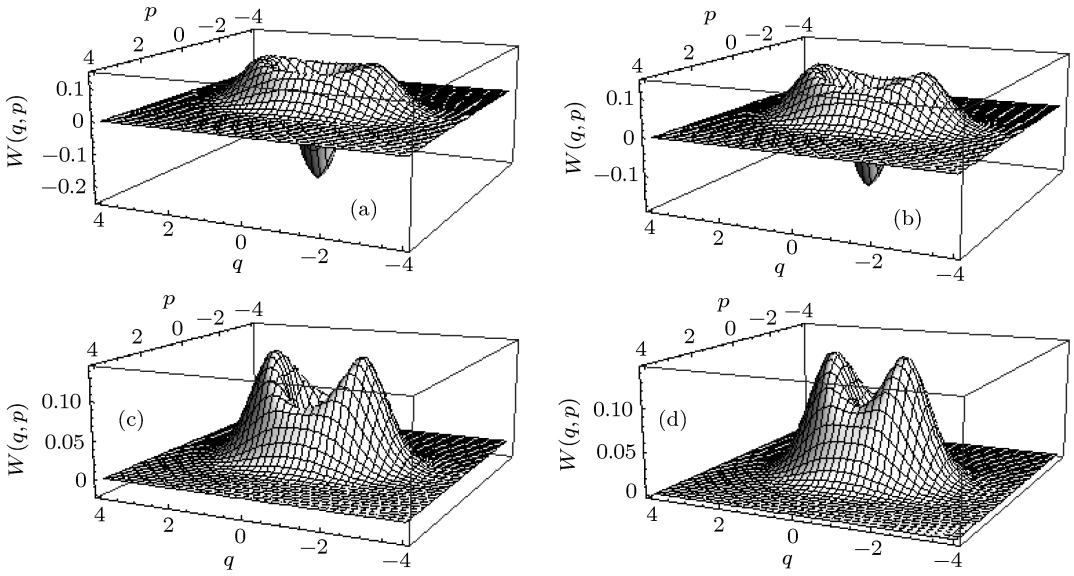


Fig. 1. Wigner function of PSSVS for $m = 1$ and $r = 0.31$ at (a) $\kappa t = 0.05$, (b) $\kappa t = 0.1$, (c) $\kappa t = 0.3$, and (d) $\kappa t = 0.5$.

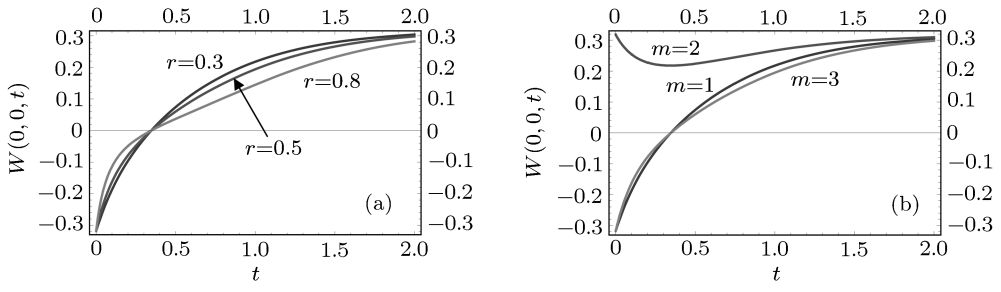


Fig. 2. Wigner function of PSSVS at the centre of the phase space as the function of the time t (a) $m = 1$ and $r = 0.3$, 0.5, 0.8; (b) $r = 0.3$ and $m = 1$, 2, 3.

5. Husimi function of $\rho_{r,m}(t)$

The WF itself is not a probability distribution since not always positive. To overcome this shortcoming, the so-called HF is introduced,[27] which is defined in a manner which guarantees it to be non-negative and gives it a probability interpretation. The theoretical calculation of HF can help experimentalists to judge the quality of the experiment. Based on this, we discuss the HF of PASVS and its effect with time evolution in a dissipative channel.

To begin with, we derive the expression of HF evolution of $\rho_{r,m}(t)$ for PASVS in a dissipative channel. Recall that Husimi operator $\Delta_h(q,p;\epsilon)$, which smoothes out the Wigner operator $\Delta_w(q',p')$, is introduced via averaging over a "coarse graining" function,

$$\Delta_h(q,p;\epsilon) = 2\int \mathrm{d}q'\mathrm{d}p'\Delta_w(q',p') \times \exp\left[-\epsilon\left(q'-q\right)^2 - \frac{1}{\epsilon}\left(p'-p\right)^2\right], \quad (43)$$

where positive ϵ is the Gaussian spatial width parameter, and may express a pure state density operator, i.e.,[28]

$$\Delta_h(q,p;\epsilon) = |q,p\rangle_{\epsilon,\epsilon}\langle q,p|, \quad (44)$$

where $|q,p\rangle_\epsilon$ is expressed as

$$|q,p\rangle_\epsilon = \sqrt{C}\exp\left(\frac{D}{2} + A^*a^\dagger + Ba^{\dagger 2}\right)|0\rangle, \quad (45)$$

in which

$$A \equiv \frac{\sqrt{2}\left(\epsilon q - \mathrm{i}\,p\right)}{1+\epsilon}, \quad B \equiv \frac{1-\epsilon}{2\left(1+\epsilon\right)},$$
$$C \equiv \frac{2\sqrt{\epsilon}}{1+\epsilon}, \quad \text{and} \quad D \equiv -\frac{\epsilon q^2 + p^2}{1+\epsilon}.$$

Thus, the HF of various density matrix is given by

$$\mathcal{H}(p,q) \equiv {}_\epsilon\langle q,p|\,\rho\,|q,p\rangle_\epsilon, \quad (46)$$

which is the expectation value of a pure state. For simplicity, here we assume $0 < \epsilon < 1$. Equation (46) provides us a good representative space for studying various properties of HF.

Substituting Eq. (15) into Eq. (46), we can describe the HF of $\rho_{r,m}(t)$ for PSSVS as

$$\mathcal{H}(p,q;t) = N_{r,m}^2 \mathrm{sech}\, r \sum_{n=0}^{\infty}\frac{T^n}{n!}\left(\frac{\tanh r}{2}\right)^{n+m}\left|{}_\epsilon\langle q,p|\,\mathrm{e}^{\mu^2 a^{\dagger 2}}\mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^\dagger\right)|0\rangle\right|^2. \quad (47)$$

Firstly we calculate the overlap ${}_\epsilon\langle q,p|\,\mathrm{e}^{\mu^2 a^{\dagger 2}}\mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^\dagger\right)|0\rangle$. Inserting the completeness relation of coherent state and using Eqs. (30) and (22), we have

$$\begin{aligned}{}_\epsilon\langle q,p|\,\mathrm{e}^{\mu^2 a^{\dagger 2}}\mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^\dagger\right)|0\rangle &= \int\frac{\mathrm{d}^2 z}{\pi}\,{}_\epsilon\langle q,p\,|z\rangle\langle z|\,\mathrm{e}^{\mu^2 a^{\dagger 2}}\mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^\dagger\right)|0\rangle \\ &= \sqrt{C}\mathrm{e}^{D/2}\frac{\partial^{n+m}}{\partial t^{n+m}}\int\frac{\mathrm{d}^2 z}{\pi}\mathrm{e}^{-|z|^2+Az+2\,\mathrm{i}\,\mu t z^*+Bz^2+\mu^2 z^{*2}-t^2}|_{t=0} \\ &= \sqrt{C}\mathrm{e}^{D/2}E^{n+m+1}\mathrm{e}^{(AE\mu)^2}\mathrm{H}_{n+m}\left(\mathrm{i}\,AE\mu\right), \end{aligned} \quad (48)$$

where $E = \sqrt{1/(1-4B\mu^2)}$, then the HF of $\rho_{r,m}(t)$ is expressed as

$$\begin{aligned}\mathcal{H}(p,q;t) &= N_{r,m}^2\mathrm{sech}\, rC\mathrm{e}^D\left|\exp\left[(AE\mu)^2\right]\right|^2 \\ &\times\sum_{n=0}^{\infty}\frac{T^n}{n!}\left(\frac{\tanh r}{2}\right)^{n+m}\left|E^2\right|^{n+m+1}\left|\mathrm{H}_{n+m}\left(\mathrm{i}\,AE\mu\right)\right|^2, \end{aligned} \quad (49)$$

which is non-negative in phase space and is related to Hermite polynomial.

Especially, when $\kappa t = 0$, $T = 0$, equation (51) becomes the HF of PSSVS

$$\begin{aligned}\mathcal{H}(p,q;0) &= N_{r,m}^2\mathrm{sech}\, rC\mathrm{e}^D\left(\frac{\tanh r}{2}\right)^m\left|\frac{1}{1-2B\tanh r}\right|^{m+1} \\ &\times\left|\exp\left(\frac{A^2\tanh r}{2-4B\tanh r}\right)\mathrm{H}_m\left(\mathrm{i}\,A\sqrt{\frac{\tanh r}{2-4B\tanh r}}\right)\right|^2, \end{aligned} \quad (50)$$

while $\kappa t \to \infty$, $T \to 1$, then

$$\mathcal{H}(p,q;\infty) = \frac{2\sqrt{\epsilon}}{1+\epsilon} \exp\left(-\frac{\epsilon q^2 + p^2}{1+\epsilon}\right), \tag{51}$$

which is also of a Gaussian form.

6. Tomogram of $\rho_{r,m}(t)$

Once the probability distributions $P_\theta(\hat{x}_\theta)$ of the quadrature amplitude are obtained, one can use the inverse Radon transformation familiar in tomographic imaging to obtain the WF and density matrix. Thus the Radon transform of the WF corresponds to the probability distributions $P_\theta(\hat{x}_\theta)$.[29] The tomogram $\mathcal{T}$ of the PSSVS in dissipative channel, denoted as the Radon transform of WF, is defined by[30,31]

$$\begin{aligned}\mathcal{T}(q,t)_{f,g} &= \int \delta(q - fq' - gp') \operatorname{Tr}\left[\Delta_w(\beta,\beta^*)\rho_{r,m}(t)\right] \mathrm{d}q'\mathrm{d}p' \\ &= \operatorname{Tr}\left[|q\rangle_{f,gf,g}\langle q|\rho_{r,m}(t)\right] = {}_{f,g}\langle q|\rho_{r,m}(t)|q\rangle_{f,g}\,,\end{aligned} \tag{52}$$

where the operator $|q\rangle_{f,gf,g}\langle q|$ is just the Radon transform of single-mode Wigner operator $\Delta_w(\beta,\beta^*)$, and

$$|q\rangle_{f,g} = M \exp\left(\frac{\sqrt{2}qa^\dagger}{G} - \frac{G^* a^{\dagger 2}}{2G}\right)|0\rangle\,, \tag{53}$$

as well as

$$G = f - \mathrm{i}\,g,\ M = \left[\pi\left(f^2+g^2\right)\right]^{-1/4} \exp\left[-\frac{q^2}{2\left(f^2+g^2\right)}\right]. \tag{54}$$

Thus the tomogram of a quantum state $\rho_{r,m}(t)$ is just the quantum average of $\rho_{r,m}(t)$ in the $|q\rangle_{f,g}$ representation, which is a kind of intermediate coordinate–momentum representation.

Substituting Eq. (15) into Eq. (52), we obtain the tomogram of $\rho_{r,m}(t)$, i.e.

$$\mathcal{T}(q,t)_{f,g} = N_{r,m}^2 \operatorname{sech} r \sum_{n=0}^{\infty} \frac{T^n}{n!} \left(\frac{\tanh r}{2}\right)^{n+m} \left|{}_{f,g}\langle q|\,\mathrm{e}^{\mu^2 a^{\dagger 2}} \mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^\dagger\right)|0\rangle\right|^2. \tag{55}$$

Firstly we will calculate the overlap ${}_{f,g}\langle q|\,\mathrm{e}^{\mu^2 a^{\dagger 2}} \mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^\dagger\right)|0\rangle$. Inserting the completeness relation of coherent state and using Eqs. (30) and (22), we obtain

$$\begin{aligned}&{}_{f,g}\langle q|\,\mathrm{e}^{\mu^2 a^{\dagger 2}} \mathrm{H}_{n+m}\left(\mathrm{i}\,\mu a^\dagger\right)|0\rangle \\ &= M \frac{\partial^{n+m}}{\partial \tau^{n+m}} \int \frac{\mathrm{d}^2 z}{\pi} \exp\left[-|z|^2 + \frac{\sqrt{2}q}{G^*}z + 2\,\mathrm{i}\,\mu\tau z^* - \frac{G}{2G^*}z^2 + \mu^2 z^{*2} - \tau^2\right]\Big|_{\tau=0} \\ &= M\left(\sqrt{\frac{G^*}{2G\mu^2 + G^*}}\right)^{n+m+1} \exp\left(\frac{2q^2\mu^2}{2\mu^2|G|^2 + G^{*2}}\right) \mathrm{H}_{n+m}\left(\frac{\mathrm{i}\sqrt{2}q\mu}{\sqrt{2|G|^2\mu^2 + G^{*2}}}\right).\end{aligned} \tag{56}$$

Then we have the tomogram of $\rho_{r,m}(t)$ as

$$\begin{aligned}\mathcal{T}(q,t)_{f,g} &= N_{r,m}^2 \operatorname{sech} r \sum_{n=0}^{\infty} \frac{M^2 T^n}{n!} \left(\frac{\tanh r}{2}\right)^{n+m} \left|\frac{G^*}{2G\mu^2 + G^*}\right|^{n+m+1} \\ &\quad \times \left|\exp\left(\frac{2q^2\mu^2}{2\mu^2|G|^2 + G^{*2}}\right) \mathrm{H}_{n+m}\left(\frac{\mathrm{i}\sqrt{2}q\mu}{\sqrt{2|G|^2\mu^2 + G^{*2}}}\right)\right|^2.\end{aligned} \tag{57}$$

Especially, when $\kappa t = 0$, $T = 0$, equation (51) becomes the tomogram of PSSVS

$$\mathcal{T}(q,0)_{f,g} = N_{r,m}^2 M^2 \operatorname{sech} r \left(\frac{\tanh r}{2}\right)^m \left|\frac{G^*}{G\tanh r + G^*}\right|^{m+1}$$

$$\times \left| \exp\left(\frac{q^2 \tanh r}{|G|^2 \tanh r + G^{*2}} \right) \mathrm{H}_{n+m}\left(\mathrm{i}\, q \sqrt{\frac{\tanh r}{|G|^2 \tanh r + G^{*2}}} \right) \right|^2, \tag{58}$$

while $\kappa t \to \infty$, $T \to 1$, then

$$\mathcal{T}(q,\infty)_{f,g} = \left[\pi\left(f^2+g^2\right)\right]^{-1/2} \exp\left(-\frac{q^2}{f^2+g^2}\right), \tag{59}$$

which is also a Gaussian distribution corresponding to the vacuum state.

7. Conclusions

In summary, we discussed the decoherence of PSSVS in dissipative channel by describing its statistical properties with time evolution, such as Wigner function, Husimi function, and tomogram. First, we calculated the normalization factor of PSSVS, related to Legendre polynomial and derived the normally ordered density operator of PSSVS in dissipative channel. Then, the explicit analytical expressions of time evolution of PSSVS's statistical distribution function were also deduced in detail. By analysing the special case, we found that these statistical distributions loss their non-Gaussian nature and become Gaussian at last in the dissipative environment, this shows the fact that dissipative quantum channels tend to deteriorate the degree of nonclassicality. At present, the combination of the photon addition and subtraction has been discussed in Ref. [32], thus studies about the combination of the photon addition and subtraction applying on the squeezed vacuum state are in progress in dissipative channel and new results will be reported later.

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