Conformal invariance and conserved quantities of Birkhoff systems under second-class Mei symmetry*

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This paper proposes a new concept of the conformal invariance and the conserved quantities for Birkhoff systems under second-class Mei symmetry. The definition about conformal invariance of Birkhoff systems under second-class Mei symmetry is given. The conformal factor in the determining equations is found. The relationship between Birkhoff system's conformal invariance and second-class Mei symmetry are discussed. The necessary and sufficient conditions of conformal invariance, which are simultaneously of second-class symmetry, are given. And Birkhoff system's conformal invariance may lead to corresponding Mei conserved quantities, which is deduced directly from the second-class Mei symmetry when the conformal invariance satisfies some conditions. Lastly, an example is provided to illustrate the application of the result.

Keywords: second-class Mei symmetry, conformal invariance, conserved quantity, Birkhoff system

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1. Introduction

In theoretical physics, the conformal invariance is a symmetry under dilatation (scale invariance) and also under the special conformal transformations[1]. In fact, the conformal invariance means that the equation obtained by the action of the infinitesimal operator equals a conformal multiplier matrix multiplied itself. It was introduced into physics by Cunningham[2] and Bateman[3] shortly after Einstein's first paper on special relativity. Afterwards, Galiullin *et al.*[4] studied the conformal invariance of Birkhoff equations and their relation with the Lie symmetry. Robert and Matthew[5] investigated the conformal Hamiltonian systems by geometric method, discussed the geometric structure of conformal systems with invariance. Recently, the conformal invariance developed rapidly in dynamical mechanical systems, and many scholars studied the conformal invariance and conserved quantities of mechanical systems.[6−15] Since Mei[16] presented a new symmetry — the form invariance, the more and more attention has been paid to research on Mei symmetry and conserved quantity, which was quickly extended to all types of mechanical systems.[17−30] Mei's form invariance is also called Mei symmetry, which means that the dynamical function in the equation of motion still satisfies the primary form of the equations after some infinitesimal transformations.

However, until now the research on the conformal invariance and conserved quantities of Birkhoff systems under second-class Mei symmetry have not been reported yet. In the present paper, the new concept of conformal invariance and conserved quantities of Birkhoff systems under second-class Mei symmetry are presented first. In fact, the aim of studying the conformal invariance and conserved quantities of Birkhoff systems under second-class Mei symmetry is to seek the one-parameter group of infinitesimal transformation of equations. By infinitesimal transformations, we can obtain the conformal factor of conformal invariance and the conserved quantities of Birkhoff system under second-class Mei symmetry.

The Birkhoffian dynamics is more general than the Hamiltonian mechanics and it is extensively applied to the fields of modern physics. Therefore, the researches on conformal invariance of Birkhoffian dynamics systems under second-class Mei symmetry is significant not only geometrically but also mechanically.

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2. Second-class Mei symmetry of Birkhoff systems

Suppose that the Birkhoffian equation can be written as

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0,\quad(\mu,\nu=1,\dots 2n),\tag{1}$$

where $B=B(t,\boldsymbol{a})$ is called Birkhoffian, $R_\mu=R_\mu(t,\boldsymbol{a})$ is called Birkhoffian functions, and a^μ are variables. The system described by Eq. (1) is called Birkhoffian system. From Eq. (1) we can obtain

$$\dot{a}^\mu=\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\right),\tag{2}$$

where $\Omega^{\mu\nu}$ is a Birkhoffian contravariant tensor, that is

$$\Omega^{\mu\nu}=\left(\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)^{-1}\right)^{\mu\nu},\tag{3}$$

where

$$\Omega^{\mu\nu}\Omega_{\nu\rho}=\delta_{\mu\rho}.\tag{4}$$

Suppose that the system is non-singular, i.e.

$$\det\Omega_{\mu\nu}\neq 0,\tag{5}$$

we can solve all generalized accelerations as

$$\ddot{a}^\mu=\sigma_\mu(t,\boldsymbol{a}).\tag{6}$$

Let us introduce the infinitesimal transformations of time and generalized coordinate as

$$t^*=t+\Delta t,\quad a^{\mu*}(t^*)=a^\mu(t)+\Delta a^\mu.\tag{7}$$

Their expanding forms are

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{a}),\quad a^{\mu*}(t^*)=a^\mu(t)+\varepsilon\xi_\mu(t,\boldsymbol{a}),\tag{8}$$

where ε is the small parameter and ξ_0, ξ_μ are the generators of the one-parameter infinitesimal transformation group. Taking the infinitesimal generators as

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_\mu\frac{\partial}{\partial a^\mu},\tag{9}$$

we find that its first expanded vector is

$$X^{(1)}=X^{(0)}+(\dot{\xi}_\mu-\dot{a}^\mu\dot{\xi}_0)\frac{\partial}{\partial\dot{a}^\mu},\tag{10}$$

and its second expanded vector is

$$X^{(2)}=X^{(1)}+(\ddot{\xi}_\mu-2\ddot{a}^\mu\dot{\xi}_0-\dot{a}^\mu\ddot{\xi}_0)\frac{\partial}{\partial\ddot{a}^\mu}.\tag{11}$$

Undergoing the transformations (8), we find that Birkhoff's functions R_μ and the Birkhoffian B become respectively R_μ^* and B^* as follows:

$$R_\mu^*=R_\mu(t^*,\boldsymbol{a}^*)=R_\mu(t,\boldsymbol{a})+\varepsilon X^{(0)}(R_\mu)+O(\varepsilon^2),$$

$$B^*=B(t^*,\boldsymbol{a}^*)=B(t,\boldsymbol{a})+\varepsilon X^{(0)}(B)+O(\varepsilon^2).\tag{12}$$

Definition 1 If the form of Birkhoff equation (1) keeps invariant when the Birkhoff's function R_μ and the Birkhoffian B are replaced respectively by R_μ^* and B^* under the infinitesimal transformations (8), then the invariance is called the second-class Mei symmetry of the Birkhoff system.

Substituting Eq. (12) into Eq. (1), by Definition 1, we have

$$\left(\frac{\partial R_\nu^*}{\partial a^\mu}-\frac{\partial R_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B^*}{\partial a^\mu}-\frac{\partial R_\mu^*}{\partial t}=0.\tag{13}$$

Neglecting the terms of $O(\varepsilon^2)$ and higher order terms, we obtain

$$\left[\frac{\partial}{\partial a^\mu}X^{(0)}(R_\nu)-\frac{\partial}{\partial a^\nu}X^{(0)}(R_\mu)\right]\dot{a}^\nu-\frac{\partial}{\partial a^\mu}X^{(0)}(B)-\frac{\partial}{\partial t}X^{(0)}(R_\mu)=0.\tag{14}$$

Equation (14) is called the criterion equation of the second-class Mei symmetry for Birkhoff system. Then, we have the following criterion.

Criterion 1 For Birkhoff equations (1), if the infinitesimal generators ξ_0 and ξ_μ satisfy Eq. (14), then invariance of Eq. (1) under the infinitesimal transformations (8) is called the second-class Mei symmetry.

3. Conformal invariance of second-class Mei symmetry for Birkhoff systems

Definition 2 In infinitesimal transformations of infinitesimal generators $\xi_0(t,\boldsymbol{a})$ and $\xi_\mu(t,\boldsymbol{a})$, if $(\ddot{a}^\mu-\sigma_\mu)$ satisfies the following expression

$$X^{(2)}(\ddot{a}^\mu-\sigma_\mu)=l_\mu^l(\ddot{a}^\mu-\sigma_\mu),\tag{15}$$

then the infinitesimal transformations have conformal invariance of the corresponding Birkhoff system (1), equation (15) is called determining equation of Eq. (6), l_μ^l is called the conformal factor.

In order to obtain the general expression of conformal factor of conformal invariance, we calculate the difference

$$X^{(2)}(\ddot{a}^\mu-\sigma_\mu)-X^{(2)}(\ddot{a}^\mu-\sigma_\mu)\big|_{\ddot{a}^\mu-\sigma_\mu=0}.\tag{16}$$

As

$$\dot{\xi}_\mu=\frac{\partial\xi_\mu}{\partial t}+\frac{\partial\xi_\mu}{\partial a^\nu}\dot{a}^\nu\quad(\mu=0,1\dots,2n),\tag{17}$$

$$\ddot{\xi}_\mu=\frac{\partial^2\xi_\mu}{\partial t^2}+2\frac{\partial^2\xi_\mu}{\partial a^\nu\partial t}\dot{a}^\nu+\frac{\partial^2\xi_\mu}{\partial a^r\partial a^\nu}\dot{a}^\nu\dot{a}^r+\frac{\partial\xi_\mu}{\partial a^\nu}\ddot{a}^\nu\quad(r,\nu=1,\dots,2n),\tag{18}$$

therefore

$$
\begin{aligned}
X^{(2)}(\ddot{a}^\mu-\sigma_\mu) &= X^{(1)}(\ddot{a}^\mu-\sigma_\mu)+\left(\ddot{\xi}_\mu-2\ddot{a}_\mu\dot{\xi}_0-\dot{a}^\mu\ddot{\xi}_0\right)\frac{\partial}{\partial\ddot{a}^\mu}(\ddot{a}^\mu-\sigma_\mu)\\
&= X^{(0)}(\ddot{a}^\mu-\sigma_\mu)+\left(\dot{\xi}_\mu-\dot{a}^\mu\dot{\xi}_0\right)\frac{\partial}{\partial\dot{a}^\mu}(\ddot{a}^\mu-\sigma_\mu)+\left(\ddot{\xi}_\mu-2\ddot{a}^\mu\dot{\xi}_0-\dot{a}^\mu\ddot{\xi}_0\right)\frac{\partial}{\partial\ddot{a}^\mu}(\ddot{a}^\mu-\sigma_\mu)\\
&= -\xi_0\frac{\partial\sigma_\mu}{\partial t}-\xi_\mu\frac{\partial\sigma_\mu}{\partial a^\mu}-\left[\left(\frac{\partial\xi_\mu}{\partial t}+\frac{\partial\xi_\mu}{\partial a^\nu}\dot{a}^\nu\right)-\dot{a}^\mu\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^\nu}\dot{a}^\nu\right)\frac{\partial}{\partial\dot{a}_\mu}\right]\sigma_\mu\\
&\quad+\frac{\partial^2\xi^\mu}{\partial t^2}+2\frac{\partial^2\xi_\mu}{\partial a^r\partial t}\dot{a}^r+\frac{\partial^2\xi_\mu}{\partial a^\nu\partial a^r}\dot{a}^\nu\dot{a}^r+\frac{\partial\xi_\mu}{\partial a^\nu}\ddot{a}^\nu-2\ddot{a}^\mu\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^\nu}\dot{a}^\nu\right)\\
&\quad-\dot{a}^\mu\left(\frac{\partial^2\xi_0}{\partial t^2}+2\frac{\partial^2\xi_0}{\partial a^r\partial t}\dot{a}^r+\frac{\partial^2\xi_0}{\partial a^\nu\partial a^r}\dot{a}^\nu\dot{a}^r+\frac{\partial\xi_0}{\partial a^\nu}\dot{a}^\nu\right)\\
&= \Omega^{\mu\nu}\left(\frac{\partial\xi_\mu}{\partial a^r}-\dot{a}^\mu\frac{\partial\xi_0}{\partial a^r}\right)\ddot{a}^r-2\Omega^{\mu\nu}\ddot{a}^\mu\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^r}\dot{a}^r\right)+X^{(0)}(\Omega^{\mu\nu})\ddot{a}^\mu+C(t,\boldsymbol{a},\dot{\boldsymbol{a}}),
\end{aligned}
\tag{19}
$$

where $C(t,\boldsymbol{a},\dot{\boldsymbol{a}})$ is the sum of items which do not contain $\ddot{\boldsymbol{a}}$.

Treating alike leads to

$$
\begin{aligned}
X^{(2)}(\ddot{a}^\mu-\sigma_\mu)\big|_{\ddot{a}^\mu-\sigma_\mu=0} &= X^{(1)}(\ddot{a}^\mu-\sigma_\mu)\big|_{\ddot{a}^\mu-\sigma_\mu=0}+\left(\ddot{\xi}_\mu-2\ddot{a}_\mu\dot{\xi}_0-\dot{a}^\mu\ddot{\xi}_0\right)\frac{\partial}{\partial\ddot{a}^\mu}(\ddot{a}^\mu-\sigma_\mu)\Big|_{\ddot{a}^\mu-\sigma_\mu=0}\\
&= -\xi_0\frac{\partial\sigma_\mu}{\partial t}-\xi_\mu\frac{\partial\sigma_\mu}{\partial a^\mu}-\left[\left(\frac{\partial\xi_\mu}{\partial t}+\frac{\partial\xi_\mu}{\partial a^\nu}\dot{a}^\nu\right)-\dot{a}^\mu\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^\nu}\dot{a}^\nu\right)\frac{\partial}{\partial\dot{a}_\mu}\right]\sigma_\mu\\
&\quad+\frac{\partial^2\xi^\mu}{\partial t^2}+2\frac{\partial^2\xi_\mu}{\partial a^\nu\partial t}\dot{a}^\nu+\frac{\partial^2\xi_\mu}{\partial a^\nu\partial a^r}\dot{a}^\nu\dot{a}^r+\frac{\partial\xi_\mu}{\partial a^\nu}\sigma_\mu-2\sigma_\mu\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^\nu}\dot{a}^\nu\right)\\
&\quad-\dot{a}^\mu\left(\frac{\partial^2\xi_0}{\partial t^2}+2\frac{\partial^2\xi_0}{\partial a^\nu\partial t}\dot{a}^\nu+\frac{\partial^2\xi_0}{\partial a^\nu\partial a^r}\dot{a}^\nu\dot{a}^r+\frac{\partial\xi_0}{\partial a^\nu}\dot{a}^\nu\right)\\
&= \Omega^{\mu\nu}\left(\frac{\partial\xi_\mu}{\partial a^r}-\dot{a}^\mu\frac{\partial\xi_0}{\partial a^r}\right)\sigma_r-2\Omega^{\mu\nu}\sigma_\mu\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^r}\dot{a}^r\right)\\
&\quad+X^{(0)}(\Omega^{\mu\nu})\sigma_\mu+C(t,\boldsymbol{a},\dot{\boldsymbol{a}}),
\end{aligned}
\tag{20}
$$

we have

$$
\begin{aligned}
&X^{(2)}(\ddot{a}^\mu-\sigma_\mu)-X^{(2)}(\ddot{a}^\mu-\sigma_\mu)\big|_{\ddot{a}^\mu-\sigma_\mu=0}\\
&= \Omega^{\mu\nu}\left(\frac{\partial\xi_\mu}{\partial a^r}-\dot{a}^\mu\frac{\partial\xi_0}{\partial a^r}\right)(\ddot{a}^r-\sigma_r)-2\Omega^{\mu\nu}(\ddot{a}^\mu-\sigma_\mu)\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^r}\dot{a}^r\right)+X^{(0)}(\Omega^{\mu\nu})(\ddot{a}^\mu-\sigma_\mu)\\
&= \left[\Omega^{\mu\nu}\left(\frac{\partial\xi_\mu}{\partial a^r}-\dot{a}^\mu\frac{\partial\xi_0}{\partial a^r}\right)\Omega_{rl}-2\delta^l_\mu\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^r}\dot{a}^r\right)+X^{(0)}(\Omega^{\mu\nu})\Omega_{\mu l}\right](\ddot{a}^r-\sigma_l)=\beta^l_\mu(\ddot{a}^r-\sigma_l),
\end{aligned}
\tag{21}
$$

where

$$
\beta^l_\mu=\Omega^{\mu\nu}\left(\frac{\partial\xi_\mu}{\partial a^r}-\dot{a}^\mu\frac{\partial\xi_0}{\partial a^r}\right)\Omega_{rl}-2\delta^l_\mu\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial a^r}\dot{a}^r\right)+X^{(0)}(\Omega^{\mu\nu})\Omega_{\mu l}\quad(\mu,\nu,r,l=1,\ldots 2n).
\tag{22}
$$

If equation (6) is of the second-class Mei symmetry by the infinitesimal point transformation of one-parameter, we must have

$$
X^{(2)}(\ddot{a}^\mu-\sigma_\mu)\big|_{\ddot{a}^\mu-\sigma_\mu=0}=0\ .
\tag{23}
$$

From Eqs. (21) and (23), we obtain

$$
(l^l_\mu-\beta^l_\mu)(\ddot{a}^\mu-\sigma_l)=X^{(2)}\ (\ddot{a}^\mu-\sigma_\mu)|_{\ddot{a}^\mu-\sigma_\mu=0}\ .
\tag{24}
$$

So the necessary and sufficient conditions of the conformal invariance, which is of simultaneous second-class Mei symmetry by the action of the infinitesimal point transformation of one-parameter group, is as follows:

$$
l^l_\mu=\beta^l_\mu.
\tag{25}
$$

Deduction Suppose the differential equation (2) can be transformed into the standard form

$$
F=\ddot{a}^\mu-\sigma_\mu(t,\boldsymbol{a},\dot{\boldsymbol{a}})
\tag{26}
$$

by the action of infinitesimal transformations of one-parameter group, the necessary and sufficient conditions of conformal invariance, which are simultane-

ously of second-class Mei symmetry, are as follows:

$$l_\mu^l = \frac{\partial \xi_\mu}{\partial a^l} - 2\dot{\xi}_0 - \dot{a}^\mu \frac{\partial \xi_0}{\partial a^l}. \tag{27}$$

In fact, we can easily obtain Eq. (27) from Eq. (22).

4. Conserved quantities of conformal invariance for second-class Mei symmetry

The conformal invariance of Birkhoff systems under the second-class Mei symmetry would also lead to conserved quantities. Let us study Mei conserved quantities given by the following proposition.

Proposition For Birkhoff systems, if the generating functions ξ_0 and ξ_μ of the second-class Mei symmetry and gauge function $G_{\rm F} = G_{\rm F}(t, \boldsymbol{a})$ satisfy the structure equation

$$\begin{aligned} & X^{(0)}\left[X^{(0)}(R_\mu)\right] \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) \\ & - X^{(0)}\left[X^{(0)}(B)\right] \\ & - X^{(0)}(B)\,\dot{\xi}_0 + X^{(0)}(R_\mu)\dot{\xi}_\mu + \dot{G}_{\rm F} = 0, \end{aligned} \tag{28}$$

then the second-class Mei symmetry can lead to the Mei conserved quantity

$$I_{\rm F} = X^{(0)}(R_\mu)\,\xi_\mu - X^{(0)}(B)\,\xi_0 + G_{\rm F}. \tag{29}$$

Proof According to the criterion equation (14) of the second-class Mei symmetry for Birkhoff system, by using the structure equation (28), we can prove that the Proposition is true.

5. Illustration example

The Birkhoff's functions and the Birkhoffian of a system are respectively

$$\begin{aligned} R_1 &= 0, \\ R_2 &= a^1 - a^4 - \left(a^2 + a^3\right) t \\ &\quad + \left(a^2 \sin t + a^4 \cos t\right) \cos t, \\ R_3 &= a^1 - a^4 - (a^2 + a^3)t, \\ R_4 &= -(a^2 \sin t + a^4 \cos t) \sin t, \\ B &= (a^2 \sin t + a^4 \cos t)^2. \end{aligned} \tag{30}$$

Find the conformal factor and the Mei conserved quantity of the system under the second-class Mei symmetry.

From Eq. (30) we have

$$\begin{aligned} X^{(0)}(R_1) &= 0, \\ X^{(0)}(R_2) &= \xi_1 - \xi_4 - (\xi_2 + \xi_3)t - (a^2 + a^3)\xi_0 \\ &\quad + (\xi_2 \sin t + a^2 \xi_0 \cos t \\ &\quad + \xi_4 \cos t - a^4 \xi_0 \sin t) \cos t \\ &\quad - (a^2 \sin t + a^4 \cos t)\xi_0 \sin t, \\ X^{(0)}(R_3) &= \xi_1 - \xi_4 - (\xi_2 + \xi_3)t - (a^2 + a^3)\xi_0, \\ X^{(0)}(R_4) &= -(\xi_2 \sin t + a^2 \xi_0 \cos t \\ &\quad + \xi_4 \cos t - a^4 \xi_0 \sin t) \sin t \\ &\quad - (a^2 \sin t + a^4 \cos t)\xi_0 \cos t, \\ X^{(0)}(B) &= 2\left(a^2 \sin t + a^4 \cos t\right)(\xi_2 \sin t \\ &\quad + a^2 \xi_0 \cos t + \xi_4 \cos t - a^4 \xi_0 \sin t). \end{aligned} \tag{31}$$

Taking infinitesimal transformation generators

$$\begin{aligned} \xi_0 &= \xi_2 = \xi_4 = 0, \\ \xi_1 &= 1 + (a^2 + a^3)t, \\ \xi_3 &= a^2 + a^3, \end{aligned} \tag{32}$$

we have

$$\begin{aligned} & X^{(0)}(B) = X^{(0)}(R_1) = X^{(0)}(R_4) = 0, \\ & X^{(0)}(R_2) = X^{(0)}(R_3) = 1. \end{aligned} \tag{33}$$

We know that the Birkhoff system is of the second-class Mei symmetry according to Eq. (14) of Criterion 1.

Using Eq. (27) of Deduction, we can obtain conformal factor

$$l_\mu^l = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

From Eq. (28), we have $G_{\rm F} = 0$. Using the Proposition, from Eq. (29) we obtain the conserved quantity

$$I_{\rm F} = a^2 + a^3 = \text{const.} \tag{35}$$

6. Conclusion

In this paper by introducing the concept of the conformal invariance and conserved quantities for Birkhoff systems under the second-class Mei symmetry, we can find the conformal factor and the conserved quantities as shown in Eqs. (22) or (27) and (29). For a Birkhoff equation, if it possesses the characteristics of conformal invariance of the second-class Mei symmetry, then it possesses the second-class Mei symmetry. The conformal invariance may lead to corresponding Mei conserved quantities when the conformal invariance satisfies some conditions. The results can be applied to other fields, and lay the foundation for the corresponding study of the holonomic system.

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