Lie symmetry and Mei conservation law of continuum system*

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Lie symmetry and Mei conservation law of continuum Lagrange system are studied in this paper. The equation of motion of continuum system is established by using variational principle of continuous coordinates. The invariance of the equation of motion under an infinitesimal transformation group is determined to be Lie-symmetric. The condition of obtaining Mei conservation theorem from Lie symmetry is also presented. An example is discussed for applications of the results.

Keywords: continuum mechanics, Lie symmetry, Mei symmetry, conservation law

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1. Introduction

The symmetry plays an important role not only in understanding the behaviours of the mechanical systems, whether discrete or continuous medium, but also in identifying conserved quantities. Many important conclusions of theoretical mechanics can be stated in the forms of conservation laws which indicate under what conditions the mechanical quantities are constant with respect to the time.[1−4] There exist three symmetries we always use for searching invariant quantities. The Lie symmetry has been determined to be an invariance of the differential equation of motion under a continuous group now known as Lie group used to unify various methods of solving ordinary differential equations as its original purpose.[5−10] The Noether symmetry is an invariance of the action integral under Lie transformation group. The Noether theory has become a useful tool in many subfields of mathematics, physics and mechanics, since Noether showed that there was a close relationship between the mathematical symmetries of a physical system and its conserved quantities.[11−15] These two types of symmetries and their applications have obtained many results in discrete medium constrained mechanical systems.[2−4] Recently, Mei and his collaborators have introduced a new type of symmetry theory referred to as Mei symmetry, related to the form invariances of the differential equations of motion under dynamical transformations in which the dynamical quantities such as Lagrangian, non-potential generalized forces, generalized constrained forces, etc. are replaced by the switched ones which are transformed by the Lie transformation groups. Investigations on the Mei symmetry and its conserved quantities in discrete medium constrained systems have evolved to being a vast field and obtained many important developments.[2−4,16−20]

The purpose of this paper is to extend the Lie symmetry and the Mei symmetry, and their relationships to the continuous medium systems. We construct the Euler–Lagrange equations of motion for continuum system and determine Lie symmetry and Mei symmetry. We also derive the criteria for obtaining Mei conservation laws from Lie symmetry.

2. Equation of motion of continuum system

Let $u_\alpha(x_i)$, $(\alpha = 1, 2, \ldots, p;\ i = 1, 2, 3, 4)$ be field quantities in a continuous medium system where $x_1 = t$ denotes time parameter and $x_2 = x$, $x_3 = y$, $x_4 = z$ refer to position parameters. We consider the continuum system denoted by its Lagrangian density $\mathscr{L}(x_i, u_\alpha, u_{\alpha,i})$ where $u_{\alpha,i} = \partial u_\alpha/\partial x_i$, and the Lagrangian of the system to be

$$L = \int \mathscr{L}\,\mathrm{d}(x_j) \quad j = 2, 3, 4. \tag{1}$$

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The action integral is defined as

$$I = \int_1^2 \mathscr{L}\,\mathrm{d}(x_k) \quad k = 1, 2, 3, 4. \tag{2}$$

The calculation of the variation of action functional (2) yields

$$\begin{aligned}\delta I &= \delta \int_1^2 \mathscr{L}\,\mathrm{d}(x_k) \\ &= \int_1^2 \left(\frac{\partial \mathscr{L}}{\partial u_\alpha}\delta u_\alpha + \frac{\partial \mathscr{L}}{\partial u_{\alpha,i}}\delta u_{\alpha,i}\right)\mathrm{d}(x_k). \end{aligned}\tag{3}$$

Using two relation equations:

$$\delta u_{\alpha,i} = \delta\frac{\partial u_\alpha}{\partial x_i} = \frac{\partial}{\partial x_i}(\delta u_\alpha) \tag{4}$$

and

$$\begin{aligned}&\frac{\partial \mathscr{L}}{\partial u_{\alpha,i}}\delta u_{\alpha,i} \\ &= \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}}{\partial u_{\alpha,i}}\delta u_\alpha\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}}{\partial u_{\alpha,i}}\right)\delta u_\alpha, \end{aligned}\tag{5}$$

equation (3) becomes

$$\begin{aligned}\delta\int_1^2 \mathscr{L}\,\mathrm{d}(x_k) = &\int_1^2 \left[\frac{\partial \mathscr{L}}{\partial u_\alpha} - \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}}{\partial u_{\alpha,i}}\right)\right]\delta u_\alpha \\ &+ \int_1^2 \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}}{\partial u_{\alpha,i}}\delta u_\alpha\right)\mathrm{d}(x_k). \end{aligned}\tag{6}$$

The quantities x_i, $(i = 1,\ 2,\ 3,\ 4)$ are independent of each other, therefore the partial differential ∂x_i are equivalent to the total differential $\mathrm{d}x_i$. Adopting the fixed boundary condition in u space $(\delta u_\alpha)_1 = (\delta u_\alpha)_2 = 0$, the last term on the right-hand side of Eq. (6) vanishes. In consequence of the variational principle of Hamilton of continuous coordinates, we obtain the Euler–Lagrange equation for continuum system to be

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial \mathscr{L}}{\partial u_{\alpha,i}}\right) - \frac{\partial \mathscr{L}}{\partial u_\alpha} = 0, \tag{7}$$

where there are as many equations as the different values of α.

3. Lie symmetry of the continuum Lagrange system

Let the infinitesimal transformations of continuous coordinates and field quantities be

$$x_i^* = x_i + \varepsilon\xi_i(x_i, u_\alpha), \quad u_\alpha^* = u_\alpha + \varepsilon\eta_\alpha(x_i, u_\alpha), \tag{8}$$

where ε is a group parameter, and ξ_i and η_α are generators of the infinitesimal transformations. The vector field of generators is

$$X^{(0)} = \xi_i\frac{\partial}{\partial x_i} + \eta_\alpha\frac{\partial}{\partial u_\alpha}, \tag{9}$$

which can be extended to the l-th-order form:

$$\begin{aligned}&X^{(l)} \\ &= \xi_i\frac{\partial}{\partial x_i} + \eta_\alpha\frac{\partial}{\partial u_\alpha} + \eta^{(1)}_{\alpha,i_1}\frac{\partial}{\partial u_{\alpha,i_1}} + \eta^{(2)}_{\alpha,i_1i_2}\frac{\partial}{\partial u_{\alpha,i_1i_2}} \\ &\quad + \cdots + \eta^{(l)}_{\alpha,i_1i_2\ldots i_l}\frac{\partial}{\partial u_{\alpha,i_1i_2\ldots i_l}}, \end{aligned}\tag{10}$$

where

$$u_{\alpha,i_1i_2\ldots i_m} = \partial^m u_\alpha/\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_m}, \tag{11}$$

$i_m = 1,\ 2,\ 3,\ 4$, for $m = 1, 2, \ldots, l$, denote all the l-th order partial derivatives of u_α with respect to x_i. In Eq. (10), there are the following recursions:

$$\eta^{(1)}_{\alpha,i_1} = D_{i_1}\eta_\alpha - (D_{i_1}\xi_n)u_{\alpha,n}, \tag{12}$$

$$\eta^{(2)}_{\alpha,i_1i_2} = D_{i_2}\eta^{(1)}_{\alpha,i_1} - (D_{i_2}\xi_n)u_{\alpha,i_1n}, \tag{13}$$

$$\cdots$$

$$\begin{aligned}\eta^{(l)}_{\alpha,i_1i_2\ldots i_l} = &D_{i_l}\eta^{(l-1)}_{\alpha,i_1i_2\ldots i_{l-1}} \\ &- (D_{i_l}\xi_n)u_{\alpha,i_1i_2\ldots i_{l-1}n},\ n = 1, 2, 3, 4, \end{aligned}\tag{14}$$

where

$$\begin{aligned}D_{i_m} &= \frac{\mathrm{d}}{\mathrm{d}x_{i_m}} \\ &= \frac{\partial}{\partial x_{i_m}} + u_{\alpha,i_m}\frac{\partial}{\partial u_\alpha} + u_{\alpha,n_1i_m}\frac{\partial}{\partial u_{\alpha,n_1}} \\ &\quad + \cdots + u_{\alpha,n_1n_2\ldots n_li_m}\frac{\partial}{\partial u_{\alpha,n_1n_2\ldots n_l}} \end{aligned}\tag{15}$$

denotes the total derivative operator.

We now rewrite Eq. (7) as

$$F_\alpha(x_i, u_\alpha, u_{\alpha,i_1}, u_{\alpha,i_1i_2}, \ldots, u_{\alpha,i_1i_2\ldots i_l}) = 0. \tag{16}$$

Applying prolongation (10) to Eq. (7), we obtain

$$X^{(l)}_\alpha[F_\alpha(x_i, u_\alpha, u_{\alpha,i_1}, u_{\alpha,i_1i_2}, \ldots, u_{\alpha,i_1i_2\ldots i_l})] = 0. \tag{17}$$

Criterion 1 If the infinitesimal generators ξ_i and η_α satisfy Eq. (17), then invariance of each equation of motion in expression (7) is the Lie symmetry of the continuum Lagrange system. Equations in expression (17) are thus the determining equations of Lie symmetry.

4. Mei conservation theorem

We perform the dynamical transformation of Lagrangian density $\mathscr{L}$ and give

$$\begin{aligned}&\mathscr{L}^*\\&=\mathscr{L}(x_i^*,u_\alpha^*,u_{\alpha,i}^*)\\&=\mathscr{L}(x_i,u_\alpha,u_{\alpha,i})+\varepsilon X^{(1)}(\mathscr{L})+o(\varepsilon^2)+\cdots,\end{aligned}\tag{18}$$

where $o(\varepsilon^2)+\cdots$ are the second and the higher order infinitesimals of $\mathscr{L}$. Substituting the transformed Lagrangian density $\mathscr{L}^*$ into Eq. (7) yields

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\mathscr{L}^*}{\partial u_{\alpha,i}}\right)-\frac{\partial\mathscr{L}^*}{\partial u_\alpha}=0.\tag{19}$$

Omit the second and the higher order infinitesimals $o(\varepsilon^2)+\cdots$, use Eq. (7) and keep the form of Eq. (7) invariant, then we will have

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}}\right)-\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_\alpha}=0.\tag{20}$$

Criterion 2 If the infinitesimal generators ξ_i and η_α satisfy Eq. (20), then the form invariance of each equation of motion in expression (7) is the Mei symmetry of a continuum system. Equations in expression (20) are thus the determining equations of Mei symmetry. If the infinitesimal group generators ξ_i and η_α satisfy Eq. (20), in addition, there exists gauge function $G_M(x_i,u_\alpha)$ such that the identity

$$X^{(1)}(\mathscr{L})\frac{\mathrm{d}\xi_i}{\mathrm{d}x_i}+X^{(1)}(X^{(1)}(\mathscr{L}))+\frac{\mathrm{d}G_M}{\mathrm{d}x_i}=0\tag{21}$$

holds, then the system has the Mei conserved law

$$X^{(1)}(\mathscr{L})\xi_i+(\eta_\alpha-u_{\alpha,n}\xi_n)\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}}+G_M=\text{const.}\tag{22}$$

Proof Using the total derivative operator (15), equation (22) becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x_i}\left[X^{(1)}(\mathscr{L})\xi_i+(\eta_\alpha-u_{\alpha,n}\xi_n)\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}}+G_M\right]\\&=X^{(1)}(\mathscr{L})\frac{\mathrm{d}\xi_i}{\mathrm{d}x_i}+\xi_i\left(\frac{\partial X^{(1)}(\mathscr{L})}{\partial x_i}+u_{\alpha,i}\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_\alpha}+u_{\alpha,ni}\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,n}}\right)\\&\quad+(\eta_\alpha-u_{\alpha,n}\xi_n)\frac{\mathrm{d}}{\mathrm{d}x_i}\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}}+\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}}\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\eta_\alpha-u_{\alpha,n}\frac{\mathrm{d}}{\mathrm{d}x_i}\xi_n-\xi_n u_{\alpha,ni}\right)\\&\quad-(\eta_\alpha-u_{\alpha,n}\xi_n)\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_\alpha}+(\eta_\alpha-u_{\alpha,n}\xi_n)\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_\alpha}+\frac{\mathrm{d}G_M}{\mathrm{d}x_i}=0.\end{aligned}\tag{23}$$

There are two relevant equations since $n=i=1,\ 2,\ 3,\ 4$, i.e.,

$$\xi_i u_{\alpha,i}\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_\alpha}=\xi_n u_{\alpha,n}\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_\alpha},\tag{24}$$

$$\xi_i u_{\alpha,ni}\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,n}}=\xi_n u_{\alpha,ni}\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}},\tag{25}$$

therefore equation (23) reduces to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x_i}\left[X^{(1)}(\mathscr{L})\xi_i+(\eta_\alpha-u_{\alpha,n}\xi_n)\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}}+G_M\right]\\&=X^{(1)}(\mathscr{L})\frac{\mathrm{d}\xi_i}{\mathrm{d}x_i}+\xi_i\frac{\partial X^{(1)}(\mathscr{L})}{\partial x_i}+\eta_\alpha\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_\alpha}+\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\eta_\alpha-u_{\alpha,n}\frac{\mathrm{d}}{\mathrm{d}x_i}\xi_n\right)\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}}\\&\quad+(\eta_\alpha-u_{\alpha,n}\xi_n)\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_{\alpha,i}}-\frac{\partial X^{(1)}(\mathscr{L})}{\partial u_\alpha}\right)+\frac{\mathrm{d}G_M}{\mathrm{d}x_i}\\&=X^{(1)}(\mathscr{L})\frac{\mathrm{d}\xi_i}{\mathrm{d}x_i}+X^{(1)}(X^{(1)}(\mathscr{L}))+\frac{\mathrm{d}G_M}{\mathrm{d}x_i}.\end{aligned}\tag{26}$$

Combining Eq. (26) with Eq. (21), we have the result Eq. (22). Given Criterion 1, there is the condition for obtaining Mei conservative theorem from Lie symmtry of the continuum system:

Criterion 3 If the Lie symmetry generators ξ_i and η_α satisfy Eq. (21), then the Lie symmetry of the system leads to Mei conserved law Eq. (22).

5. Example

We take the transmission line equation as an example to apply this procedure. The Lagrangian density of the system is

$$\mathscr{L} = \frac{1}{2}Lu_t^2 - \frac{1}{2}\frac{1}{C}u_x^2, \tag{27}$$

where L represents the inductance and C denotes the capacitance. The equation of motion of the system is

$$Lu_{tt} - \frac{1}{C}u_{xx} = F = 0, \tag{28}$$

where we have used Euler–Lagrange equation (7). Substituting Eq. (28) into Eq. (17) and using Eqs. (10)–(15) lead to determining equation of Lie symmetry

$$L\eta_{tt} - \frac{1}{C}\eta_{xx} = 0. \tag{29}$$

Now we will solve Eq. (29) to obtain the Lie symmetry generators. Suppose that the generators are in the forms of

$$\xi_1 = \xi_1(x,t), \quad \xi_2 = \xi_2(x,t), \tag{30}$$

$$\eta(x,t,u) = f(x,t)u + g(x,t). \tag{31}$$

Substituting Eqs. (30) and (31) into Eq. (29) leads to the determining equation

$$\begin{aligned} & L\left[\frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 f}{\partial t^2}u - \frac{\partial^2 \xi_1}{\partial t^2}u_x + \left(2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi_2}{\partial t^2}\right)u_t - 2\frac{\partial \xi_1}{\partial t}u_{xt} + \left(f - 2\frac{\partial \xi_2}{\partial t}\right)u_{tt}\right] \\ = & \frac{1}{C}\left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}u - \frac{\partial^2 \xi_2}{\partial x^2}u_t + \left(2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi_1}{\partial x^2}\right)u_x - 2\frac{\partial \xi_2}{\partial x}u_{xt} + \left(f - 2\frac{\partial \xi_1}{\partial x}\right)u_{xx}\right], \end{aligned} \tag{32}$$

and setting the coefficient of each term of Eq. (32) to be zero yields

$$\begin{aligned} & \eta(x,t,u) \\ = & \left(CC_1x^2 + \frac{1}{L}C_1t^2 + C_2xt + C_3\right)u + CC_4x^2 \\ & + \frac{1}{L}C_4t^2 + C_5xt + C_6x + C_7t + C_8, \end{aligned} \tag{33}$$

$$\xi_1(x,t) = C_9x + \frac{1}{L}C_{10}t + C_{11}, \tag{34}$$

$$\xi_2(x,t) = CC_{10}x + C_9t + C_{12}, \tag{35}$$

where C_1–C_{12} are constants.

As for the Mei conservative law, after substituting Eqs. (33)–(35) into Mei symmetry determining equation (21)

$$X^{(1)}(\mathscr{L})\frac{\mathrm{d}\xi_i}{\mathrm{d}x_i} + X^{(1)}(X^{(1)}(\mathscr{L})) + \frac{\mathrm{d}G_M}{\mathrm{d}x_i} = 0, \tag{36}$$

we obtain that

$$\eta(x,t,u) = C_8, \tag{37}$$

$$\xi_1(x,t) = -\frac{1}{2}x + C_{11}, \tag{38}$$

$$\xi_2(x,t) = -\frac{1}{2}t + C_{12}, \tag{39}$$

then

$$G_M(x,t,u) = 0. \tag{40}$$

Use Eqs. (37)–(40) and Criteria 2 and 3, we can obtain the Mei conservative law

$$\begin{aligned} I_M = & \left(\frac{1}{2}Lu_t^2 - \frac{1}{2}\frac{1}{C}u_x^2\right)\left(-\frac{1}{2}x - \frac{1}{2}t + C_{11} + C_{12}\right) \\ & + \left[C_6x + C_7t + C_8 - \left(-\frac{1}{2}x + C_{11}\right)u_x \right. \\ & \left. - \left(-\frac{1}{2}t + C_{12}\right)u_t\right] \\ & \times \left(Lu_t - \frac{1}{C}u_x\right) = \text{const}. \end{aligned} \tag{41}$$

6. Conclusions

We obtained four main results in this paper: (i) the determining equation (20) of Mei symmetry of the continuum system; (ii) the conditions of Mei conservative theorems (21) and (22); (iii) Criterion 3 for obtaining Mei conservative law from Lie symmetry; (iv) the Lie symmetry generators (33)–(35) and the Mei conserved quantity (41) of the transmission line equation.

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