Consensus of high-order continuous-time multi-agent systems with time-delays and switching topologies*

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Consensus problems of high-order continuous-time multi-agent systems with time-delays and switching topologies are studied. The motivation of this work is to extend second-order continuous-time multi-agent systems from the literature. It is shown that consensus can be reached with arbitrarily bounded time-delays even though the communication topology might not have spanning trees. A numerical example is included to show the theoretical results.

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1. Introduction

Consensus problems of multi-agent systems have received considerable attention due to broad applications of multiagent systems in many areas including the cooperative control of unmanned air vehicles (UAVs), formation control, flocking, distributed sensor networks, attitude alignment of clusters of satellites, and congestion control in communication networks. A great number of results have been obtained in Refs. [1]–[14]. For example, Hong *et al.*[3] used a Lyapunov-based approach to study second-order multi-agent systems with jointly-connected topologies. Also, Xie and Wang[4] considered consensus problems in networks of second-order agents without relative velocity measurement and gave sufficient conditions for state consensus of the systems. Moreover, Xiao and Wang[7] studied first-order multi-agent systems with nonuniform time-delays and dynamically changing topologies and showed that the time-delays could be arbitrarily bounded. Lin and Jia[8,10] investigated second-order multi-agent systems with time-delays and switching topology, and neighbour-based protocols were proposed with the consideration of arbitrarily bounded communication time-delays. Recently, many researches have been focused on high-order multi-agent systems, which can be used to explain the behaviours of flocks of the birds, in which the consensus on not only their positions and velocities but also on their accelerations needs to be built. For example, Ren *et al.*[9] studied high-order multi-agent systems with fixed topology and derived sufficient conditions under which all agents reach consensus. Also, Lin *et al.*[11] considered discrete-time high-order multi-agent systems and proposed a high-order protocol with the consideration of time-delays and switching topologies.

The motivation of this work is to extend the work of Ref. [10] to high-order multi-agent systems with dynamically changing topologies and nonuniform time-delays. The dynamics of each agent is in the form of continuous-time, different from that in Ref. [11] where discrete-time agents were considered. The approach adopted in the present paper is to make model transformations so as to use the property of the Meltzer matrix with row sums. It is shown that the consensus can be achieved in the presence of arbitrarily bounded time-delays under the given conditions. To our best knowledge, this is the first time to study high-order

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continuous-time multi-agent systems with nonuniform time-delays where the communication topologies dynamically change and might have no spanning trees.

The following notations will be used throughout this paper. Let **1** and **0** denote the $n \times 1$ column vectors each with all entries equal to one and zero respectively, I_n denotes the $n \times n$ identity matrix, $\mathbb{R}^m$ denotes the set of all m dimensional real column vectors, and $\otimes$ denotes the Kronecker product.

2. Graph theory

Let $G = \{V, E, A\}$ be a weighted direct graph of n nodes $V = \{v_1, v_2, \ldots, v_n\}$, a set of edges $E \subseteq V \times V$ and a weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements a_{ij} for all $i, j \in I = \{1, 2, \ldots, n\}$. The node indexes belong to a finite index set. A directed edge of G is denoted by $e_{ij} = (v_i, v_j)$. The set of neighbours of node v_i is the set of all nodes which communicate with v_i, denoted by $N_i = \{v_j \in V : (v_j, v_i) \in E\}$. The adjacency matrix $A = [a_{ij}]$ is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(v_i, v_j) \in E$ for $i \neq j$.

The Laplacian matrix of the weighted directed graph is defined as $L = [l_{ij}]$, where $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ with $i \neq j$. Note that L has an important property that all the row sums of L are zero.

A directed path in G is a sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), (v_{i_3}, v_{i_4}) \ldots$ in that graph. Graph G is called strong connection if there is a directed path between any arbitrary node pair. We say that a graph contains a spanning tree if there exists a node such that there is a directed path from every other node to this node.

3. Formulation and main results

3.1. Formulation

Consider a multi-agent system consisting of n continuous-time agents. Suppose that the ℓ^{th}-order dynamics of the i-th ($i \in \mathscr{I}$) agent is given by

$$\begin{aligned}
&\dot{\xi}_i^{(0)}(t) = \xi_i^{(1)}(t),\\
&\dot{\xi}_i^{(1)}(t) = \xi_i^{(2)}(t),\\
&\qquad\vdots\\
&\dot{\xi}_i^{(\ell-2)}(t) = \xi_i^{(\ell-1)}(t),\\
&\dot{\xi}_i^{(\ell-1)}(t) = u_i(t),
\end{aligned} \tag{1}$$

where $\xi_i^{(k)} \in \mathbb{R}$, $k = 0, 1, \ldots, \ell - 1$, are the states, $u_i \in \mathbb{R}$ is the protocol (control input), and $\xi_i^{(k)}$ denotes the k^{th} component of ξ_i.

We say that the protocol u_i asymptotically solves the consensus problem, if the states of agents satisfy

$$\lim_{t \to +\infty} [\xi_i(t) - \xi_j(t)] = 0, i, j \in \mathscr{I}. \tag{2}$$

To solve the consensus problem, we use the following delayed consensus protocol:

$$\begin{aligned}
u_i(t) = &-\sum_{k=1}^{\ell-1} p_k \xi_i^{(k)}(t) - \sum_{v_j \in N_i(t)} a_{ij}(t)(\xi_i^{(0)}(t)\\
&- \xi_j^{(0)}(t - \tau_{ij}(t))),
\end{aligned} \tag{3}$$

where $i \in \mathscr{I}$, $p_k > 0$, $a_{ij}(t) > 0$ that is an edge weight chosen from a finite set, and $\tau_{ij}(t) \leq \tau_{\max}$ that is the communication time-delay from v_i to v_j, with $\tau_{\max}$ denoting the maximal time-delay. It should be emphasized that the time-delays can be bounded in any large size.

Let

$$\xi(t) = [\xi_1^{\mathrm{T}}(t), \xi_2^{\mathrm{T}}(t), \ldots, \xi_n^{\mathrm{T}}(t)]^{\mathrm{T}},$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \ldots & 0 & 0 & 1\\ 0 & -p_1 & \ldots & -p_{\ell-2} & -p_{\ell-1} \end{bmatrix} \in \mathbb{R}^{\ell \times \ell}$$

and

$$B = \begin{bmatrix} 0 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & 0\\ 1 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{\ell \times \ell}.$$

Then under protocol (3), the network dynamics is

$$\begin{aligned}
\dot{\xi}(t) = &[I_n \otimes A - (L_0(t) - Q_0(t)) \otimes B]\xi(t)\\
&+ (Q_1(t) \otimes B)\xi(t-1) + \cdots\\
&+ (Q_{\tau_{\max}}(t) \otimes B)\xi(t - \tau_{\max}),
\end{aligned} \tag{4}$$

where $L_0(t) = \mathrm{diag}(L(t))$, and the ij-th element of $Q_m(t)$ ($m = 0, 1, \ldots, \tau_{\max}$) is either zero or a_{ij} if $\tau_{ij} = m$. If at some point one agent receives the information at a different time from another agent, the lastest information should be taken into consideration. It is clear that $L(t) = L_0(t) - \sum_{m=0}^{\tau_{\max}} Q_m(t)$.

3.2. Main results

To analyse the stability of continuous-time multi-agent systems with switching topologies, a common approach is to employ the Lyapunov function or use the properties of Metzler matrix with zero row sums. As the closed-loop system matrix of system (4) is not a Metzler with zero row sums, we should perform some transformations to help us analyse the stability of system (4).

Let $\gamma_i(t) = \Omega\xi_i(t)$ and $\gamma(t) = [\gamma_1^{\mathrm{T}}(t), \gamma_2^{\mathrm{T}}(t), \ldots, \gamma_n^{\mathrm{T}}(t)]^{\mathrm{T}}$, where

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & S_1 & 0 & 0 & 0 \\ 1 & \sum\limits_{j_0=1}^{2} S_{j_0} & \prod\limits_{j_0=1}^{2} S_{j_0} & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & \sum\limits_{j_0=1}^{\ell-1} S_{j_0} & \sum\limits_{j_0=1}^{\ell-2} \prod\limits_{j_1=j_0+1}^{\ell-1} S_{j_0}S_{j_1} & \cdots & \prod\limits_{j_0=1}^{\ell-1} S_{j_0} \end{bmatrix} \in \mathbb{R}^{\ell\times\ell}, \quad \text{and} \quad S_j > 0, j = 1, 2, \ldots, \ell-1.$$

By the above variable substitution, system (4) can be given by

$$\dot{\gamma}(t) = [I_n \otimes C - (L_0(t) - Q_0(t)) \otimes D]\gamma(t) + (Q_1(t) \otimes D)\gamma(t-1) + \cdots + (Q_{\tau_{\max}}(t) \otimes D)\gamma(t-\tau_{\max}), \tag{5}$$

where $D = B\prod\limits_{i=1}^{\ell-1} S_i \in \mathbb{R}^{\ell\times\ell}$ and

$$C = \Omega A\Omega^{-1} = \begin{bmatrix} -\dfrac{1}{S_1} & \dfrac{1}{S_1} & 0 & 0 & 0 \\ 0 & -\dfrac{1}{S_2} & \dfrac{1}{S_2} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\dfrac{1}{S_{\ell-1}} & \dfrac{1}{S_{\ell-1}} \\ C_{\ell 1} & C_{\ell 2} & \cdots & C_{\ell(\ell-1)} & C_{\ell\ell} \end{bmatrix} \in \mathbb{R}^{\ell\times\ell},$$

$$\begin{aligned} [C_{\ell 1}, C_{\ell 2}, C_{\ell 3}, \ldots, C_{\ell\ell}] &= \left[1 \sum_{j_0=1}^{\ell-1} S_{j_0} \sum_{j_0=1}^{\ell-2} \prod_{j_1=j_0+1}^{\ell-1} S_{j_0}S_{j_1} \cdots \prod_{j_0=1}^{\ell-1} S_{j_0}\right][(A-I)+I]\Omega^{-1} \\ &= \left[-1, 1 - \sum_{j_0=1}^{\ell-1} S_{j_0} - p_1 \prod_{j_0=1}^{\ell-1} S_{j_0}, \sum_{j_0=1}^{\ell-1} S_{j_0} - \sum_{j_0=1}^{\ell-2} \prod_{j_1=j_0+1}^{\ell-1} S_{j_0}S_{j_1} - p_2 \prod_{j_0=1}^{\ell-1} S_{j_0}, \ldots, \right. \\ &\quad \left. \sum_{j_0=1}^{\ell-1} \prod_{m\neq j_0}^{\ell-1} S_m - p_{\ell-1} \prod_{j_0=1}^{\ell-1} S_{j_0}\right]\Omega^{-1} + [\mathbf{0}_{\ell-1}^{\mathrm{T}}, 1]. \end{aligned}$$

First, we consider system (5) in the absence of communication time-delays, i.e., the network dynamics of system (5) is

$$\dot{\gamma}(t) = \Theta\gamma(t), \tag{6}$$

where $\Theta = I_n \otimes C - L(t) \otimes D$. Note that in Eq. (6), the row sums of Ω^{-1} are all zero except in the first row, and the first element is equal to 1 and other elements are all equal to 0 in the first row of Ω^{-1}. Hence $[C_{\ell 1}, C_{\ell 2}, C_{\ell 3}, \ldots, C_{\ell\ell}]\mathbf{1} = 0$ and obviously $\Theta\mathbf{1} = [I_n \otimes C - L(t) \otimes D]\mathbf{1} = 0$ holds. Meanwhile, to ensure that Θ is a Metzler matrix, we should let

$$\begin{aligned} &S_i > 0, \quad C_{\ell 1} < d_{\max} \prod_{m=1}^{\ell-1} S_m, \\ &C_{\ell j} > 0, \quad \text{and} \quad C_{\ell\ell} < 0, \end{aligned} \tag{7}$$

where $i = 1, 2, \ldots, \ell-1$, $j = 2, 3, \ldots, \ell-1$ and $d_{\max}$ is the largest entry of all possible $L(t)$.

Under condition (7), the diagonal entries of Θ are negative and the off-diagonal entries are nonnegative. Thus, Θ is a Metzler matrix with zero row sums.

Then we consider system (5) in the presence of time-delays. We can see that the time-delays only affect some off-diagonal terms in Eq. (6). Hence according to Theorem 1 of Ref. [12] and Theorem 3 of Ref. [7], we can easily draw the conclusion that consensus of systems (5) can be achieved with arbitrary bounded time-delay and switching topologies. Summarizing the above analysis, we have the following results.

Theorem 1 Suppose that each communication topology of multi-agent system (1) keeps unchanged for at least d_1 time, and the union of the communication topologies has spanning trees every bounded time interval whose length is smaller than d_2 for given constants d_1 and d_2. Then multi-agent system (1) with protocol (3) under condition (7) achieves consensus. Specifically, $\lim_{t\to+\infty}[\xi_i^{(0)}(t)-\xi_j^{(0)}(t)]=0$ and $\lim_{t\to+\infty}\xi_i^{(j)}(t)=0$ for all $i,j=1,2,\cdots,\ell-1$.

Remark 1 Theorem 1 requires that the union of communication topologies in every bounded interval has spanning tree, and the time-delay can be arbitrarily bounded.

Corollary 1 Under the assumption of Theorem 1, if all time-delays are zero, then multi-agent system (1) with protocol (3) under condition (7) achieves consensus. Specifically, $\lim_{t\to+\infty}[\xi_i^{(0)}(t)-\xi_j^{(0)}(t)]=0$ and $\lim_{t\to+\infty}\xi_i^{(j)}(t)=0$ for all $i,j=1,2,\cdots,\ell-1$.

Remark 2 In some practical applications, for example, in the flocking behaviour of the birds, the communication time-delays can be omitted and Corollary 1 gives an excellent explaination why birds fly in a specific shape for a long time.

Remark 3 In practical applications, due to unreliable link or limited communication capacity, the credibility of information each agent receives often changes over time, which makes the weight of each edge in the communication graph time-varying. But if the values of edge weight each are adopted from a finite set, then the multi-agent systems can reach consensus under the conditions given by Theorem 1 and Corollary 1.

4. Simulation results

In the following, numerical simulations are given to illustrate the obtained theoretical results. Four communication topologies of directed networks with 10 three-order agents are shown in Fig. 1. Note that all graphs in this figure do not have spanning trees but their union has a spanning tree.

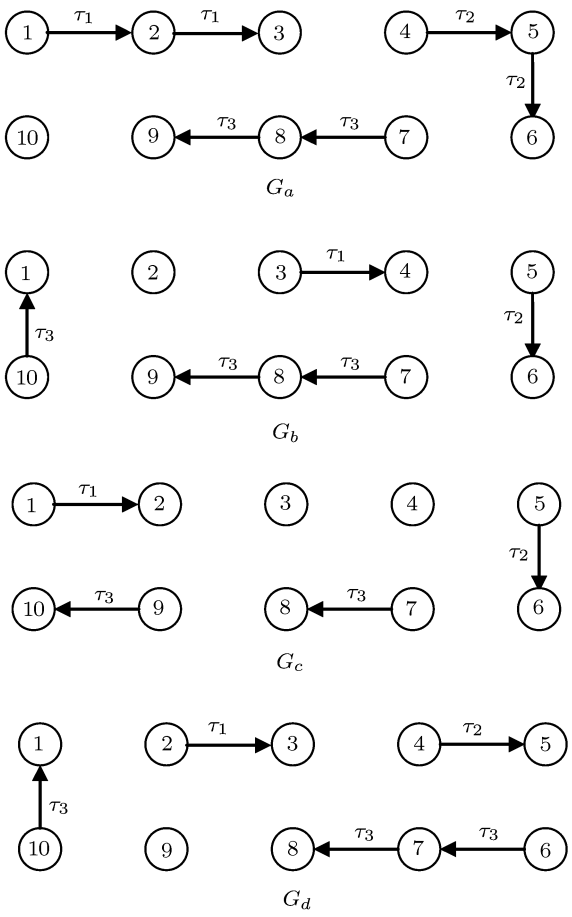


Fig. 1. Four communication topologies of directed networks.

In this simulation, we consider two scenarios. First, we consider a scenario where each edge in the topologies has a fixed weight. For simplicity, all weights are set as 1. Second, as mentioned in Remark 2, due to unreliable link or limited communication capacity, the weight of each edge in the communication graph may be time-varying. Hence we consider a scenario that the weight of each edge in the topologies switches between 1 and 0.5.

In the two scenarios, the time-delay for edges (1, 2), (2, 3), (3, 4) is $\tau_1 = 1$, for edges (4, 5), (5, 6) is $\tau_2 = 2$, and (6, 7), (7, 8), (8, 9), (9, 10), (10, 1) is $\tau_3 = 3$. In scenario 1, the communication topology of the multi-agent system switches every 0.5 s in the sequence of $\langle G_a, G_b, G_c, G_d, G_a, G_b, \ldots\rangle$, namely, the period of topology change is 2 s. In scenario 2, the communication topology switches as the same as in scenario 1. The weight of each edge is set to be 1 or 0.5 in each period of topology change, and it switches

between 1 and 0.5 every 2 s. To meet condition (7), we choose $S_1 = S_2 = 1$. As $d_{\max} = 1$ here, we choose $C_{31} = 0$, $C_{32} = 1$ and $C_{33} = -1$. Thus the protocol parameters are taken as $(p_1, p_2) = (2, 3)$. The simulation results of scenario 1 are shown in Fig. 2, and the simulation results of scenario 2 are shown in Fig. 3.

From Figs. 2 and 3, we can see that this system can achieve consensus asymptotically, and the trajectories are not smooth perfectly as the communication topology changes.

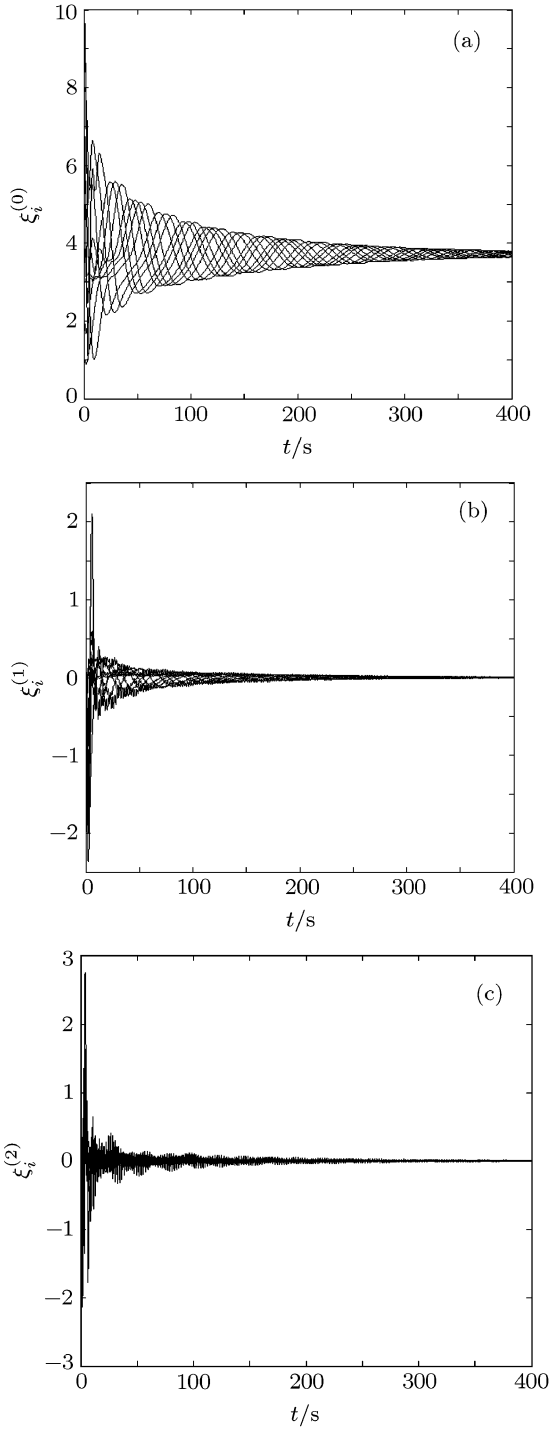


Fig. 2. Trajectories of the first (a), second (b) and third (c) variables of ten agents in the topologies with fixed weights.

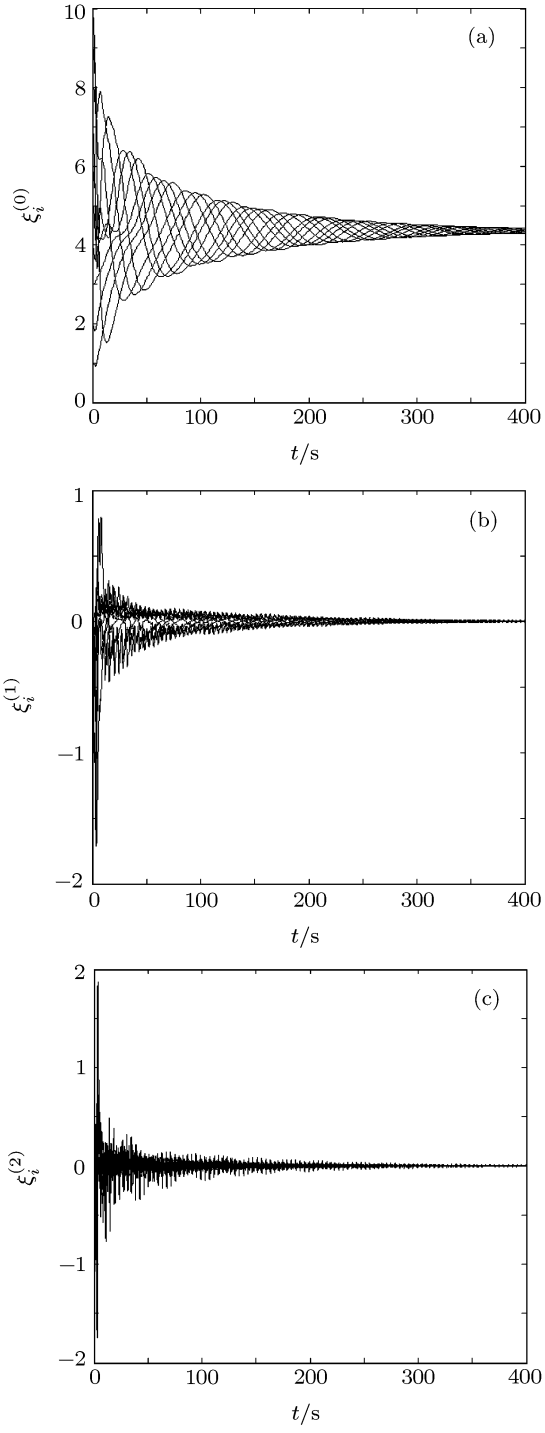


Fig. 3. Trajectories of the first (a), second (b) and third (c) variables of ten agents in the topologies with time-varying weights.

5. Conclusions

We extend second-order continuous-time multi-agent systems to high-order systems. It is shown that consensus can be reached with aribitratrily bounded time-delays even though the communication topology changes. A numerical example is provided to illustrate the obtained theoretical results.

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