# Robust fault-sensitive synchronization of a class of nonlinear systems*

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Aiming at enhancing the quality as well as the reliability of synchronization, this paper is concerned with the fault detection issue within the synchronization process for a class of nonlinear systems in the existence of external disturbances. To handle such problems, the concept of robust fault-sensitive (RFS) synchronization is proposed, and a method of determining such a kind of synchronization is developed. Under the framework of RFS synchronization, the master and the slave systems are robustly synchronized, and at the same time, sensitive to possible faults based on a mixed $H_-/H_\infty$ performance. The design of desired output feedback controller is realized by solving a linear matrix inequality, and the fault sensitivity $H_-$ index can be optimized via a convex optimization algorithm. A master-slave configuration composed of identical Chua's circuits is adopted as a numerical example to demonstrate the effectiveness and applicability of the analytical results.

**Keywords:** nonlinear system, synchronization, disturbance rejection, fault detection

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## 1. Introduction

Synchronization means that two or more systems adjust each other for giving rise to a common dynamical behaviour, induced by both globally or locally coupling the systems and forcing them.[1,2] Within interdisciplinary fields such as applied physics, mathematical biology and control science, the notion of synchronization of complex or chaotic systems has received considerable attention in recent years, since it eliminates certain degrees of freedom of the coupled system and so produces a significant reduction of complexity.[3−5] Except for the main motivation that lies in potential applications regarding secure communications, chaos synchronization appears in numerous different ways, most notably in biological applications, physics and engineering. Various types of synchronization schemes based on control approach have been introduced into the systems under a master-slave configuration,[6] including the observer based control,[7,8] time-delay feedback approach,[9] adaptive control,[10,11] and so on. Although most of the existing schemes concentrate on the master-slave synchronization of nonlinear Lur'e systems,[12] meanwhile, several other attempts have been made to extend the synchronization between two systems to an array of uniformly coupled Lur'e systems.[13−15] The main reason is that, in various fields of theory and engineering application, a large class of nonlinear systems such as Chua's circuit,[16,17] Goodwin model,[18] toggle switch[19] and swarm model[20] can be represented as Lur'e systems. Primary methods of dealing with such problems, among others, are developed within the frame of absolute stability theory.[21]

Traditional efforts to improve the synchronous reliability and dependability are to enhance the quality and the robustness of the synchronization scheme. Since noises or disturbances are ubiquitous in real-world systems which may cause instability and poor performance, it is practical and reasonable to take the noise phenomenon into account in synchronization.[22] In this regard, the $H_\infty$ control concept is adopted and the $H_\infty$ synchronization problem has been extensively discussed for disparate systems.[23−25] Even so, due to

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the inevitability of fault, a fault-free synchronization process cannot always be guaranteed.

Research on fault diagnosis (FD) has been gaining increasing consideration world wide in both theory and application.[26−28] The strong impetus comes mainly from the trend of automation towards more complexity and the growing demand of higher availability, reliability and operating safety in modern control systems, such as nuclear reactors, chemical plants, aircrafts, and rapid transit trains.[26] The 'fault' here is to be understood as an unexpected change of system function, which hampers or disturbs the normal operation of an automatic system, thus an unacceptable deterioration of the performance of the system or even dangerous situations will be caused.[26] Among the vast majority of relevant jobs, the FD problem of a linear dynamic system is in the spotlight, and hardly any of previous studies has brought the concept of fault into physical aspects such as synchronization between two coupled nonlinear systems.

Based on the consideration that faults may also bring degradation of the synchronous quality, the present paper initially introduces the notion of FD into the synchronization process and proposes the robust fault-sensitive (RFS) synchronization framework. It shows that nonlinear Lur'e systems form a natural class for considering RFS synchronization between perturbed master systems with possible faults and coupled slave systems under control. The main challenge in RFS synchronization is to distinguish failures from other disturbances, and accordingly, the $H_\infty/H_-$ paradigm is introduced.[29,30] For the purpose of description, the robustness objectives during synchronization are considered in virtue of the $H_\infty$ norm, while the fault sensitivity specifications are expressed by utilizing the formulation of $H_-$ index. In this manner, the closed-loop error system is asymptotically stable, with the $H_\infty$-norm from the disturbance input to controlled output reduced to a prescribed level, and at the same time, with the $H_-$ performance index maximized. Results derived under such a control scheme extend previous work in a substantial way, and sufficient conditions to the RFS synchronization problem for chaotic Lur'e dynamical systems are developed in terms of sets of linear matrix inequalities (LMI).[31]

The rest of the present paper is organized as follows. In Section 2 we summarize some key facts that will be used in the later sections, introduced the definition of $H_\infty$ norm and $H_-$ index in time domain, and give the mathematical formulations of the RFS synchronization problem to be solved in this paper. Criteria on RFS synchronization between two identical Lur'e systems are proposed in Section 3. Chua's circuit is adopted as a numerical example in Section 4 in order to demonstrate the effectiveness of our design approaches. Finally, some conclusions and future work are presented in Section 5.

The notations used in this paper are presented as follows. $\mathbb{R}^{n\times n}$ is the set of $n\times n$ real matrices. For a matrix $A$, $A^{\mathrm{T}}$ denotes its transpose matrix. If $A$ is a real symmetric negative-definite matrix, it is shown by $A<0$, while $B\geq 0$ means that $B$ is a semi-positive-definite matrix. The diag$(\cdot)$ implies a diagonal or block-diagonal matrix.

## 2. Preliminaries and problem formulation

The subject of our research is to demonstrate the existence of RFS synchronization regimes of a class of nonlinear systems. To make the problem statement clearer, we start our discussion with some preliminary results on RFS synchronization. Consider two identical nonlinear systems coupled in the following form:

$$
\begin{aligned}
(\mathscr{M})\ &\begin{cases}\dot{x} &= Ax+B\varphi(y)+B_d\ d+B_f f,\ y=Cx,\\ z &= Hx+Df,\end{cases}\\
(\mathscr{S})\ &\begin{cases}\dot{\hat{x}} &= A\hat{x}+B\varphi(\hat{y})+u,\quad \hat{y}=C\hat{x},\\ \hat{z} &= H\hat{x},\quad u=L(z-\hat{z}),\end{cases}
\end{aligned}
\tag{1}
$$

with master system $(\mathscr{M})$ and slave system $(\mathscr{S})$, where $\mathscr{M}$ is subjected to an input noise and possible faults. The master system and the slave system are both Lur'e systems where $x,\ \hat{x}\in\mathbb{R}^n$ are the state vectors, $y,\ \hat{y}\in\mathbb{R}^m$ represent the control output vectors, and $z,\ \hat{z}\in\mathbb{R}^m$ are the measurement output vectors. $d\in\mathbb{R}^p$ is an unknown input vector (including disturbance, uninterested fault as well as some norm-bounded unstructured model uncertainty) belonging to $L_2[0,+\infty)$, $f\in\mathbb{R}^q$ denotes the process, sensor or actuator fault vector to be detected and isolated, and $u$ is an output feedback control, in which the gain matrix $L$ is to be designed. $f$ and $d$ can be modeled as different types of signals, depending on specific situations under consideration. The matrices $A,B,C,D,B_d,B_f$ and $H$ are known constant matrices with appropriate dimensions. Suppose that the nonlinearity $\varphi=(\varphi_1^{\mathrm{T}}(y_1),\ldots,\varphi_m^{\mathrm{T}}(y_m))^{\mathrm{T}}:\mathbb{R}^m\to\mathbb{R}^m$

is a continuously differentiable vector valued function with $\varphi_i(\cdot)$ satisfying sector conditions:

$$0 \leq \varphi_i(y_i)y_i \leq k_i y_i^2,\ 0 \leq \varphi_i'(y_i) \leq k_i,\ i = 1, \ldots, m, \tag{2}$$

where $k_i \in \mathbb{R}^+,\ i = 1, \ldots, m$. System (1) represents a large class of nonlinear systems in various fields of physical and engineering applications, such as the Chua's circuit, the Goodwin model, the toggle switch as well as the swarm model.

**Remark 1** Since synchronization has a strong connection with the observer design method,[32] it is natural to find that, in virtue of output feedback control scheme, the RFS synchronization process falls in the class of fault-detection observers. Accordingly, the design of controller gain matrix is parallel to that of observer parameters. Influenced by disturbance and fault, the state variable of ($\mathscr{M}$) may not be derived completely. As a consequence, the synchronization is realized by utilizing output feedback control method. On the other hand, when the state feedback control approach is adopted, the master system must be connected with a state observer, hence a new synchronization problem will be formulated in Section 4, which is in some sense an extension of the RFS synchronization scheme.

Define the error signal $e = x - \hat{x}$ and residual signal $r = z - \hat{z}$, then the dynamics of synchronization residual error between the master and the slave systems can be represented as

$$\begin{aligned} \dot{e} &= (A - LH)e + B\sigma(Ce; C\hat{x}) \\ &\quad + B_d\ d + (B_f - LD)f, \\ r &= He + Df. \end{aligned} \tag{3}$$

Here, $\sigma(Ce; C\hat{x}) = \varphi(Ce + C\hat{x}) - \varphi(C\hat{x})$ satisfies the sector restriction

$$\begin{aligned} 0 \leq\ & \frac{\sigma_i(c_i^{\mathrm{T}}e, x)}{c_i^{\mathrm{T}}e} \\ =\ & \frac{\varphi_i(c_i^{\mathrm{T}}e + c_i^{\mathrm{T}}\hat{x}) - \varphi_i(c_i^{\mathrm{T}}\hat{x})}{c_i^{\mathrm{T}}e} \leq k, \\ & i = 1, 2, \ldots, m \quad (c_i^{\mathrm{T}}e \neq 0), \end{aligned}$$

which leads to the following inequalities:

$$\sigma_i(c_i^{\mathrm{T}}e; \hat{x})(\sigma_i(c_i^{\mathrm{T}}e; \hat{x}) - kc_i^{\mathrm{T}}e) \leq 0, \quad i = 1, 2, \ldots, m. \tag{4}$$

Hence the residual error dynamics (3) also possesses a standard form of nonlinear Lur'e system. In allusion to this system, several definitions are presented as follows.

**Definition 1** If the transfer function $d \mapsto r$ of the error dynamics (3) is given as

$$G_{rd}(s) \triangleq H(sI - A + LH)^{-1}B_d, \tag{5}$$

then its $\mathscr{H}_\infty$ norm is defined by $\|G_{rd}\|_\infty = \sup_{d \in L_2}(\|r\|_2/\|d\|_2) = \sup_{d \in L_2}(\|G_{rd}d\|_2/\|d\|_2)$.

**Definition 2** Consider the following transfer function from the input $f$ to output $r$:

$$G_{rf}(s) \triangleq H(sI - A + LH)^{-1}(B_f - LD) + D. \tag{6}$$

The $\mathscr{H}_-$ index of the transfer function matrix $G_{rf}(s)$ is defined as

$$\|G_{rf}(s)\|_-^{[0,\bar{\omega}]} \triangleq \inf_{\omega \in [0,\bar{\omega}]} \underline{\sigma}[G_{rf}(j\omega)], \tag{7}$$

where $\underline{\sigma}$ represents the minimum singular value and $\bar{\omega}$ denotes the frequency band $[0, \bar{\omega}]$. In addition, this performance criterion beyond the frequency domain can be extended from the viewpoint of signal as $\|G_{rf}\|_- = \inf_{f \in L_2}(\|G_{rf}f\|_2/\|f\|_2)$.

**Remark 2** Initially introduced by Hou and Patton,[30] the $\mathscr{H}_-$ index defined has been widely adopted to measure the sensitivity of residual to fault, and there are various kinds of $\mathscr{H}_-/\mathscr{H}_\infty$ performance measurement criteria proposed by Ding.[29] In this study, for the sake of simplicity, we shall consider the case of maximizing the fault sensitivity $\|G_{rf}(s)\|_-$ with disturbance attenuation $\|G_{rd}(s)\|_\infty$ being a prescribed constant.

According to the basic knowledge of synchronization, the residual error dynamics must be asymptotically stable in order for the whole process to work. Note that the dynamics of the residual error signal $r$ depends not only on $f$, $d$ and $u$, but also on the states of the master and slave systems $x$ and $\hat{x}$. In consequence, this study specifically aims at designing an appropriate feedback term $L$ such that

i) the residual error dynamics (3) is asymptotically stable;

ii) the generated residual $r$ is robust to the unknown input $d$ and as sensitive as possible to the fault signal $f$.

Under such circumstances, the master system ($\mathscr{M}$) and slave system ($\mathscr{S}$) are said to achieve RFS synchronization with a guaranteed $\mathscr{H}_-/\mathscr{H}_\infty$ performance. In other words, the goal is to determine the gain matrix $L$ such that the following conditions hold:

$$\|G_{rf}(s)\|_- > \beta, \quad \|G_{rd}(s)\|_\infty < \gamma, \tag{8}$$

where $\gamma$ is a prescribed positive constant, and $\beta$ is a constant to be optimized. The basic idea behind this

design scheme is that the reference model ensures a maximal fault detection rate and a given disturbance attenuation in the case where the system model is perfectly known (i.e. no modeling errors).

By applying the well-known Parseval theorem to the ratios $\mathscr{H}_\infty$ norm and $\mathscr{H}_-$ index stated in Definition 1 and Definition 2, respectively, it is shown that relation (8) is equivalent to the following statements.

**Definition 3** The master system ($\mathscr{M}$) and the slave one ($\mathscr{S}$) in Eq. (1) are said to achieve robust fault-sensitive (RFS) synchronization, if the following conditions are satisfied for the residual error synchronization dynamics (3):

(i) with zero disturbance and zero fault, the dynamics of system (3) is asymptotically stable;

(ii) with zero initial condition and a given constant $\gamma > 0$, the following condition holds:

$$J_1 = \int_0^\infty [r(t)^{\mathrm{T}} r(t) - \gamma^2 d^{\mathrm{T}}(t)\mathrm{d}(t)]\mathrm{d}t < 0; \quad (9)$$

(iii) with zero initial condition and a certain constant $\beta > 0$, the following condition holds:

$$J_2 = \int_0^\infty [r(t)^{\mathrm{T}} r(t) - \beta^2 f^{\mathrm{T}}(t) f(t)]\mathrm{d}t > 0, \quad (10)$$

and makes the scalar $\beta$ maximized. Then the disturbance attenuation $\gamma$ is the $\mathscr{H}_\infty$-norm bound of this synchronization scheme, while $\beta$ denotes the sensitivity to possible faults.

## 3. RFS synchronization design scheme

The design of an RFS synchronization configuration can be treated as a multiple objective design task, that is, the design objective is not only as sensitive as possible to faults such that early detection of faults is possible, but on the other hand, whilst the sensitivity of possible faults is maximized, but also suppressing the effect of disturbances and modeling errors on the synchronization error and subsequently on the residual, in order to prevent the synchronization process from being destroyed. Next theorem gives an LMI formulation for the solutions of controller gain $L$ to the RFS synchronization.

**Theorem 1** Suppose that $\gamma > 0$ and $\beta > 0$ are prescribed constant scalars. For a given constant $\alpha$, if there exist a positive-definite matrix $P = P^{\mathrm{T}} > 0$, diagonal matrices $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_m) > 0$, $\Pi = \mathrm{diag}(\pi_1, \ldots, \pi_m) > 0$ and matrices $S, U$ such that the following LMIs hold:

$$\Psi_1 = \begin{bmatrix} \Psi_{11} + H^{\mathrm{T}}H & C^{\mathrm{T}}k\Delta - SB & \Psi_{13} & C^{\mathrm{T}}k\Lambda & -SB_d \\ * & -\mathrm{He}\,\Delta & -\alpha B^{\mathrm{T}}S^{\mathrm{T}} & 0 & 0 \\ * & * & \mathrm{He}\,\alpha S & C^{\mathrm{T}}\Pi & -\alpha SB_d \\ & * & * & -\mathrm{He}\,\Lambda & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (11)$$

$$\Psi_2 = \begin{bmatrix} \Psi_{11} - H^{\mathrm{T}}H & C^{\mathrm{T}}k\Delta - SB & \Psi_{13} & C^{\mathrm{T}}k\Lambda & UD - SB_f - H^{\mathrm{T}}D \\ * & -\mathrm{He}\,\Delta & -\alpha B^{\mathrm{T}}S^{\mathrm{T}} & 0 & 0 \\ * & * & \mathrm{He}\,\alpha S & C^{\mathrm{T}}\Pi & \alpha UD - \alpha SB_f \\ * & * & * & -\mathrm{He}\,\Lambda & 0 \\ * & * & * & * & \beta^2 I - D^{\mathrm{T}}D \end{bmatrix} < 0, \quad (12)$$

where $\Psi_{11} = \mathrm{He}\,(UH - SA)$ and $\Psi_{13} = P + S + \alpha H^{\mathrm{T}}U^{\mathrm{T}} - \alpha A^{\mathrm{T}}S^{\mathrm{T}}$, then the master system ($\mathscr{M}$) and the slave one ($\mathscr{S}$) in Eq. (1) achieve robust fault-sensitive synchronization with a guaranteed $H_\infty$ performance $\gamma$ as well as a fault detection sensitivity $\beta$, and the feedback controller is given by $L = S^{-1}U$.

**Proof** The proof of this theorem is divided into two parts. First, we shall show the global asymptotical stability of the residual error dynamics (3) with $d = f = 0$. Under the given conditions, the performance indexes $J_1$ and $J_2$ in Definition 3 are then proven to be satisfied.

When $d = f = 0$, equation (3) is represented as

$$\dot{e} = (A - LH)e + B\sigma(Ce; C\hat{x}). \quad (13)$$

Choose a Lyapunov functional candidate in the form of

$$V = e^{\mathrm{T}}Pe + 2\sum_{j=1}^{m} \pi_j \int_0^{c_j^{\mathrm{T}}e} \varphi(s)\mathrm{d}s. \quad (14)$$

Calculating the time derivative of $V$ along the trajectory of the residual error dynamics (3) yields

$$\begin{aligned}\dot{V} &= 2e^{\mathrm{T}}P\dot{e} + 2\sum_{j=1}^{m}\pi_j\varphi(c_j^{\mathrm{T}}e)c_j^{\mathrm{T}}\dot{e}\\ &= 2e^{\mathrm{T}}P\dot{e} + 2\varphi^{\mathrm{T}}(Ce)\Pi C\dot{e}. \end{aligned} \tag{15}$$

Then consider the sector restrictions (2) and (4) that are satisfied by nonlinearities $\varphi(\cdot)$ and $\sigma(\cdot)$ respectively. Namely, for any diagonal matrix $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_m) > 0$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) > 0$

$$\begin{aligned}&2\sum_{i=1}^{m}\delta_i\sigma_i(c_i^{\mathrm{T}}e;\hat{x})(\sigma_i(c_i^{\mathrm{T}}e;\hat{x}) - kc_i^{\mathrm{T}}e)\\ &= 2\sigma^{\mathrm{T}}(Ce;\hat{x})\Delta\sigma(Ce;\hat{x}) - 2\sigma^{\mathrm{T}}(Ce;\hat{x})\Delta kCe \le 0,\\ &2\sum_{i=1}^{m}\lambda_i\varphi_i(c_i^{\mathrm{T}}e)(\varphi_i(c_i^{\mathrm{T}}e) - kc_i^{\mathrm{T}}e)\\ &= 2\varphi^{\mathrm{T}}(Ce)\Lambda\varphi(Ce) - 2\varphi^{\mathrm{T}}(Ce)\Lambda kCe \le 0. \end{aligned} \tag{16}$$

Moreover, it is known from Eq. (13) that there exists any block-partitioned matrix $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ such that the following equality holds:

$$\begin{aligned}\Phi &= \left\langle \begin{bmatrix} e \\ \dot{e} \end{bmatrix}, 2Q[\dot{e} - (A - LH)e - B\sigma(Ce;C\hat{x})] \right\rangle\\ &= 0. \end{aligned} \tag{17}$$

Here $Q_1$ and $Q_2$ are considered as free-weighting matrices. For the sake of simplicity, denote $\sigma \triangleq \sigma^{\mathrm{T}}(Ce;\hat{x})$, $\varphi \triangleq \varphi(Ce)$ in the following contexts. Incorporating formulations (16) and (17) into Eq. (15) derives

$$\begin{aligned}\dot{V} \le\ & 2e^{\mathrm{T}}P\dot{e} + 2\varphi^{\mathrm{T}}\Pi C\dot{e} - 2\sigma^{\mathrm{T}}\Delta\sigma + 2\sigma^{\mathrm{T}}\Delta kCe - 2\varphi^{\mathrm{T}}\Lambda\varphi + 2\varphi^{\mathrm{T}}\Lambda kCe + 2e^{\mathrm{T}}Q_1\dot{e} - 2e^{\mathrm{T}}Q_1(A - LH)e\\ &- 2e^{\mathrm{T}}Q_1B\sigma + 2\dot{e}^{\mathrm{T}}Q_2\dot{e} - 2\dot{e}^{\mathrm{T}}Q_2(A - LH)e - 2\dot{e}^{\mathrm{T}}Q_2B\sigma\\ =\ & \eta^{\mathrm{T}}\Psi\eta, \end{aligned} \tag{18}$$

where

$$\eta = \begin{bmatrix} e \\ \sigma \\ \dot{e} \\ \varphi \end{bmatrix}, \quad \Psi = \begin{bmatrix} \mathrm{He}\, Q_1(LH - A) & C^{\mathrm{T}}k\Delta - Q_1B & P + Q_1 + (LH - A)^{\mathrm{T}}Q_2^{\mathrm{T}} & C^{\mathrm{T}}k\Lambda \\ * & -\mathrm{He}\,\Delta & -B^{\mathrm{T}}Q_2^{\mathrm{T}} & 0 \\ * & * & \mathrm{He}\, Q_2 & C^{\mathrm{T}}\Pi \\ * & * & * & -\mathrm{He}\,\Lambda \end{bmatrix}.$$

To facilitate the design of the coupling feedback term $L$, we designate $Q_1 = S$ and $Q_2 = \alpha S$, where $\alpha$ is a constant scalar, also it can be seen from inequality (11) that $\alpha(S + S^{\mathrm{T}}) < 0$ and thus $S$ is nonsingular. By substituting these representations into $\Psi$ and applying the change of variable $L = S^{-1}U$, $\Psi < 0$ is guaranteed by the upper left block of LMI (11), hence the synchronization residual error dynamics (3) is globally asymptotically stable.

In what follows, we will show that the restriction on performance index $J_1$ given in inequality (9) is satisfied under zero initial conditions for all nonzero $d \in L_2[0, \infty)$. In this case, the error dynamics (3) is expressed as

$$\dot{e} = (A - LH)e + B\sigma(Ce;C\hat{x}) + B_d d, \quad r = He. \tag{19}$$

Based on expressions (18) and (19), it is not difficult to obtain

$$r^{\mathrm{T}}r - \gamma^2 d^{\mathrm{T}}d + \dot{V} \le \eta_1^{\mathrm{T}}\Psi_1\eta_1,$$

where $\eta_1 = [e^{\mathrm{T}}\ \sigma^{\mathrm{T}}(Ce;\hat{x})\ \dot{e}^{\mathrm{T}}\ d^{\mathrm{T}}]^{\mathrm{T}}$, and $\Psi_1$ is described in the theorem. It further implies that for any $d \ne 0$, $r(t)^{\mathrm{T}}r(t) - \gamma^2 d^{\mathrm{T}}(t)d(t) + \dot{V}(t) < 0$ according to condition (11). Under zero initial condition, the Lyapunov function $V$ defined in Eq. (14) satisfies $V(0) = 0$ and inequality $V(t) \ge 0$ for $t > 0$, hence

$$\begin{aligned}J_1 \le\ & \int_0^{\infty}[r(t)^{\mathrm{T}}r(t) - \gamma^2 d^{\mathrm{T}}(t)\,\mathrm{d}(t)]\,\mathrm{d}t\\ &+ V(t)|_{t\to\infty} - V(0)\\ =\ & \int_0^{\infty}[r(t)^{\mathrm{T}}r(t) - \gamma^2 d^{\mathrm{T}}(t)\,\mathrm{d}(t) + \dot{V}(t)]\\ <\ & 0, \end{aligned}$$

and inequality (9) is satisfied. As for the verification of condition (10), the error dynamics is given as

$$\begin{aligned}&\dot{e} = (A - LH)e + B\sigma(Ce;C\hat{x}) + (B_f - LD)f,\\ &r = He + Df. \end{aligned}$$

Following the same line of the proof of $J_1 < 0$, we know that if

$$\begin{aligned}-J_2 \le\ & \int_0^{\infty}[\beta^2 f^{\mathrm{T}}(t)f(t) - r(t)^{\mathrm{T}}r(t)]\,\mathrm{d}t\\ &+ V(t)|_{t\to\infty} - V(0)\end{aligned}$$

$$= \int_0^\infty [\beta^2 f^{\mathrm{T}}(t)f(t) - r(t)^{\mathrm{T}}r(t) + \dot{V}(t)] < 0 \quad (20)$$

holds, then constraint (10) will be satisfied, and further, inequality condition (20) is guaranteed by

$$\beta^2 f^{\mathrm{T}} f - r^{\mathrm{T}} r + \dot{V} \leq \eta_2^{\mathrm{T}} \Psi_2 \eta_2^{\mathrm{T}} < 0,$$

where $\eta_2 = [e^{\mathrm{T}}\ \sigma^{\mathrm{T}}(Ce;\hat{x})\ \dot{e}^{\mathrm{T}}\ f^{\mathrm{T}}]^{\mathrm{T}}$ with $\Psi_2 < 0$ given in inequality (12). Accordingly, the performance index $J_2 > 0$ is ensured, which completes the proof. □

**Remark 3** Theorem 1 provides an efficient method to guarantee that the synchronization process can keep sensitive to possible faults with performance index $\beta$, and at the same time, suppress the effect of external disturbance to a prescribed level $\gamma$. When $f = 0$, the RFS synchronization scheme becomes the $H_\infty$ synchronization addressed in Refs. [23]–[25]. From the proof of Theorem 1, if there are proper matrices $P > 0, \Delta, \Pi, \Lambda, S$ and $U$ such that the LMI (11) holds with a prescribed disturbance attenuation $\gamma$, then a sufficient condition for the $H_\infty$ synchronization will be derived, which is consistent with previous endeavors.

The following convex optimization algorithm offers a way of deriving the optimized value of fault sensitivity for the RFS synchronization.

**Corollary 1** Given a constant $\alpha$ and prescribe a scalar $\gamma > 0$. The master system and the slave system $(\mathscr{M})$, $(\mathscr{S})$ in Eq. (1) are said to achieve robust fault-sensitive synchronization with guaranteed $H_\infty$ performance $\gamma$ and fault detection sensitivity $\beta_0 = \sqrt{-\mu}$, where $\mu$ is the global minimum of the following generalized eigenvalue minimization problem with respect to $P = P^{\mathrm{T}} > 0$, diagonal matrices $\Delta > 0$, $\Pi > 0$ and matrices $S, U$:

$$\begin{aligned} &\min \quad \mu \\ &\text{s.t.} \quad \text{LMI (11)} \end{aligned}$$

and

$$\begin{bmatrix} \Psi_{11} - H^{\mathrm{T}}H & C^{\mathrm{T}}k\Delta - SB & \Psi_{13} & C^{\mathrm{T}}k\Lambda & UD - SB_f - H^{\mathrm{T}}D \\ * & -\mathrm{He}\,\Delta & -\alpha B^{\mathrm{T}}S^{\mathrm{T}} & 0 & 0 \\ * & * & \mathrm{He}\,\alpha S & C^{\mathrm{T}}\Pi & \alpha UD - \alpha SB_f \\ * & * & * & -\mathrm{He}\,\Lambda & 0 \\ * & * & * & * & -\mu I - D^{\mathrm{T}}D \end{bmatrix} < 0$$

hold, where $\Psi_{11}, \Psi_{13}$ are described in Theorem 1, and the controller is given as $L = S^{-1}U$.

## 4. Numerical example

The examples adopted in this section are based on concrete examples of the Chua's circuits, the most important nonlinear circuits, which is frequently observed in various fields of theory and engineering application.[33] The state equations of a single Chua's oscillator as shown in Fig. 1, are described by

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i}_3 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{C_1}\left(\dfrac{v_2 - v_1}{R} - g(v_1)\right) \\ \dfrac{1}{C_2}\left(\dfrac{v_1 - v_2}{R} + i_3\right) \\ -\dfrac{1}{L}(v_2 + R_0 i_3) \end{bmatrix}. \quad (21)$$

Here, the voltages across the capacitors $C_1$ and $C_2$ are denoted by $v_1$ and $v_2$, while $i_3$ is the current flowing through the inductance $L$. The term $g(v_1)$ represents the current flowing through the nonlinear resistor $N_R$ which is a piecewise-linear function expressed as $g(v_1) = G_{b1}v_1 + (1/2)(G_{a1} - G_{b1})[|v_1 + 1| - |v_1 - 1|]$, and it satisfies $\min\{G_{a1}, G_{b1}\} \leq g'(v_1) \leq \min\{G_{a1}, G_{b1}\}$. $R_0$ and $R$ are the linear resistors.

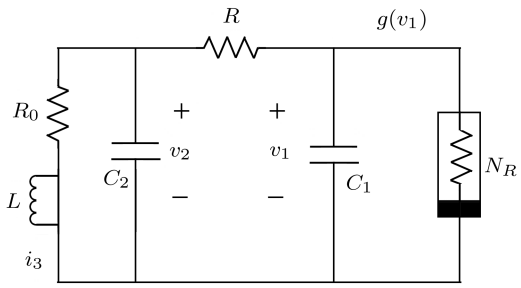



**Fig. 1.** A single Chua's oscillator.

Introducing the dimensionless variables $\tau = 1/RC_2, p = 1/RC_1, q = 1/L, s = R_0/L$, and $\epsilon = R$, then system (21) can be reformulated into the following dimensionless Lur'e form:

$$\dot{x} = Ax + B\varphi(Cx), \quad (22)$$

where

$$x = \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix},\ A = \begin{bmatrix} -p(G_{b1}+1) & p & 0 \\ \tau & -\tau & \tau\epsilon \\ 0 & -q & -s \end{bmatrix},$$

$$B = \begin{bmatrix} -\epsilon p(G_{b1}-G_{a1}) \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad (23)$$

and the nonlinear function $\varphi(Cx) = (1/2)(|x_1+1| - |x_1-1|)$ satisfies the sector condition in $[0, 1]$.

Consider that the master system and the slave system developed by Eq. (1) are both Chua's circuits in the form of Eq. (22), and take the external noise as well as possible faults into account during the synchronization of Chua's circuit. As shown in Fig. 2, the coupling between those two circuits serves as a feedback controller, indicating that there is a connection from $(\mathscr{S})$ to $(\mathscr{M})$ but none from $(\mathscr{M})$ to $(\mathscr{S})$. The external disturbance is assumed to exist in the master system, which is depicted as the current $i_d$ in the figure. In this regard, the parameters of external disturbance is taken as $B_d = [0\ 1/C_2\ 0]^{\mathrm{T}}$. $F$ plays a role of uni-communication and the undirected connection is realized by removing the element $F$. Depending on the value of the controller gain, the resistor $R_1$ can be adjusted. The process fault $f$ is assumed to be a faulty current flowing in the same direction as $i_{\mathscr{M}3}$ along the leftmost branch of the master circuit, which leads to $B_f = [0\ 1/C_2\ -R_0/L]^{\mathrm{T}}$ and $D = 1$. Furthermore, the output parameter matrices of the master circuit and the slave circuit are both assumed to be $H = [1\ 0\ 1]$.

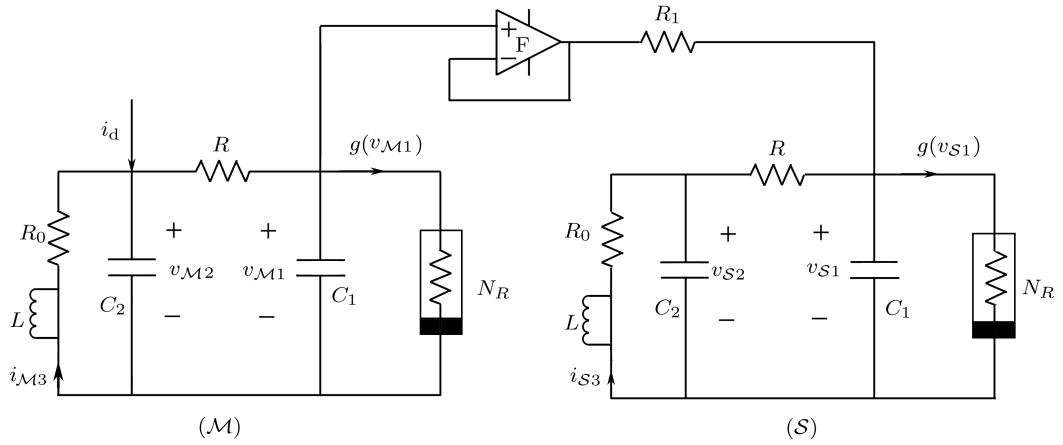



**Fig. 2.** Coupling of two identical master and slave Chua's circuits with disturbance signal.

In the following, it will be shown that how the results derived in the above sections can be used to guarantee the RFS synchronization between the two circuits. By choosing the parameters as $\tau = \epsilon = 1$, $p = 5.5, q = 7.3,\ s = 4,\ G_{a1} = -1/7$, and $G_{b1} = 2/7$ and prescribing $\alpha = 3$, we arrive at the following feasible solutions by solving LMIs (11) and (12):

$$U = \begin{bmatrix} -0.0455 \\ -4.3945 \\ -0.2094 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.2282 & 0.3037 & 0.0377 \\ 0.3037 & -3.4032 & 0.1036 \\ 0.0377 & 0.1036 & -0.1517 \end{bmatrix},$$

$$L = S^{-1}U = \begin{bmatrix} 2.9316 \\ 1.6514 \\ 3.2355 \end{bmatrix}, \quad (24)$$

with the sensitivity to faults and disturbance attenuation picked as $\beta = 0.6$, and $\gamma = 0.79$. Based on Theorem 1, it implies that the master-slave Chua's circuits achieve RFS synchronization with this derived controller gain $L$.

Simulation results also confirm the effectiveness of the design. Figure 3 depicts the time response of synchronization error of fault-free systems without disturbance signal $d(t)$, and it shows that the synchronization error converges to zero exponentially. Herein, initial values are taken as $x(0) = [-0.23\ 0.2\ 0.8]^{\mathrm{T}}$ and $\hat{x}(0) = [-0.6\ -0.67\ -0.9]^{\mathrm{T}}$. To observe the $H_\infty$ performance with disturbance attenuation, two circum-

stances are considered here according to concrete fault and disturbance types.

(I) Assume the unknown input noise disturbance $d_1$ to be

$$d_1(t) = 0.5\sin(2t), \quad t \geq 0.$$

I) Pick $\beta = 0.6$, and $\gamma = 0.79$. First simulate the fault signal $f_1(t)$ as a pulse of unit amplitude occurring from 5 s to 10 s (and is zero otherwise), and the corresponding generated residual signal $r(t)$ is illustrated in Fig. 4(a). It can be observed from this sub-figure that the effect of disturbance input $d_1(t)$ on residual error signal $r(t)$ is greatly reduced, and the residual has a rather large amplitude so that the synchronization process can remain sensitive to the fault $f_1(t)$.

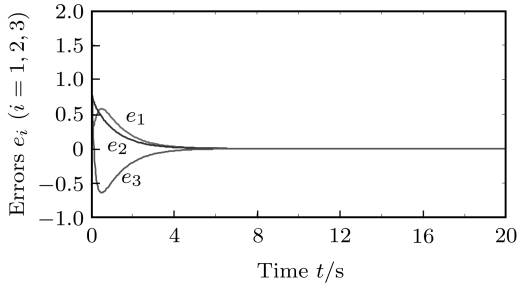



**Fig. 3.** Time responses of synchronization errors without disturbance signal $d(t)$ and fault $f(t)$.

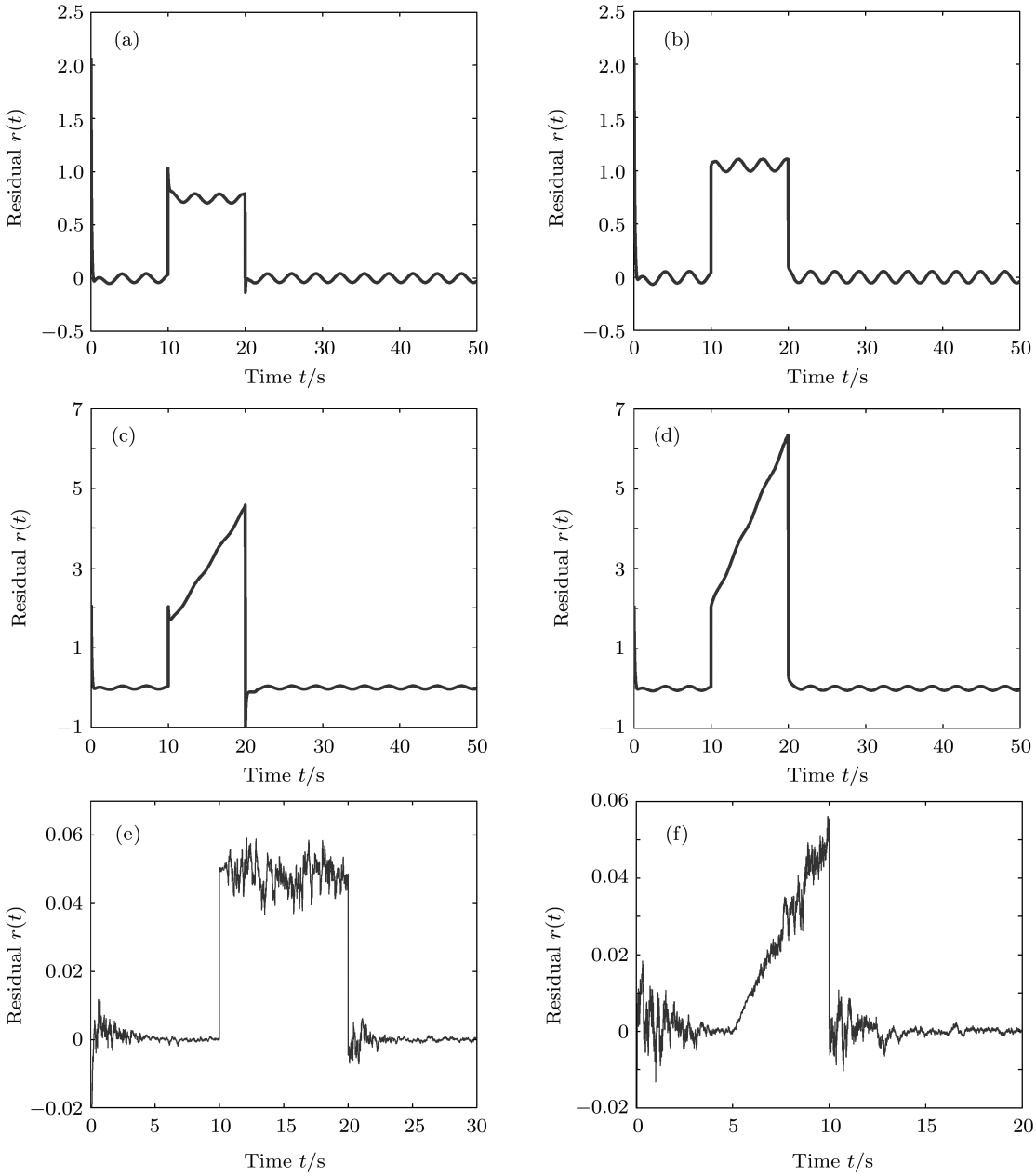



**Fig. 4.** Residual responses with $d_1(t)$, $\gamma = 0.79$ and (a) $f_1(t)$, $\beta = 0.6$; (b) $f_1(t)$, $\beta_0 = 0.9466$; (c) $f_2(t)$, $\beta = 0.6$; (d) $f_2(t)$, $\beta_0 = 0.9466$; and the residual responses with white noise disturbance, $\beta = 0.6$, $\gamma = 0.79$ and (e) $f_3(t)$; (f) $f_4(t)$.

With the same disturbance $d_1(t)$, redo the simulation for a different fault signal $f_2(t)$, then the corresponding result will be plotted in Fig. 4(c), where

$$f_2(t) = \begin{cases} 0.4(t-5), & 10 \le t < 20, \\ 0, & \text{elsewhere.} \end{cases}$$

II) Fix $\gamma = 0.79$ and solve the generalized eigenvalue problem corresponding to the minimization problem given in Corollary 1, then we will obtain an estimate of the maximum value of fault sensitivity at $\beta_0 = 0.9466$, which also guarantees in terms of Corollary 1 that the master system and the slave system achieve the RFS synchronization with $\beta < \beta_0$. In this situation, the feedback gain matrix is solved as

$$L = S^{-1}U = \begin{bmatrix} 2.0315 \\ 1.7074 \\ 1.9801 \end{bmatrix},$$

and accordingly, the generated residual signal with indices $\gamma$ and $\beta_0$ are depicted in Figs. 4(b) and 4(d), corresponding to $f_1(t)$ and $f_2(t)$, respectively. Note that with the variation of $\beta$, the magnitudes of the residual signal in these two sub-figures are significantly increased in comparison with those of Figs. 4(a) and 4(c), which reveals that the synchronization process is more sensitive to the input faults.

(II) Since sensor noise is frequently encountered in practical situations, the disturbance here is assumed to be band-limited white noise with power 0.0005, while the faults are assumed to be

$$f_3(t) = \begin{cases} 0.1(t-5), & 5 \le t < 10, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$f_4(t) = \begin{cases} 0.05, & 5 \le t < 10, \\ 0, & \text{elsewhere.} \end{cases}$$

Take the initial values as $x(0) = [-0.23\ 0.2\ 0.79]^{\mathrm{T}}$ and $\hat{x}(0) = [-0.26\ 0.27\ 0.89]^{\mathrm{T}}$. For $\beta = 0.6$ and $\gamma = 0.79$, the generated residual signals with the feasible solutions of $L$ presented in Eq. (24) are depicted in Figs. 4(e) and 4(f), which shows the effectiveness of our theoretical results.

## 5. Conclusion and future work

In this study, we have discussed the fault detection issue in the synchronization process for a class of nonlinear systems in the existence of external disturbances. The crucial distinction from synchronization schemes studied before (see Refs. [7]–[11] and [23]–[25] and the references therein) is that both the external disturbance and the system fault are introduced, and hence the RFS synchronization scheme proposed here may be more practical than the synchronization in the above-mentioned literature. In virtue of the LMI technique, we have presented an efficient algorithm to the design of desired feedback controller, where the fault sensitivity $H_-$ index is optimized via a convex optimization algorithm. A group of two coupled Chua's circuits has been used as an appropriate example to demonstrate the effectiveness and applicability of the derived results.

Generally speaking, an RFS synchronization scheme comprises two stages: synchronization residual error generation and the decision making stage. Once the synchronization scheme is determined, the remaining tasks for fault detection during the synchronization process is the evaluation of generated residual signal, thus the design of an adaptive evaluation threshold will be stressed in the future. Besides, due to the ubiquity of model uncertainties and time delay in the synchronization process, the influences of these two factors are also worth while considering for the further study.

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