Stochastic impulsive control for the stabilization of Lorenz system*

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This paper derives some sufficient conditions for the stabilization of Lorenz system with stochastic impulsive control. The estimate of the upper bound of impulse interval for asymptotically stable control is obtained. Some differences between the system with stochastic impulsive control and with deterministic impulsive control are presented. Computer simulation is given to show the effectiveness of the proposed method.

Keywords: stabilization, Lorenz system, stochastic impulsive control

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1. Introduction

In recent years, the scheme of chaos control has attracted increasing attention. A wide variety of approaches have been proposed for the control of chaotic systems, such as OGY (Ott, Grebogi, Yorke) method,[1] adaptive control method,[2−4] adaptive-feedback control method,[5] impulsive control method,[6−17] and so on. The importance of the impulsive control is that it can deal with the systems which cannot endure continuous disturbance. And these systems exist widely in practice.[6] Thus, the impulsive control will be extensively used to stabilize chaotic systems. The main idea of impulsive control method is to change the state variables of the systems to some given values whenever some conditions are satisfied. In the past several years, many researches about impulsive control have been carried out, and many useful conclusions have been proposed.[6−17] Here reference [7] is an important article in this field. In that paper the authors used the theory of impulsive differential equation to find conditions under which impulsively controlled Lorenz system is asymptotically stable, and they also gave the estimate of the upper bound of impulsive interval for asymptotically stable control.

As we know, there are many conclusions about impulsive control, but most of these results are suited for the deterministic systems. The research about the systems with random factor is seldom reported. In this paper, we consider the stabilization of the Lorenz system[18] with stochastic impulsive control. Based on Ref. [7], for the stochastic system we give some conditions under which the origin is asymptotically stable. Combining with numerical simulation, we also give the differences of the stable region between the stochastic and the deterministic systems, which show the effects of the random perturbation on the stabilization of the impulsive controlled Lorenz system.

The rest of the paper is organized as follows. In Section 2, we introduce some conclusions for the stabilization of the Lorenz system with deterministic impulsive control. Based on the work shown in this part, we give some results for random case in Section 3. The numerical simulations are implemented in Section 4. Finally, concluding remarks are drawn in Section 5.

2. The stabilization of the Lorenz system with deterministic impulsive control

The impulsive control of the Lorenz system[7,9−11] is given by

$$\begin{cases} \dot{x} = Ax + \Phi(x), & t \neq \tau_i, \\ \Delta x = x(\tau_i^+) - x(\tau_i^-) = B_i x, & t = \tau_i, \\ x(t_0^+) = x_0, & (i = 1, 2, \ldots), \end{cases} \tag{1}$$

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where $x^{\mathrm{T}} = (x_1, x_2, x_3)$,

$$A = \begin{bmatrix} -a & a & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix}, \qquad (2)$$

and B_i, named control gains, are 3×3 real matrices. Here a, r, and b are three real positive parameters, τ_i denotes the moment when the impulsive control occurs, $x(\tau_i^+) = \lim_{t \to \tau_i^+} x(t)$ and $x(\tau_i^-) = \lim_{t \to \tau_i^-} x(t)$. For simplicity, it is assumed that $x(\tau_i^-) = x(\tau_i)$. If there exists a constant $\delta > 0$ such that $\tau_{i+1} - \tau_i = \delta$, $(i = 1, 2, \ldots)$, then we say that $\{\tau_i\}$ is equidistant.

According to Refs. [7], [9]–[11], we have the following theorem which can guarantee that system (1) will be asymptotically stable at origin.

Theorem 1[7,9−11]

Let λ_i be the largest eigenvalue of $(I + B_i^{\mathrm{T}})(I + B_i)$, $(i = 1, 2, \ldots)$, B_i are Hermitian, the spectral radii of $I + B_i$, $\rho(I + B_i) \leq 1$, and let λ be the largest eigenvalue of $(A + A^{\mathrm{T}})$. If there exists a constant $\xi > 1$, such that

$$0 \leq \tau_{i+1} - \tau_i \leq \frac{-\ln(\xi \lambda_i)}{\lambda}, \quad (i = 1, 2, \ldots). \qquad (3)$$

Then the origin of the impulsive controlled Lorenz system in Eq. (1) is asymptotically stable.

In practice, for convenience, $\{\tau_i\}$ are always set to be equidistant. Then the following corollary will be born.

Corollary 1[7,9−11]

Assume $\tau_{i+1} - \tau_i = \delta > 0$ and matrices $B_i = B$, $\rho(I + B) \leq 1$, $(i = 1, 2, \ldots)$. If there exists a constant $\xi > 1$, such that

$$0 \leq \delta \leq \frac{-\ln(\xi \lambda_1)}{\lambda}, \qquad (4)$$

then the origin of the impulsive controlled Lorenz system in Eq. (1) is asymptotically stable. Here λ_1 is the largest eigenvalue of $(I + B^{\mathrm{T}})(I + B)$.

From Eq. (4) we can obtain an estimate of the upper bound of δ, $\delta_{\max}$,[7]

$$\delta_{\max} = -\frac{\ln(\xi \lambda_1)}{\lambda} > 0, \quad \xi \to 1^+. \qquad (5)$$

Theorem 1 and Corollary 1 give the sufficient conditions for controlling the systems to the origin, but these conditions are not necessary. In other words, we may find some cases that the system (1) is asymptotically stable while the conditions are not satisfied. In Section 4 we will give some figures to illustrate these cases in detail.

3. Stabilization of the Lorenz system with stochastic impulsive control

In this section we will give some conclusions for the stability of the Lorenz system with stochastic impulsive control based on the results shown in Section 2. First, we give the equations of the stochastic system, which is

$$\begin{cases} \dot{x} = Ax + \Phi(x), & t \neq \tau_i, \\ \Delta x = x(\tau_i^+) - x(\tau_i^-) = B_i x, & t = \tau_i, \\ x(t_0^+) = x_0, & (i = 1, 2, \ldots), \end{cases} \qquad (6)$$

where B_i are random matrices.

As we know, if B_i are random matrices, the largest eigenvalue of $(I + B_i^{\mathrm{T}})(I + B_i)$ and the spectral radii of $I + B_i$, namely λ_i and $\rho(I + B_i)$, will be random variables. Hence, we cannot employ Theorem 1 to guarantee that system (6) is asymptotically stabilized. Now we introduce the notations $\overline{\lambda_i}$ and $\overline{\rho_i}$, such that $\overline{\lambda_i} = E\lambda_i$, $\overline{\rho_i} = E(\rho(I + B_i))$, $(i = 1, 2, \ldots)$, where $E\lambda_i$ and $E(\rho(I + B_i))$ denote the expectation of λ_i and $\rho(I + B_i)$, respectively. Based on the derivation of the impulsive control method,[7,9−11] we can see that it is available to replace λ_i and $\rho(I + B_i)$ by $\overline{\lambda_i}$ and $\overline{\rho_i}$ in Theorem 1. Then we have the following results.

Remark 1

When B_i $(i = 1, 2, \ldots)$ are random matrices, let random variables λ_i and $\rho(I + B_i)$ be the largest eigenvalue of $(I + B_i^{\mathrm{T}})(I + B_i)$ and the spectral radii of $I + B_i$, respectively, $\overline{\lambda_i} = E\lambda_i$, $\overline{\rho_i} = E(\rho(I + B_i)) \leq 1$. And let λ be the largest eigenvalue of $(A + A^{\mathrm{T}})$. If there exists a constant $\xi > 1$, such that

$$0 \leq \tau_{i+1} - \tau_i \leq \frac{-\ln\left(\xi \overline{\lambda_i}\right)}{\lambda}, \quad i = 1, 2, \ldots, \qquad (7)$$

then the origin of the stochastic impulsive controlled Lorenz system in Eq. (6) is asymptotically stable.

Remark 2

Assume $\tau_{i+1} - \tau_i = \delta > 0$ and random matrices $B_i = B$, $\overline{\rho_1} = E(\rho(I + B)) \leq 1$, $(i = 1, 2, \ldots)$. If there exists a constant $\xi > 1$, such that

$$0 \leq \delta \leq \frac{-\ln\left(\xi \overline{\lambda_1}\right)}{\lambda}, \qquad (8)$$

then the origin of the stochastic impulsive controlled Lorenz system in Eq. (6) is asymptotically stable.

Here $\overline{\lambda_1} = E\lambda_1$, and random variable λ_1 is the largest eigenvalue of $(I + B^{\mathrm{T}})(I + B)$.

Furthermore, for system (6), we can give an estimate of the upper bound of δ,

$$\delta_{\max} = -\frac{\ln\left(\xi\overline{\lambda_1}\right)}{\lambda} > 0, \quad \xi \to 1^{+}. \tag{9}$$

Similarly, the conditions given in Remarks 1 and 2 are also sufficient conditions, not necessary conditions.

Some numerical simulations will be performed in the following section to show the effectiveness of the scheme we have presented above.

4. Numerical simulations

In the simulations, the parameters of the system (6) are chosen as $a = 10$, $b = 8/3$, $r = 28$. Then we have

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}, A+A^{\mathrm{T}} = \begin{bmatrix} -20 & 38 & 0 \\ 38 & -2 & 0 \\ 0 & 0 & -16/3 \end{bmatrix}. \tag{10}$$

We find $\lambda = 28.051249$. With these parameters, the motion of the Lorenz system is chaotic.[7,9−11,14]

We choose

$$B_i = B = \begin{bmatrix} K & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (i = 1, 2, \ldots), \tag{11}$$

where K is a random variable, for additive noise, $K = k+\sigma N$, for multiplicative noise, $K = (1 + \sigma N)k$. And N is a noise subjected to the normal distribution with the intensity σ.

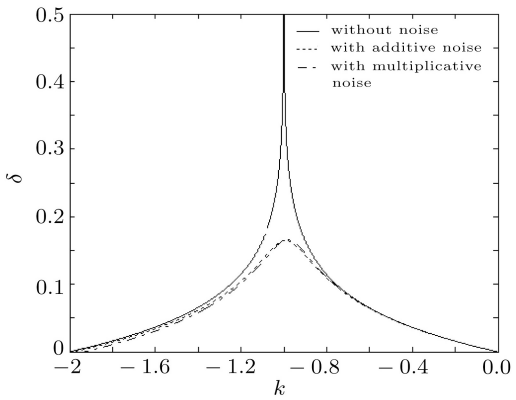


Fig. 1. The estimates of boundaries of stable region for $B_i = B = \mathrm{diag}(K, -1, -1)$.

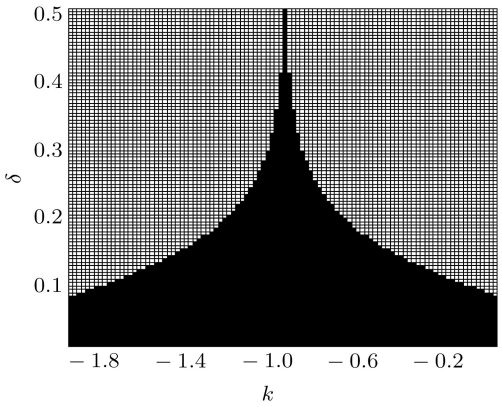


Fig. 2. The stable region of the deterministic system for $B_i = B = \mathrm{diag}(K, -1, -1)$.

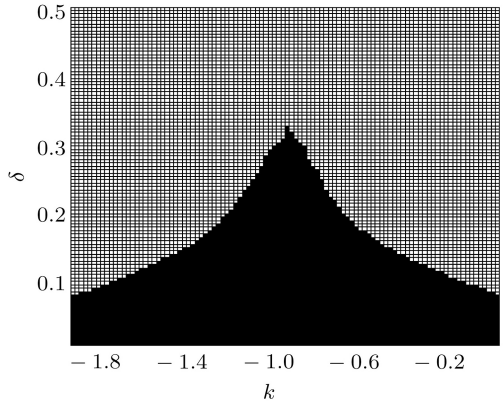


Fig. 3. The stable region of the stochastic system (with additive noise) for $B_i = B = \mathrm{diag}(K, -1, -1)$.

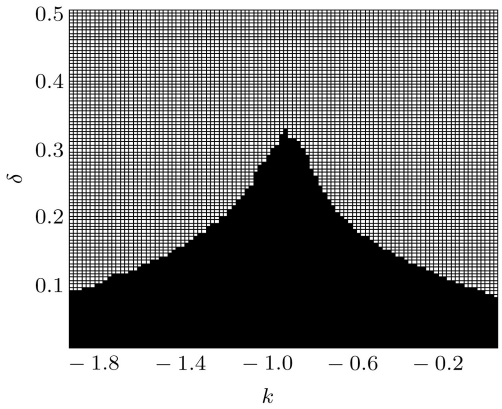


Fig. 4. The stable region of the stochastic system (with multiplicative noise) for $B_i = B = \mathrm{diag}(K, -1, -1)$.

When $\sigma = 0$ in Eq. (11), system (6) will degenerate into a deterministic system, namely, system (1). Then we find $\rho(I + B) = (k + 1)^2$, $\lambda_1 = (k + 1)^2$. Since $\rho(I + B) \leq 1$, we have $-2 \leq k \leq 0$. When $\sigma \neq 0$, (in the simulations we let $\sigma = \frac{|k|}{10}$,) system (6) is a stochastic impulsive controlled Lorenz system. When $K = k + \sigma N$, we have $\overline{\rho_1} = E(\rho(I + B)) =$

$|k+1|$, $\overline{\lambda_1} = E\lambda_1 = (k+1)^2 + \sigma^2$. For $\overline{\rho_1} \leq 1$, then $-2 \leq k \leq 0$. When $K = (1+\sigma N)k$, we have $-2 \leq k \leq 0$ and $\overline{\lambda_1} = (k+1)^2 + \sigma^2 k^2$.

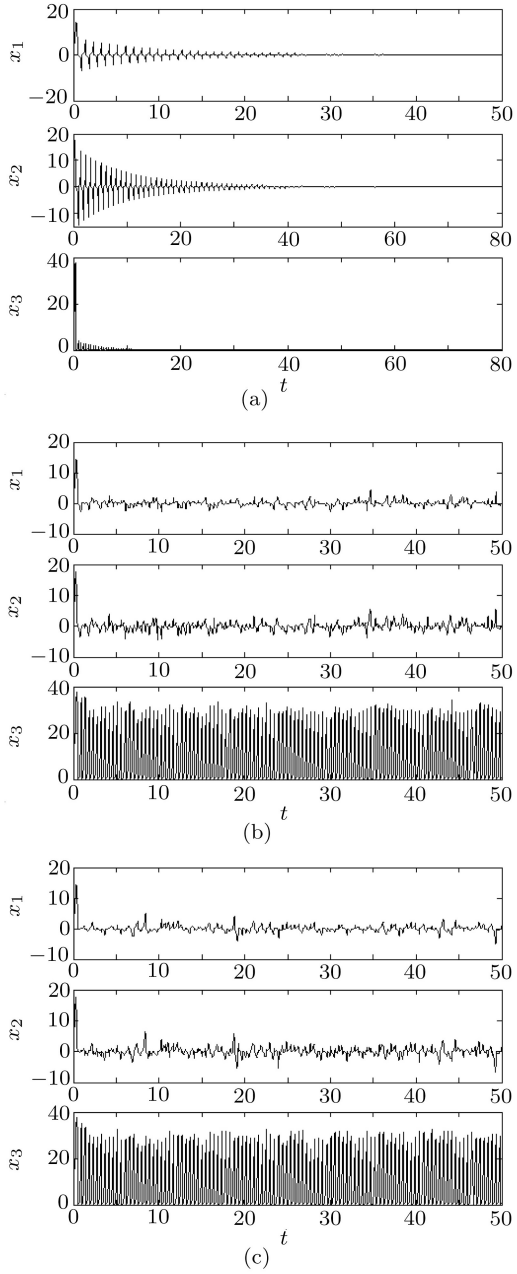


Fig. 5. Simulation results for $B_i = B = \mathrm{diag}(K, -1, -1)$: (a) the stable result of the deterministic system; (b) the unstable result of the stochastic system (with additive noise); (c) the unstable result of the stochastic system (with multiplicative noise).

According to Corollary 1 and Remark 2, we can give the estimate of boundary of stable region for each system with equidistant impulse; see Fig. 1. The curves in the figure are obtained by Eqs. (5) and (9). Obviously, the whole region under each curve is the stable region. We can see that the stable region of each stochastic system is smaller than that of deterministic system, which implies that both additive noise and multiplicative noise will break the stability of the controlled system. In other words, noise will make some stable results, for deterministic system, unstable. In order to validate this opinion, we show the stable region of each system obtained by numerical simulations in Figs. 2–4. The black region in each figure is the stable region. Comparing Fig. 1 with Figs. 2–4, one can find that the result derived by Fig. 1 is effective. Let $k = -1.01$, $\delta = 0.47$, we show the controlled results in Fig. 5. We can see that the system with deterministic impulsive control is asymptotically stable, but the systems with stochastic impulsive control, for additive noise or multiplicative noise, are unstable.

As shown in Figs. 1–4, one can find that the stable region of each system shown in Figs. 2–4 contains the corresponding region shown in Fig. 1, which implies that the estimate of each stable region obtained from Corollary 1 and Remark 2 are effective.

To illustrate our results more sufficient, the control gains $B_i = B = \mathrm{diag}(K, -0.6, -0.6)$, $(i = 1, 2, \ldots)$ are chosen as another example, the meaning of the notation K has been explained above. Similarly, if $K = k + \sigma N$, we have $-2 \leq k \leq 0$ and

$$\bar{\lambda}_1 = \begin{cases} (k+1)^2 + \sigma^2, & (k+1)^2 + \sigma^2 \geq 0.16, \\ 0.16, & \text{otherwise.} \end{cases} \tag{12}$$

If $K = (1+\sigma N)k$, we have $-2 \leq k \leq 0$ and

$$\bar{\lambda}_1 = \begin{cases} (k+1)^2 + \sigma^2 k^2, & (k+1)^2 + \sigma^2 k^2 \geq 0.16, \\ 0.16, & \text{otherwise.} \end{cases} \tag{13}$$

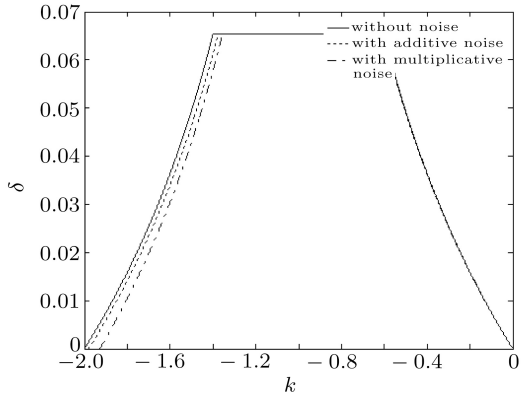


Fig. 6. The estimates of boundaries of stable region for $B_i = B = \mathrm{diag}(K, -0.6, -0.6)$.

Then the estimate of boundary of stable region can be obtained in Fig. 6. The whole region under each curve is the stable region. We can also

see a smaller stable region of each stochastic system, in comparison with the deterministic condition. Let $k = -1.6$, $\delta = 0.01$, we can see the stable results in Fig. 7. While let $k = -1.65$, $\delta = 0.3$, we can see the stable results of deterministic system, and the unstable results of stochastic systems in Fig. 8, these outcomes are consistent with the conclusions derived above.

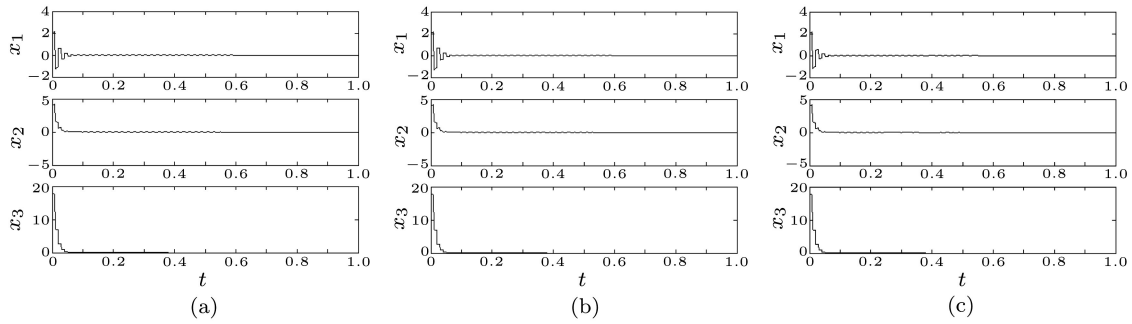


Fig. 7. Simulation results for $B_i = B = \mathrm{diag}(K, -0.6, -0.6)$: (a) the stable result of the deterministic system; (b) the stable result of the stochastic system (with additive noise); (c) the stable result of the stochastic system (with multiplicative noise).

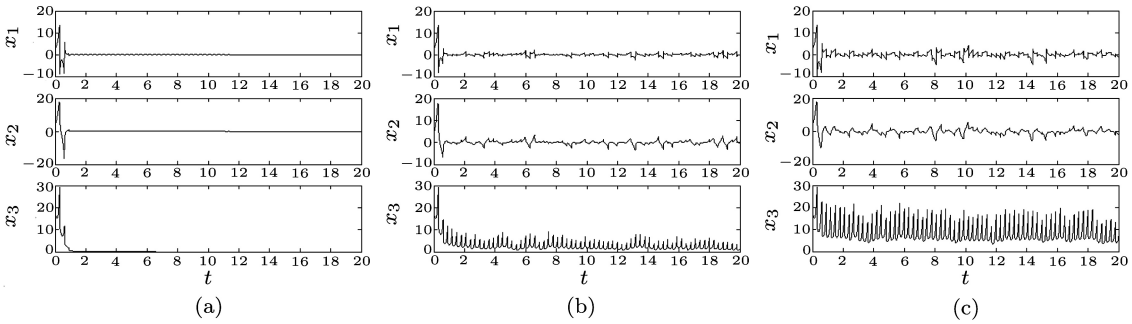


Fig. 8. Simulation results for $B_i = B = \mathrm{diag}(K, -0.6, -0.6)$: (a) the stable result of the deterministic system; (b) the unstable result of the stochastic system (with additive noise); (c) the unstable result of the stochastic system (with multiplicative noise).

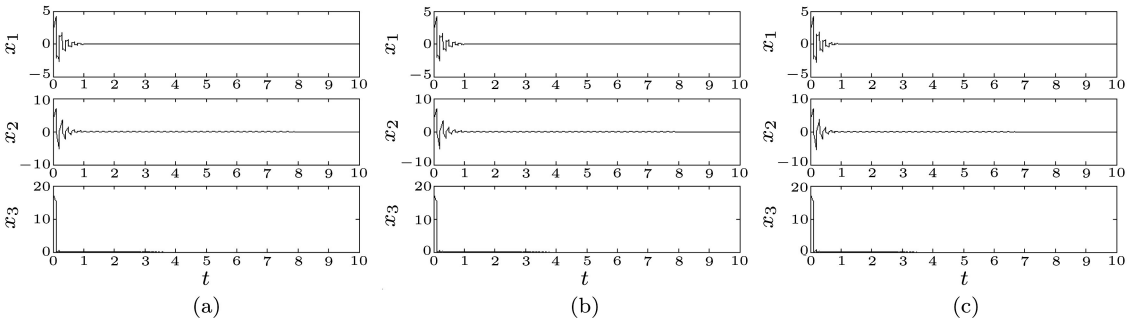


Fig. 9. Simulation results for $B_i = B = \mathrm{diag}(K, -1, -1)$: (a) the stable result of the deterministic system; (b) the stable result of the stochastic system (with additive noise); (c) the stable result of the stochastic system (with multiplicative noise).

Furthermore, according to the comparison of Fig. 1 with Figs. 2–4 we can also derive a conclusion that the conditions given in Theorem 1, Corollary 1, Remark 1 and Remark 2 are sufficient conditions, not necessary conditions. Since there exist some cases which do not satisfy the conditions while the corresponding system is asymptotically stable. Let $k = -1.5$, $\delta = 0.1$, we can easily find that condi-

tions (4) and (8) are not satisfied, but each system is asymptotically stable, as shown in Fig. 9.

5. Conclusions

Based on Refs. [6]–[11], we have obtained some sufficient conditions for asymptotic stability of the Lorenz system with stochastic impulsive control. And we gave the estimate of the upper bound of impulse interval for asymptotically stable control. We also presented the idea that the noise will break the stability of the controlled system. The effectiveness of the given results has been validated by numerical simulations. Although the proposed method is only used for Lorenz system in this article, according to the deduction process, we believe that this scheme can be effectively used for other similar systems, such as Chen system, Lü system and so on.

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