Phase space reconstruction of chaotic dynamical system based on wavelet decomposition*

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In view of the disadvantages of the traditional phase space reconstruction method, this paper presents the method of phase space reconstruction based on the wavelet decomposition and indicates that the wavelet decomposition of chaotic dynamical system is essentially a projection of chaotic attractor on the axes of space opened by the wavelet filter vectors, which corresponds to the time-delayed embedding method of phase space reconstruction proposed by Packard and Takens. The experimental results show that, the structure of dynamical trajectory of chaotic system on the wavelet space is much similar to the original system, and the nonlinear invariants such as correlation dimension, Lyapunov exponent and Kolmogorov entropy are still reserved. It demonstrates that wavelet decomposition is effective for characterizing chaotic dynamical system.

Keywords: chaotic dynamical system, phase space reconstruction, wavelet decomposition

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1. Introduction

Chaotic system can be found widely in nonlinear field. Many applications in chaotic dynamics, chaotic control, chaotic encryption and chaotic prediction are involved with many fields such as physics, biology, medicine, communication and the like.[1] These applications are always started from the study of chaotic time series. The characteristics of the chaotic system are usually characterized by the dynamical invariants such as correlation dimension, Lyapunov exponent and Kolmogorov entropy and the like. These invariants can reflect objectively the intrinsic characteristics of chaotic system. However, one of the indispensable process for the solution of these invariants is the phase space reconstruction for any chaotic system. The purpose of the phase space reconstruction is to construct the system state by using its history and viewing it in a higher dimensional space. A popular method used for phase space reconstruction is known as the time-delayed embedding method proposed by Packard and Takens.[2,3] It is usually the external characteristics of many chaotic systems which are only represented by a time series of a certain single variable. The time-delayed embedding method can extract the nonlinear dynamical characteristics from the system by reconstructing the single time series of chaotic system. In the reconstructed space, each system invariants can be restored or reversed. However, for the time-delayed reconstruction, the reconstruction parameters (time lag τ and embedding dimension m) are difficult to determine for many chaotic system. The improper parameter estimates may make a great impact on system reconstruction due to individual difference in the structure of chaotic system. Therefore, various methods for determining the reconstruction parameters are developed by many researchers,[4−9] for example, the nearest neighbour method, the singular value decompose method, the autocorrelation method and the mutual information method. The most prevalent method for determining the time lag τ is based on the number of sampling points from the beginning to the first zero cross-point of the autocorrelation function of time series.[10] However, for Lorenz system, there is almost not any zero cross point can be obtained from the autocorrelation function. For electroencephalogram (EEG) time series, the first zero cross-point will be found after 4∼8 sampling points. It is obvious that there may be a great difference in reconstruction parameters for different chaotic systems, therefore, it can not be determined by only one method. In this paper, we illustrate the physical essence of wavelet decompo-

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sition of chaotic system from the viewpoint of phase space reconstruction. It is indicated that, the wavelet decomposition of chaotic dynamical system is essentially a projection of chaotic attractor on the axes of the space opened by the wavelet filter vectors, which is consistent with the time-delayed embedding method of phase space reconstruction proposed by Packard and Takens.

2. Phase space reconstruction based on time-delayed embedding method

Assume that a continuous time dynamical system is given as $\mathrm{d}x(t)/\mathrm{d}t = F(x(t))$, $x(t) \in R^n$, it becomes a discrete time system after discretization. $F : R^n \to R^m$, $x(t+1) = f(x(t)), t = 0, 1, 2, \ldots$. Let $x(t+m) = f^m(x(t))$ and $y(t)$ be a component of system state $x(t)$ and $h : R^n \to R$, so as to $y(t) = h(x(t))$. Let $y^m(t) = (y(t+m-1), y(t+m-2), \ldots, y(t))$. If U is a compact manifold, then for $m \geq 2n+1$, $J^m(x(t)) = y^m(t) = (h(f^{m-1}(x(t))), h(f^{m-2}(x(t))), \ldots, h(x(t)))$ is an embedding from U to $J^m(U)$. Generally, there exists a function $g : R^m \to R^m$, which makes $y^m(t+1) = g(y^m(t)) = (y(t+m), y(t+m-1), \ldots, y(t+1))$. Take a note of $y^m(t+1) = J^m(x(t+1)) = J^m(f(x(t)))$, so $J^m(f(x(t))) = g(J^m(x(t)))$. Under the condition that J^m is a homeomorphism, f and g is topological isomorphic, so that dynamical characteristics of f can be reserved in g. It is obvious that, some dynamical invariants of the original system can be computed and analysed by observing a component $y(t)$ of system state $x(t)$.

Let X denote a matrix $X = [X_1 \ \ X_2 \ \ \ldots \ \ X_N]^{\mathrm{T}}$, each row of X denotes a vector of the embedding space, i.e. X_i represents the system state at discrete time point i. For a time series $x = \{x_1, x_2, x_3, \ldots, x_L\}$ with length of L, the phase space will be R^m after time-delayed reconstruction, then, the dynamical trajectory of system in phase space is determined by

$$X_i = (x_i \ \ x_{i+\tau} \ \ \ldots \ \ x_{i+(m-1)\tau}), \tag{1}$$

where $i = 1, 2, 3, \ldots, N$, τ denotes the time lag, m denotes the embedding dimension, and the vector number $N = L - (m-1)\tau$, that is, there are N points in the embedding space. There are a large number of researches on phase space reconstruction, indicating that many chaotic systems, such as Lorenz, Rossler and EEG, are not sensitive to the selection of embedding dimension m, but sensitive to the selection of time lag τ, moreover, the selection of τ is not as simple as some reports, it depends on either the selection of the computing step length or the initial value when we solve the system equations by using Runge–Kutta method. Generally, the smaller the computing step length is, the larger τ should be chosen.

3. Projection of chaotic time series onto the wavelet space

Given a time series $x = \{x_1, x_2, x_3, \ldots, x_N\}$ with length of N, which is sampled from a chaotic system. For discrete wavelet decomposition, and at the scale j, the approximation component a_k^j and the detail component d_k^j of x are given respectively by[11]

$$a_k^j = \sum_n h(n-2k) x_n^{j-1}, \tag{2}$$

$$d_k^j = \sum_n g(n-2k) x_n^{j-1}, \tag{3}$$

where $h(\cdot)$ are lowpass filter coefficients, $g(\cdot)$ are bandpass filter coefficients, $n = 1, 2, \ldots, N$, $k = 1, 2, \ldots, N$. Equations (2) and (3) can be expressed as projection of chaotic attractor on the axes of the wavelet space opened by filter vectors $\boldsymbol{H}$ and $\boldsymbol{G}$ as

$$\begin{aligned} A^j &= H \cdot X^{j-1}, \\ D^j &= G \cdot X^{j-1}, \end{aligned} \tag{4}$$

where the filter vectors $\boldsymbol{H}$ and $\boldsymbol{G}$ are given by

$$\begin{aligned} \boldsymbol{H} &= (h_{-M} h_{-(M-1)} \ldots h_{-1} h_0 h_1 \ldots h_{(M-1)} h_M), \\ \boldsymbol{G} &= (g_{-M} g_{-(M-1)} \cdots g_{-1} g_0 g_1 \cdots g_{(M-1)} g_M), \end{aligned}$$

When we perform the computation of Eqs. (2) and (3), we usually use a special algorithm called as "à trous algorithm",[12] that is, at each scale j, a_k^j or d_k^j is the sum of N dot products between the filter coefficient of $h(\cdot)$ or $g(\cdot)$ and x_n^{j-1}, which is the multiscale approximation component of x at lower scale $j-1$, then, shifted it every time by a time lag $\tau_j = 2^{j-1}$, such that expression (4) can be expressed as the following matrix form:

$$\begin{bmatrix} a_1^j \\ a_2^j \\ a_3^j \\ \vdots \\ a_N^j \end{bmatrix} = \begin{bmatrix} x_1^{j-1} & x_{1+\tau_j}^{j-1} & \cdots & x_{1+(m-1)\tau_j}^{j-1} \\ x_2^{j-1} & x_{2+\tau_j}^{j-1} & \cdots & x_{2+(m-1)\tau_j}^{j-1} \\ x_3^{j-1} & x_{3+\tau_j}^{j-1} & \cdots & x_{3+(m-1)\tau_j}^{j-1} \\ \vdots & \vdots & \cdots & \vdots \\ x_N^{j-1} & x_{N+\tau_j}^{j-1} & \cdots & x_{N+(m-1)\tau_j}^{j-1} \end{bmatrix} \begin{bmatrix} \vdots \\ h_{-1} \\ h_0 \\ h_1 \\ \vdots \\ \vdots \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} d_1^j \\ d_2^j \\ d_3^j \\ \vdots \\ d_N^j \end{bmatrix} = \begin{bmatrix} x_1^{j-1} & x_{1+\tau_j}^{j-1} & \cdots & x_{1+(m-1)\tau_j}^{j-1} \\ x_2^{j-1} & x_{2+\tau_j}^{j-1} & \cdots & x_{2+(m-1)\tau_j}^{j-1} \\ x_3^{j-1} & x_{3+\tau_j}^{j-1} & \cdots & x_{3+(m-1)\tau_j}^{j-1} \\ \vdots & \vdots & \cdots & \vdots \\ x_N^{j-1} & x_{N+\tau_j}^{j-1} & \cdots & x_{N+(m-1)\tau_j}^{j-1} \end{bmatrix} \begin{bmatrix} \vdots \\ g_{-1} \\ g_0 \\ g_1 \\ \vdots \\ \vdots \end{bmatrix}, \tag{6}$$

where $j = 1, 2, \ldots, J$. Then the k-th vector of Eq. (5) or Eq. (6) can be written as

$$X_k = (x_k \quad x_{k+\tau_j} \quad \ldots \quad x_{k+(m-1)\tau_j}). \tag{7}$$

By comparing Eq. (7) with Eq. (1), we can see that each vector of Eq. (7) is actually the phase space reconstruction of the chaotic time sereis, which represents the k-th point of chaotic attractor trajectory in phase space R^m, but the time-lag τ_j is varied with the scale j, and the embedding dimension m is determined by the length of wavelet filter, i.e. $m = 2 * M + 1$. As long as the fractal dimension of dynamical system is not greater than M, then the embedding dimension m can meet the Takens's theorem.[3] However, if the fractal dimension of dynamical system is greater than M, then, the length of wavelet filter must be increased. There are two methods which can be used directly, one of them is to increase finite zeros on the two ends of filter, which has hardly any effect on the result of wavelet decomposition. The other method is to choose a proper wavelet basis function, for example, the cubic B-spline wavelet or Daubechies wavelet and the like.

4. Results and discussion

In this section, we illustrate the phase trajectories of Rössler system and EEG in both the time-delayed embedding space and the wavelet space for the sake of comparison between the two reconstruction methods.

4.1. Dynamical trajectory of the Rössler system

The Rössler system consists of three coupled first order differential equations as

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= -(z + y), \\ \mathrm{d}y/\mathrm{d}t &= x + ay, \\ \mathrm{d}z/\mathrm{d}t &= b + xz - cz. \end{aligned} \tag{8}$$

In the present paper, equation (8) is numerically integrated by the fourth-order Runge–Kutta algorithm with $a = b = 0.2, c = 5$, and computing step length of 0.01. The length of each sample (time series) of (x, y, z) is $N = 40000$, then, reconstructed by Eq. (1) with embedding dimension $m = 8$ and time lag $\tau = 80$. Figure 1 shows the dynamical trajectory (attractor) sections of Rössler system in the X_1–X_2, X_2–X_3 and X_3–X_4 planes of 6-dimensional embedded space.

After making wavelet decomposition of each sample (x, y, z) of Rössler system by Eqs. (2) and (3), the approximation component a_k^j and the detail component d_k^j of each sample can be obtained, that is the projections of Rössler system on the axes of the wavelet space are opened by filter vectors $\boldsymbol{H}$ and $\boldsymbol{G}$ in terms of the scale j. From a_k^j and d_k^j of each sample (x, y, z), we can obtian their dynamical trajectory sections in wavelet space. In the present paper the quadratic B-spline wavelet with 6 scales is chosen. Figure 2 shows the dynamical trajectory sections of Rössler system at scales 2∼4 in the A_2–D_2, A_3–D_3 and A_4–D_4 planes of wavelet space.

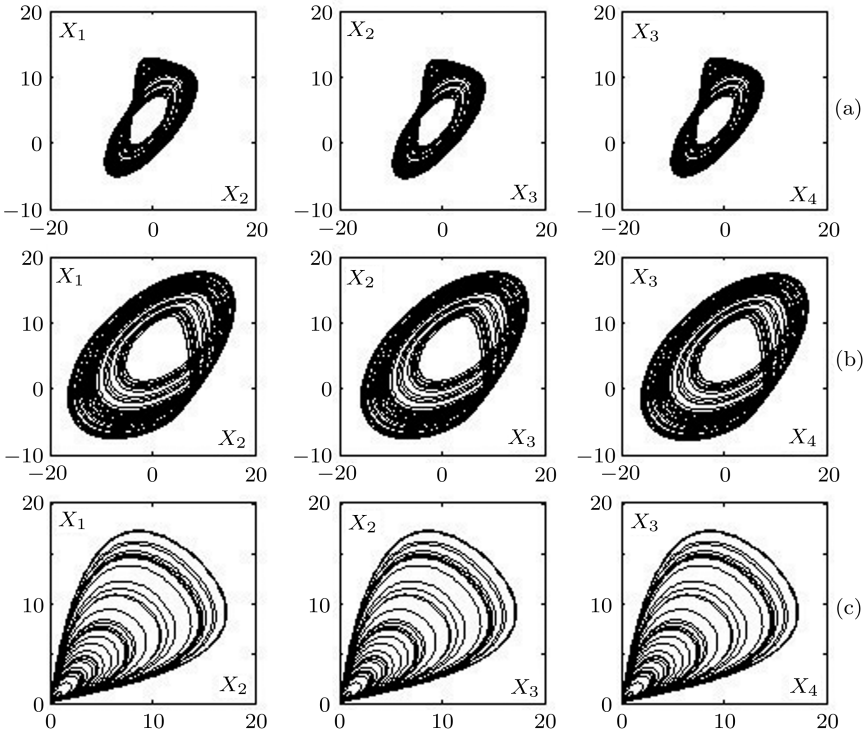


Fig. 1. Dynamical trajectories of Rössler system in the X_1–X_2, X_2–X_3 and X_3–X_4 planes of 6-dimensional embedded space for x (a), y (b) and z (c) variables.

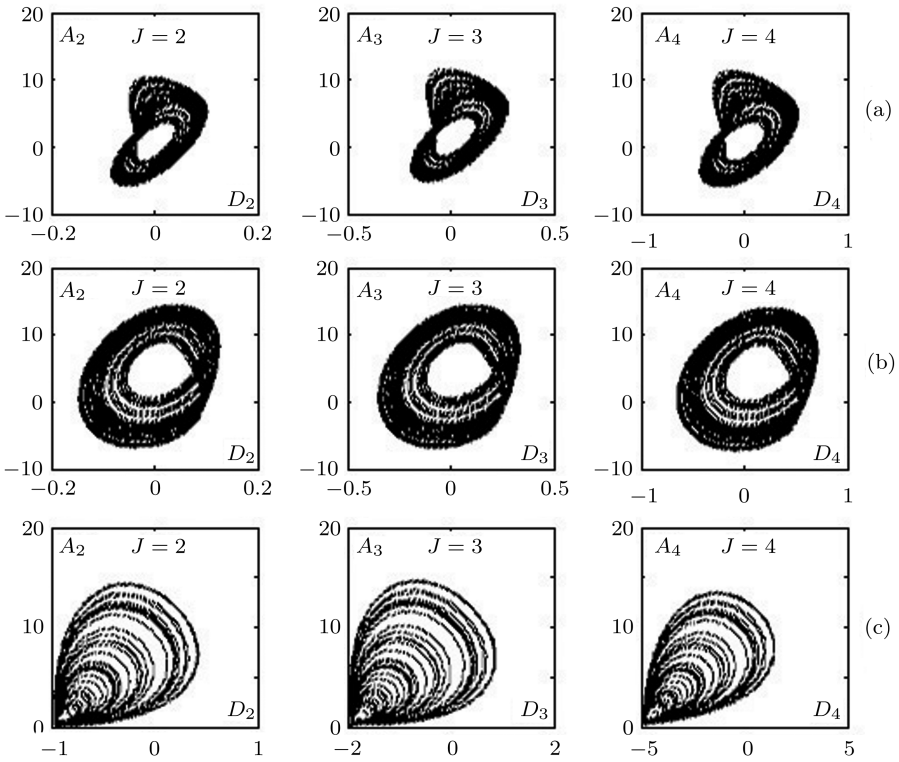


Fig. 2. Dynamical trajectories of Rössler system at the scales $J = 2 \sim 4$ in the A_2–D_2, A_3–D_3 and A_4–D_4 planes of wavelet space for x (a), y (b) and z (c) variables.

4.2. Dynamical trajectory of EEG

In the present work, a total of 200 samples of EEG time series are recorded from epileptic patients with sampling frequency of 173.61 Hz, 12 bits A/D transformation, and sampling duration of 23.6 s, so the time

series length of each sample is 4096. Each sample is then reconstructed by Eq. (1) with embedding dimension $m = 6 \sim 10$ and time lag $\tau = 4 \sim 8$. Figure 3(a) shows the dynamical trajectory (attractor) sections of EEG in the X_1–X_2, X_2–X_3 and X_3–X_4 planes of 6-dimensional embedded space. Figure 3(b) shows the dynamical trajectory sections of EEG at scales 2∼4 in the A_2–D_2, A_3–D_3 and A_4–D_4 planes of wavelet space.

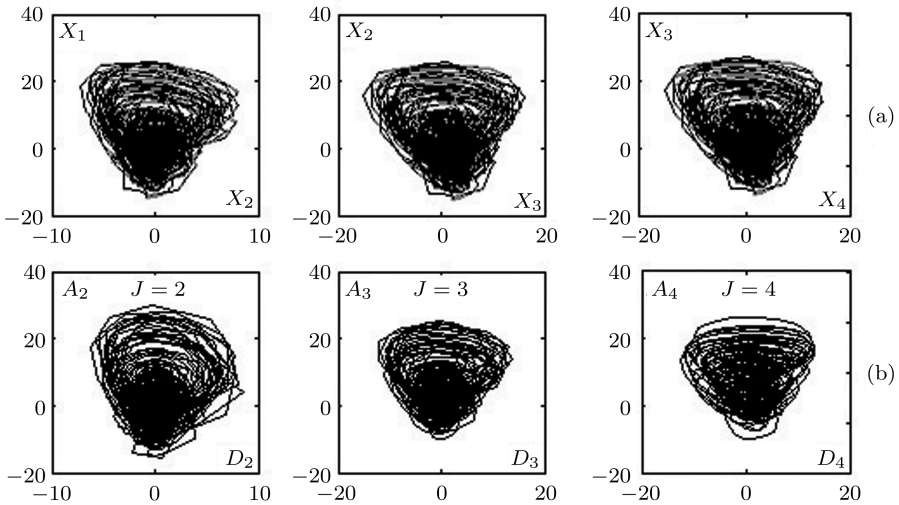


Fig. 3. Dynamical trajectories of EEG: (a) in the X_1–X_2, X_2–X_3 and X_3–X_4 planes of 6-dimensional embedded space; (b) at the scales $J = 2 \sim 4$ in the A_2–D_2, A_3–D_3 and A_4–D_4 planes of wavelet space.

From Figs. 1 and 2 we can see clearly that, the structure of each trajectory in Fig. 1 is much similar to the corresponding trajectory in Fig. 2, it is indicated that the characteristics of chaotic attractor are reserved in the wavelet space. The same results can be seen in Figs. 3(a) and 3(b).

5. Conclusion

As illustrated above, in both time-delayed embedding space and wavelet space, the dynamical trajectory structure of chaotic system is almost identical. It demonstrates that the wavelet space and the embedding space are equivalent in topology. In order to determine the nonlinear invariants of a chaotic system (e.g. Rössler system or EEG), such as correlation dimension and Kolmogorov entropy etc., the most commonly used method is just the space embedding. However, the embedding parameters, for example, the embedding dimension m and the time delay τ, are difficult to determine for most situations, and theoretically, there is not any basis for this solution so far. The wavelet decomposition has the advantage of decomposing the data into different scales, that is, different levels of fineness, such that we can reconstruct phase space at each scale. It is useful to observe the variations of trajectory structure with the scale change. Moreover, from Eq. (4) the wavelet decomposition can be considered as a projection of the chaotic attractor onto the orthogonal spaces opened by filter family, which does not need any additional parameters like the space embedding. Therefore, those methods which combine wavelet decomposition with chaotic theory will be effective for analysing a complex system.

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