Solution and transcritical bifurcation of Burgers equation

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Burgers equation is reduced into a first-order ordinary differential equation by using travelling wave transformation and it has typical bifurcation characteristics. We can obtain many exact solutions of the Burgers equation, discuss its transcritical bifurcation and control dynamical behaviours by extending the stable region. The transcritical bifurcation exists in the $(2+1)$-dimensional Burgers equation.

Keywords: Burgers equation, transcritical bifurcation, exact solution, bifurcation control

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1. Introduction

Burgers equation is a well-know nonlinear equation which has been studied extensively as a model of fluid turbulence, acoustic and shock waves, galaxy formation, and other physical phenomena separately.[1,2] Burgers equation on semiline with flux boundary condition and general initial condition has been solved exactly.[3,4] A general method of solving the Burgers equation with a moving boundary has been introduced.[5] A generalized Burgers equation has been obtained for a far field of weakly nonlinear waves and numerical results of strongly nonlinear waves were achieved.[6] The result from the self-similar reduction of a generalized Burgers equation with nonlinear damping was obtained.[7]

Burgers equation is a simple, but important physical equation.[8−13] Bifurcations of this equation and its derivative equation were less studied. Jordan[8] analysed the exact traveling wave solution for Kuznetsov's equation and determined a bifurcation value of the Mach number. The dimensionless transport speed was used as the bifurcation parameter and the bifurcation diagram of the Kelvin–Voigt versus Burgers system with internal damping was presented in Ref. [9]. Dang-Vu and Delcare[10] applied the numerical method to the solution of the Burgers equation and discussed the existence of a Hopf bifurcation. Detailed studies of bifurcations of Burgers equation are only in the beginning.

With the aid of parameter transformation, Burgers equation can be reduced into a first-order ordinary differential equation with travelling wave solution.[11] For that case, Burgers equation becomes a one-dimensional system with the transcritical bifurcation. On the basis of the concept some partial differential equations can also become one-dimensional systems with typical bifurcation characteristics.[14−16] Some outstanding results of the bifurcation analysis were published in Refs. [17]–[22].

The transcritical bifurcation mentioned above is a common type of bifurcation existing in nonlinear systems. In this paper, we discuss exact solutions and their transcritical bifurcation of the Burgers equation, and the $(2+1)$-dimensional Burgers equation.

2. Burgers equation

2.1. Travelling wave transformation

The Burgers equation with a physical field $u(x,t)$ with two variables x and t can be written as

$$u_t + \gamma u u_x - \nu u_{xx} = 0. \tag{1}$$

Applying a travelling wave transformation

$$u(x,t) = U(\xi),$$
$$\xi = -kx + \lambda t \tag{2}$$

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to Eq. (1) yields

$$\lambda\frac{\mathrm{d}U}{\mathrm{d}\xi}-\gamma kU\frac{\mathrm{d}U}{\mathrm{d}\xi}-\nu k^2\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}=0. \tag{3}$$

After integration, equation (1) becomes

$$\frac{\mathrm{d}U}{\mathrm{d}\xi}=\frac{\lambda}{\nu k^2}U-\frac{\gamma}{2\nu k}U^2+c. \tag{4}$$

When $c=0$, equation (4) is reduced into

$$U'=\mu U-U^2, \tag{5}$$

where

$$U'=\frac{\mathrm{d}U}{\mathrm{d}\varsigma},\quad \varsigma=\frac{\gamma}{2\nu k}\xi,\quad \mu=\frac{2\lambda}{\gamma k}. \tag{6}$$

2.2. Solution of equation

Equation (5) has exact solutions as follows.

(i) When $\mu>0$, the exact solution can be expressed as

$$U=\frac{C\mu\,\mathrm{e}^{\mu\varsigma}}{C\,\mathrm{e}^{\mu\varsigma}-1}. \tag{7}$$

The solution of Eq. (1) is given as

$$u(x,t)=\frac{C(2\lambda/\gamma k)\exp[(\lambda/\nu k^2)(-kx+\lambda t)]}{C\exp[(\lambda/\nu k^2)(-kx+\lambda t)]-1}, \tag{8}$$

and the evolution of solution (8) is shown in Fig. 1.

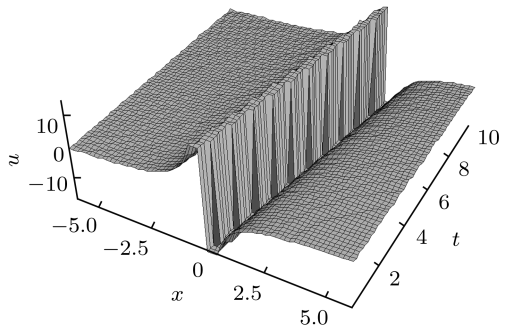


Fig. 1. Structure of solution for $u(x,t)$ expressed by Eq. (8).

(ii) When $\mu<0$, the exact solution can be written as

$$U=\frac{\mu}{1-C\,\mathrm{e}^{-\mu\varsigma}}. \tag{9}$$

The solution of Eq. (1) is given as

$$u(x,t)=\frac{2\lambda/\gamma k}{1-C\exp[-(\lambda/\nu k^2)(-kx+\lambda t)]}, \tag{10}$$

and the evolution of solution (10) is shown in Fig. 2.

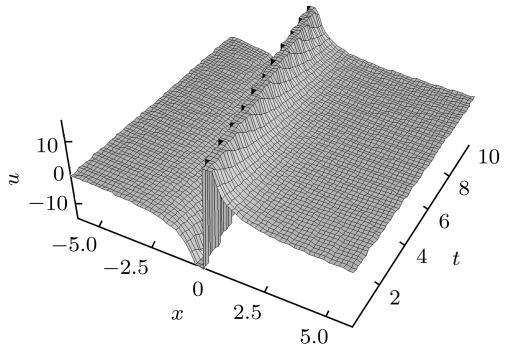


Fig. 2. Structure of solution for $u(x,t)$ expressed by Eq. (10).

(iii) When $\mu=0$, the exact solution is in the form

$$U=\frac{1}{\varsigma+C}. \tag{11}$$

The solution of Eq. (1) is given as

$$u(x,t)=\frac{1}{(\gamma/2\nu k)(-kx+\lambda t)+C}, \tag{12}$$

and the evolution of solution (12) is shown in Fig. 3.

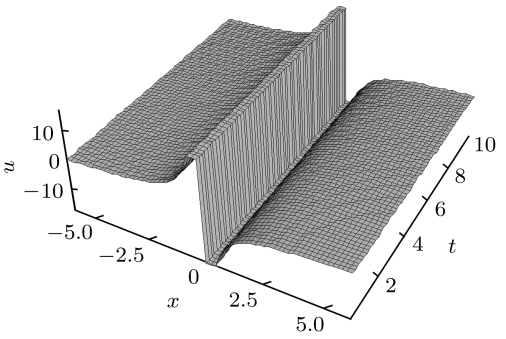


Fig. 3. Structure of solution for $u(x,t)$ expressed by Eq. (12).

3. The $(2+1)$-dimensional Burgers equation

The $(2+1)$-dimensional Burgers equation is written as

$$\begin{aligned}&u_t=uu_y+avu_x+bu_{yy}+abu_{xx},\\&v_y=u_x.\end{aligned} \tag{13}$$

Applying the travelling wave transformation

$$\begin{aligned}&u(x,y,t)=U(\xi),\quad v(x,y,t)=V(\xi),\\&\xi=kx+ly+\lambda t\end{aligned} \tag{14}$$

to Eq. (13) yields

$$\begin{aligned}&\lambda\frac{\mathrm{d}U}{\mathrm{d}\xi}=lU\ \frac{\mathrm{d}U}{\mathrm{d}\xi}+akV\frac{\mathrm{d}U}{\mathrm{d}\xi}+(bl^2+abk^2)\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2},\\&l\frac{\mathrm{d}V}{\mathrm{d}\xi}=k\frac{\mathrm{d}U}{\mathrm{d}\xi},\end{aligned} \tag{15}$$

after integration, equation (15) becomes

$$\frac{\mathrm{d}U}{\mathrm{d}\xi} = \frac{\lambda}{bl^2 + abk^2}U - \frac{l^2 + ak^2}{2l(bl^2 + abk^2)}U^2 + C. \quad (16)$$

When $C = 0$, equation (15) is reduced into

$$U' = \mu U - U^2,$$

which is the same as Eq. (5), where

$$U' = \frac{\mathrm{d}U}{\mathrm{d}\varsigma}, \ \varsigma = \frac{l^2 + ak^2}{2l(bl^2 + abk^2)}\xi, \ \mu = \frac{2\lambda l}{l^2 + ak^2}. \quad (17)$$

Hence the exact solution of the $(2 + 1)$-dimensional Burgers equation can be obtained.

4. Transcritical bifurcation

Equation (5) may be rewritten as

$$\dot{u} = f(u; \mu) = \mu u - u^2. \quad (18)$$

The one-dimensional system (18) shows a typical bifurcation which is called the transcritical bifurcation.[14] It has two equilibria: $u_1^* = 0$ and $u_2^* = \mu$. The Jacobian of the system is

$$J = \left.\frac{\partial f(u; \mu)}{\partial u}\right|_{u=0} = \mu. \quad (19)$$

For $\mu < \mu_0 = 0$, the equilibrium $u_1^* = 0$ is stable, and for $\mu > \mu_0 = 0$, it is unstable. Hence, the equilibrium point $(u_1^*, \mu_0) = (0, 0)$ is a bifurcation point (see Fig. 4). In the same way, equilibrium point (u_2^*, μ_0) is another bifurcation point.

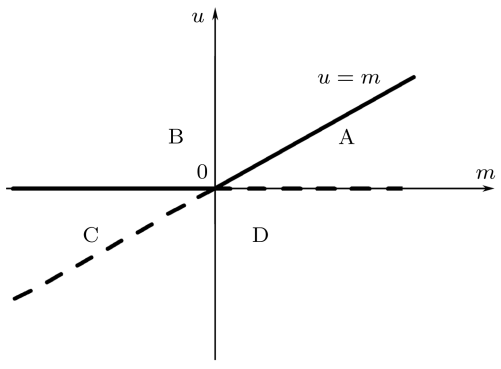


Fig. 4. Transcritical bifurcation.

The lines $u = 0$ and $u = \mu$ divide the plane μ–u into four parts A, B, C and D. In the regions A, B and C, the curves of u are shown in Figs. 5(a), 5(b) and 5(c) respectively. The solutions of Eq. (18) are stable in the region B. In the region D, the solutions of Eq. (18) are unstable.

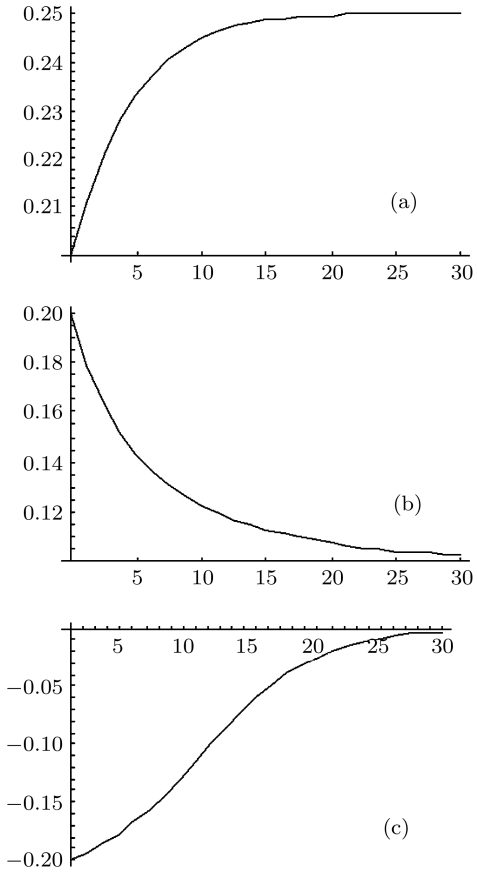


Fig. 5. Curves of u in regions A (a), B (b) and C (c).

5. Bifurcation control

Finally we discuss the control of the transcritical bifurcation for Burgers equation. To introduce the linear controller $v(u, \dot{u})$ into Eq. (18), we obtain the controlled system in the form of

$$\dot{u} = \mu u - u^2 + v(u, \dot{u}). \quad (20)$$

The controller is assumed to be in the form of

$$v = e(\mu - u), \quad (21)$$

where e is the control gain. Equation (20) becomes the following form:

$$\dot{u} = \mu u - u^2 + e(\mu - u) = (u + e)(\mu - u). \quad (22)$$

As the control gain changes, the region of stable solutions changes (Fig. 6). The linear controller does not change the characteristics of the original system but it can extend the stable region B. Similarly, we can also control the pitchfork and saddle-node bifurcations.

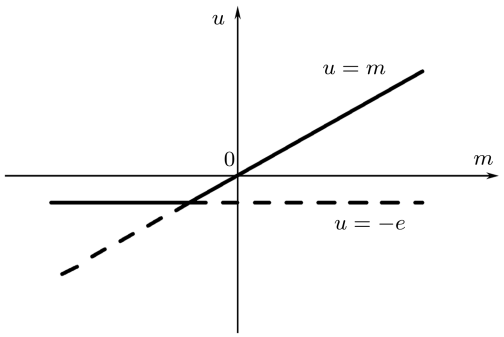


Fig. 6. Controlled transcritical bifurcation.

6. Conclusions

A method of analysing the bifurcations of some partial differential equations is proposed. The transcritical bifurcation of Burgers equation is discussed first. The transcritical bifurcation of Burgers equation can be controlled by using a linear controller. The linear feedback control is employed to move the bifurcation point. Hence the stable region of solutions in the parameter plane is changed. It is found that the pitchfork and saddle-node bifurcations exist in nonlinear systems represented by partial differential equations.

References

[1] Foursov M V 2000 *Phys. Lett.* A **272** 57

[2] Gear C W, Li J and Kevrekidis I G 2003 *Phys. Lett.* A **316** 190

[3] Calogero F and Lillo S D 1991 *J. Math. Phys.* **32** 99

[4] Joseph K T and Sachdev P L 1993 *Int. J. Nonlinear Mech.* **28** 627

[5] Biondini G and Lillos D 2001 *Phys. Lett.* A **279** 194

[6] Yoshinori Inoue, Noriko Kubo and Daisuke Mutou 2004 *Fluid Dyn. Res.* **34** 99

[7] Srinivasa R C, Sachdev P L and Ramaswamy M 2003 *Nonlinear Anal. Real World Appl.* **4** 723

[8] Marynowski K and Kapitaniak T 2002 *Int. J. Nonlinear Mech.* **37** 1147

[9] Jordan P M 2004 *Phys. Lett.* A **326** 77

[10] Dang-Vu H and Delcarte C 1995 *Appl. Math. Comput.* **73** 99

[11] Feng Z S 2002 *Phys. Lett.* A **293** 57

[12] Bigolin F and Cassano F S 2010 *J. Math. Anal. Appl.* **366** 561

[13] Goldys B and Neklyudov M 2009 *J. Math. Anal. Appl.* **354** 397

[14] Chen G, Moiola J L and Wang H O 2000 *Int. J. Bifur. Chaos* **10** 511

[15] Tang J S, Zhao M H, Han F and Fu W B 2010 *Mod. Phys. Lett.* B **24** 567

[16] Tang J S, Han F, Xiao H and Wu X 2009 *Nonlinear Anal.* **71** 2491

[17] Zou H L, Xu J X and Jiang J 2008 *Chin. Phys.* B **17** 117

[18] Zhang Q C, Wang W and Liu F H 2008 *Chin. Phys.* B **17** 4123

[19] Wang Z and Mao P W 2008 *J. Dyn. Control* **6** 16 (in Chinese)

[20] Qian C Z 2008 *J. Dyn. Control* **6** 126 (in Chinese)

[21] Zhao D M and Zhang Q C 2010 *Chin. Phys.* B **19** 030518

[22] Li Q H, Tan J Y, Xi J Z and Ding X L 2010 *J. Dyn. Control* **8** 39 (in Chinese)