The universal sound velocity formula for the strongly interacting unitary Fermi gas*

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Due to the scale invariance, the thermodynamic laws of strongly interacting limit unitary Fermi gas can be similar to those of non-interacting ideal gas. For example, the virial theorem between pressure and energy density of the ideal gas $P = 2E/3V$ is still satisfied by the unitary Fermi gas. This paper analyses the sound velocity of unitary Fermi gases with the quasi-linear approximation. For comparison, the sound velocities for the ideal Boltzmann, Bose and Fermi gas are also given. Quite interestingly, the sound velocity formula for the ideal non-interacting gas is found to be satisfied by the unitary Fermi gas in different temperature regions.

Keywords: unitary Fermi gas, sound velocity, universal thermodynamics

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1. Introduction

In recent years, strongly interacting fermion physics has attracted much attention in the quantum many-body field.[1] Through Feshbach resonance,[2] one can control the scattering length a or interaction strength of atomic fermions by tuning the external magnetic field. At the resonance point, the scattering length a can be $\pm\infty$ and the scattering between atomic fermions can have a zero-energy bound state. The infinite scattering length system can exhibit the Bardeen–Cooper–Schrieffer (BCS) regime to Bose–Einstein condensation (BEC) (BCS-BEC crossover).[3,4] The scattering cross section will be saturated as $\sigma \sim 4\pi/k^2$ (where k is the relative momentum between atomic fermions) at the resonance due to the fundamental unitary property limit. The relevant thermodynamics of infinite scattering length fermions system is also called the unitary Fermi gas physics, which attracts much attention in the literature.[5,6]

For the dilute Fermi gas system, the thermodynamical properties are determined by the two-body scattering length a, the particle number n, and temperature T. In the unitary limit with $a = \pm\infty$, the divergent scattering length should drop out in the thermodynamical quantities; i.e., the thermodynamical quantities of the unitary Fermi system are universal.[7−9]

For example at $T = 0$, the ground state energy per particle of unitary Fermi gas can be proportional to the Fermi kinetic energy through $E/N = \xi^{3/5}\varepsilon_{\rm f}$ (where $\varepsilon_{\rm f} = k_{\rm f}^2/(2m)$ with m being the bare fermion mass and $k_{\rm f}$ being the Fermi momentum). The fundamental universal coefficient ξ has excited many theoretical and experimental efforts in recent years.[1,10−16] The finite temperature thermodynamics of unitary Fermi gas is as intriguing as the ground state energy. Several Monte Carlo calculations have been done on unitary Fermi gas of finite temperature.[17−19]

Furthermore, it is pointed out that the Virial theorem for the ideal gas (i.e., $P = \frac{2}{3}E/V$) is still satisfied by the strongly interacting limit system.[8,10] This can be attributed to the scale invariance. One can naturally worry about whether if other thermodynamic formulae of ideal gas are satisfied by the unitary Fermi gas.

In exploring the thermodynamic properties of dilute atomic system, the sound velocity is an important characteristic quantity. It can be experimentally measured and is closely related to the equation of state (EOS) or pressure. In this work, we will analyse the sound velocity formula of the strongly interacting limit unitary Fermi gas. Within EOS obtained in the quasi-linear approximation method, a very simple formula of the sound velocity is given for the unitary Fermi

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gas. Especially, we find that the sound velocity formula is similar to those of the ideal non-interacting gases.

This article is formed as follows. In Section 2, we discuss the derivation method of the velocity formulae in terms of of Jacobian variable. In this section, the sound velocity expressions for the classical ideal Boltzmann gas, the ideal quantum Bose and Fermi gases are given, respectively. In Section 3, the sound velocity for the unitary Fermi gas is calculated from the EOS given by the quasi-linear approximation formalism. The numerical results are given in Section 4. Section 5 is presented for the conclusion remarks.

2. Sound velocity formulae for the ideal gases

According to statistical mechanics, the sound velocity propagating in the medium is determined by the equilibrium EOS. As analysed first by Laplace, the propagation of sound wave should be considered as an adiabatic process, i.e., the sound velocity a is given by[20,21]

$$a^2 = \left(\frac{\partial P}{\partial \rho}\right)_S = \frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_S, \tag{1}$$

where P is pressure, ρ is mass density $\rho = mn$ of the medium and n is the particle number density $n = N/V$. From Eq. (1), one can see that the central task for determining the sound velocity is to give the derivative of pressure P according to the particle number density n for fixed total entropy S. It is not a trivial task for performing the relevant derivation for the sound velocity. The pressure P and the particle density n are intensive properties. Oppositely, the total entropy S is an extensive quantity.

The properties of Jacobian formulae for the multiple variables can be very useful for performing the derivation conveniently. For example, the partial derivative in the above equation can be rewritten as

$$\left(\frac{\partial P}{\partial n}\right)_S = \left(\frac{\partial P}{\partial n}\right)_T - \left(\frac{\partial P}{\partial T}\right)_n \left(\frac{\partial (s/n)}{\partial n}\right)_T \Big/ \left(\frac{\partial (s/n)}{\partial T}\right)_n. \tag{2}$$

Through the above equation, we have changed the calculation with the extensive quantity S into that with intensive one $s = S/V$ and $n = N/V$. Therefore with the intensive quantities, the alternative formalism Eq. (2) contributes to give the sound velocity. The following discussion is based on this formalism.

The natural units $k_{\rm B} = \hbar = 1$ are used throughout this paper.

2.1. Sound velocity of classical ideal Boltzmann gas

For completeness, let us rederive the sound velocity of classical ideal Boltzmann gas in terms of statistical viewpoint.

For the Boltzmann gas, the pressure and entropy density are, respectively

$$P = nT, \tag{3}$$

$$s = \frac{S}{V} = n\left\{\ln\left[\frac{1}{n}\left(\frac{mT}{2\pi}\right)^{3/2}\right] + \frac{5}{2}\right\}. \tag{4}$$

With

$$\left(\frac{\partial P}{\partial n}\right)_T = T, \tag{5}$$

$$\left(\frac{\partial P}{\partial T}\right)_n = n, \tag{6}$$

$$\left(\frac{\partial (s/n)}{\partial n}\right)_T = -n^{-1}, \tag{7}$$

$$\left(\frac{\partial (s/n)}{\partial T}\right)_n = \frac{3}{2}T^{-1}, \tag{8}$$

and from Eq. (2), we can obtain

$$a = \sqrt{5T/3m} = \sqrt{\frac{5}{3}\frac{P}{mn}}. \tag{9}$$

The result equation (9) is consistent with that of thermodynamics.[21]

2.2. Ideal Bose gas occasion

In this subsection, we will discuss the sound velocity of ideal quantum Bose gas.

It is well known that the ideal Bose gas will exhibit (BEC) phenomenon below the transition temperature $\left(T_{\rm C} = \frac{2\pi}{m}\left(\frac{n}{\zeta[\frac{3}{2}]}\right)^{2/3}\right)$. The sound velocity of ideal Bose gas should be discussed for two cases, i.e., the regions for $T < T_{\rm C}$ and $T > T_{\rm C}$ should be analysed separately.

For $T > T_{\rm C}$ occasion, the pressure, entropy density and particle density are given by[21]

$$\begin{aligned} P &= \frac{T}{\lambda^3} g_{5/2}(z), \\ s &= n\left[\frac{5}{2}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \ln z\right], \\ n &= \frac{1}{\lambda^3} g_{3/2}(z). \end{aligned} \tag{10}$$

In the above equations, $\lambda = \sqrt{2\pi/mT}$ is the thermal de Broglie wavelength. The Bose integral functions $g_\nu(z)$ are defined by

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}{\rm d}x}{z^{-1}{\rm e}^x - 1}.$$

Here, $\Gamma(\nu)$ is the gamma function, and $z = {\rm e}^{\mu/T}$ is the fugacity. From the properties $z\partial/\partial z[g_\nu(z)] = g_{\nu-1}(z)$ of the the Bose integral functions, we can verify

$$\left(\frac{\partial P}{\partial n}\right)_T = \frac{nT}{z}\left(\frac{\partial z}{\partial n}\right)_T, \tag{11}$$

$$\left(\frac{\partial P}{\partial T}\right)_n = \frac{5}{2}\frac{1}{\lambda^3} g_{5/2}(z) + \frac{nT}{z}\left(\frac{\partial z}{\partial T}\right)_n, \tag{12}$$

$$\left(\frac{\partial (s/n)}{\partial n}\right)_T = \left[-\frac{5}{2n}\frac{g_{5/2}(z)}{g_{3/2}(z)} + \frac{3}{2z}\left(\frac{\partial z}{\partial n}\right)_T\right], \tag{13}$$

$$\left(\frac{\partial (s/n)}{\partial T}\right)_n = \frac{3}{2}\left[\frac{5}{2T}\frac{g_{5/2}(z)}{g_{3/2}(z)} + \frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_n\right]. \tag{14}$$

Substituting them into Eq. (2), one can have the final result of sound velocity for $T > T_{\rm C}$

$$a = \sqrt{\frac{5}{3}\frac{P}{mn}}. \tag{15}$$

Even with the quantum degeneracy effect, the formalism is quite similar to that of classical ideal Boltzmann gas.

Now let us further analyse the sound velocity for $T < T_{\rm C}$. For the low temperature region with the BEC, the system chemical potential vanishes. The fugacity is $z = {\rm e}^{\mu/T} = 1$, therefore the pressure and entropy density are respectively given by

$$P = \frac{T}{\lambda^3} g_{5/2}(1), \tag{16}$$

$$s = \frac{S}{V} = \frac{5}{2}\frac{1}{\lambda^3} g_{5/2}(1). \tag{17}$$

With

$$\left(\frac{\partial P}{\partial n}\right)_T = 0, \tag{18}$$

$$\left(\frac{\partial P}{\partial T}\right)_n = \frac{5}{2}\frac{1}{\lambda^3} g_{5/2}(1), \tag{19}$$

$$\left(\frac{\partial (s/n)}{\partial n}\right)_T = -\frac{5}{2}\frac{1}{n^2\lambda^3} g_{5/2}(1), \tag{20}$$

$$\left(\frac{\partial (s/n)}{\partial T}\right)_n = \frac{15}{4}\frac{1}{n\lambda^3 T} g_{5/2}(1), \tag{21}$$

one can have the expression for $T < T_{\rm C}$

$$a = \sqrt{\frac{5}{3}\frac{T}{mn\lambda^3} g_{5/2}(1)} = \sqrt{\frac{5}{3}\frac{P}{mn}}. \tag{22}$$

Therefore, in the whole temperature region for $T < T_{\rm C}$ and $T > T_{\rm C}$, the sound velocity expressions (15) and (22) according to P and n are consistent with Eq. (9) of classical ideal Boltzmann gas.

2.3. Sound velocity of ideal Fermi gas

Analogous to the discussion of the ideal Bose gas for $T > T_{\rm C}$, we analyse the sound velocity of ideal Fermi gas. For the ideal Fermi gas, the expressions for pressure, entropy density and particle number density can be given in terms of the fugacity $z = {\rm e}^{\mu/T}$[21]

$$P = \frac{2T}{\lambda^3} f_{5/2}(z), \tag{23}$$

$$s = \frac{S}{V} = n\left[\frac{5}{2}\frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z\right], \tag{24}$$

$$n = \frac{2}{\lambda^3} f_{3/2}(z). \tag{25}$$

In the above equations, the functions

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}{\rm d}x}{z^{-1}{\rm e}^x + 1}$$

are the Fermi integrals. They have the important properties $z\partial/\partial z[f_\nu(z)] = f_{\nu-1}(z)$.

With the similar derivation procedure for the ideal Bose gas of $T > T_{\rm C}$, we can obtain the final expression for the sound velocity of ideal Fermi gas

$$a = \sqrt{\frac{10T}{3m}\frac{f_{5/2}(z)}{f_{3/2}(z)}} = \sqrt{\frac{5}{3}\frac{P}{mn}}. \tag{26}$$

From Eqs. (9), (15), (22) and (26), one can see that the sound velocity has the unique expression either for quantum or classical ideal gas.

3. Sound velocity for unitary Fermi Gas

We now turn to the discussion of the sound velocity for the strongly interacting limit unitary Fermi gas.

In addressing the unitary fermion thermodynamics, the central task is how to include the nonlinear quantum fluctuation/correlations contribution, which is usually considered to be very challenging.

In our previous works, we have attempted a quasi-linear approximation method to obtain the thermodynamics of the unitary Fermi gas.[10,22−26] In the developed analytical approach, the implicit variable effective chemical potential is introduced to characterize the significant nonlinear fluctuation/correlation effects. The obtained pressure P and entropy per particle s/n can be presented as

$$P = \frac{2T}{\lambda^3}\left(f_{5/2}(z') - \frac{f_{3/2}^2(z')}{2f_{1/2}(z')} + \frac{f_{3/2}^3(z')f_{-1/2}(z')}{2f_{1/2}^3(z')}\right), \tag{27}$$

$$\frac{s}{n} = \frac{5}{2}\frac{f_{5/2}(z')}{f_{3/2}(z')} - \ln z' + \frac{3f_{-1/2}(z')f_{3/2}^2(z')}{4f_{1/2}^3(z')} - \frac{f_{3/2}(z')}{4f_{1/2}(z')}, \tag{28}$$

where the function

$$f_\nu(z') = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}\mathrm{d}x}{z'^{-1}\mathrm{e}^x + 1},$$

$z' = \mathrm{e}^{\mu^*/T}$ is the *effective fugacity*, with which the particle number density is expressed as

$$n = \frac{2}{\lambda^3}f_{3/2}(z'), \tag{29}$$

which is quite similar to that of ideal Fermi gas. In doing the numerical calculation, the effective chemical potential is determined by the particle number density n according to Eq. (29).

In order to simplify the partial derivative calculation in deriving the sound velocity, let us further introduce an intermediate variable P^* according to

$$P^* = f_{5/2}(z') - \frac{f_{3/2}^2(z')}{2f_{1/2}(z')} + \frac{f_{3/2}^3(z')f_{-1/2}(z')}{2f_{1/2}^3(z')}, \tag{30}$$

with which the pressure or EOS can be reexpressed as

$$P = \frac{2T}{\lambda^3}P^*. \tag{31}$$

Therefore, we can obtain

$$\left(\frac{\partial P}{\partial T}\right)_n = \frac{5}{\lambda^3}P^* + \frac{2T}{\lambda^3}\frac{\mathrm{d}P^*}{\mathrm{d}z'}\left(\frac{\partial z'}{\partial T}\right)_n, \tag{32}$$

$$\left(\frac{\partial P}{\partial n}\right)_T = \frac{2T}{\lambda^3}\frac{\mathrm{d}P^*}{\mathrm{d}z'}\left(\frac{\partial z'}{\partial n}\right)_T, \tag{33}$$

$$\left(\frac{\partial (s/n)}{\partial T}\right)_n = \frac{\mathrm{d}(s/n)}{\mathrm{d}z'}\left(\frac{\partial z'}{\partial T}\right)_n, \tag{34}$$

$$\left(\frac{\partial (s/n)}{\partial n}\right)_T = \frac{\mathrm{d}(s/n)}{\mathrm{d}z'}\left(\frac{\partial z'}{\partial n}\right)_T. \tag{35}$$

Substituting them into Eq. (2), one can have

$$\left(\frac{\partial P}{\partial n}\right)_S = -\frac{5}{\lambda^3}P^*\left(\frac{\partial z'}{\partial n}\right)_T\left(\frac{\partial T}{\partial z'}\right)_n. \tag{36}$$

Furthermore, it is easy to verify

$$\left(\frac{\partial z'}{\partial n}\right)_T\left(\frac{\partial T}{\partial z'}\right)_n\left(\frac{\partial n}{\partial T}\right)_{z'} = -1, \tag{37}$$

from which the sound velocity of unitary Fermi gas is calculated to be

$$a = \sqrt{\left(\frac{\partial P}{\partial n}\right)_S} = \sqrt{\frac{10}{3}\frac{T}{n\lambda^3}P^*} = \sqrt{\frac{5}{3}\frac{P}{mn}}. \tag{38}$$

It is very interesting that the expression of sound velocity for the unitary Fermi gas according to pressure P and particle density n is similar to that of ideal gas. For the strongly interacting Fermi gas the dynamical and quantum fluctuation/correlation information is implicitly included in the pressure Eq. (27).

4. Numerical results and discussion

With the analytical expressions for the classical, quantum ideal gases and unitary Fermi gas, we will give the numerical results.

For comparison, all the numerical results are rescaled and shown in a single figure. The temperatures are all rescaled by the Fermi characteristic temperature or the phase transition temperature T_C. The sound velocities are all rescaled with the Fermi velocity $v_\mathrm{f} = (3\pi^2 n)^{1/3}/m$, which is determined by the particle number density n and particle mass m.

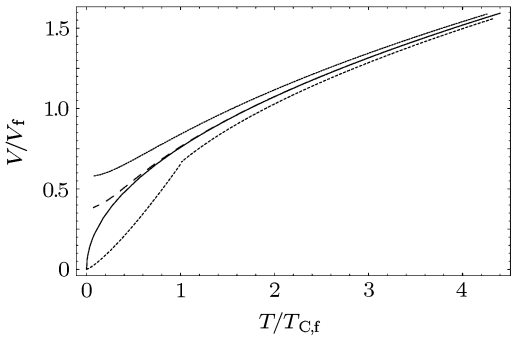


Fig. 1. The rescaled velocity versus temperature. The solid curve denotes the case for the classical Boltzmann gas. The long-dashed curve represents the unitary Fermi gas while the short-dashed curve above the solid one denotes that for the ideal Fermi gas. The short-dashed one below the solid denotes that for the ideal Bose gas.

As we can see in Fig. 1, the sound velocities of quantum gases increase with the rescaled temperature. In the weak degenerate Boltzmann region, they all approach that of classical Boltzmann gas as expected. In the low temperature region, the sound velocities of unitary Fermi gas and ideal Fermi gas will not vanish at zero temperature due to the Pauli exclusion principle. For ideal Bose gas, there exists the BEC in the low temperature region. It is BEC that makes the sound velocity curve of Bose gas versus temperature is quite different from the sound velocity curves of other occasions.

5. Conclusion

Summarizing, we have discussed the sound velocity for the unitary Fermi gas. With EOS obtained in the framework of quasi-linear approximation, we have given the explicit expression for the sound velocity. For comparison, we have also investigated the sound velocity expressions for the ideal quantum gases and classical Boltzmann gas.

It is found that the sound velocity formula (38) for the unitary Fermi gas is similar to that of ideal gas $a = \sqrt{\frac{5}{3}P/mn}$ in different temperature regions. This interesting characteristic can be rooted in the scale invariance. It can contribute to the further understanding for the relevant universal thermodynamic properties of unitary Fermi gas.

Note added: a recent updated work based on the thermodynamic arguments also found similar characteristics for the sound velocity of unitary Fermi gas.[27]

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