Temperature and thermodynamic geometry of the Kerr–Sen black hole*

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This paper studies the thermodynamic properties of the Kerr–Sen black hole from the viewpoint of geometry. It calculates the temperature and heat capacity of the black hole, Weinhold metric and Ruppeiner metric are also obtained respectively. It finds that they are both curved and the curvature scalar of Weinhold curvature implies no information about the phase transition while the Ruppeiner one does. But they both carry no information about the second-order phase transition point reproduced from the capacity. Besides, the Legendre invariant metric of the Kerr–Sen black hole has been discussed and its scalar curvature gives the information about the second-order phase transition point.

Keywords: black hole, thermodynamic geometry, phase transition

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1. Introduction

In 1972, Bekenstein proposed that a black hole is a thermodynamic system and the surface gravity corresponds to its temperature whereas the horizon area corresponds to its entropy.[1,2] Two years later, Hawking made a striking discovery that a classical black hole emits thermal spectrum, which leads to a disturbing and difficult problem about information conservation during black hole evapouration.[3] Based on their works, the four laws of black hole thermodynamics have been established, Hawking radiation from all kinds of black holes has also been studied, it is still one of the focuses in theoretical physics today.[4−8] In 1975, Weinhold proposed a geometrical way of studying thermodynamics and gave a Riemannian metric defined as the second derivatives of internal energy,[9] but his metric had not much physical meanings. As a modification, Ruppeiner defined the line element of the Riemannian metric as the second derivative of the entropy (here, the entropy is a function of the internal energy and other extensive variables of a thermodynamic system).[10] Based on Weinhold's and Ruppeiner's theories, there were a lot of geometrical descriptions about the equilibrium thermodynamics. But studies have found that Weinhold's and Ruppeiner's thermodynamic metrics are not invariant under Legendre transformations.[11,12] In recent years, Quevedo *et al.* used a thermodynamic metric which is invariant with respect to Legendre transformations and determines the geometry of the space of equilibrium states.[13−15] Considering the Legendre invariant, they presented a unified geometry where the metric structure can give a well description of various types of black hole thermodynamics. An aim of the application of different thermodynamic geometries is to describe phase transitions in terms of curvature singularities. For a thermodynamic system, it is quite interesting to investigate the corresponding relationship between the curvature of Weinhold metric, Ruppeiner metric and the Legendre invariant metric and the phase transitions. In fact the above viewpoints have been applied to various black holes.[16−30] Even though, it is widely believed that the thermodynamic geometry of a black hole is still a most fascinating and unresolved subject at present.

In this paper, we study the thermodynamic properties of the Kerr–Sen black hole from the viewpoint of geometry and calculate the corresponding curvature scalars in Weinhold metric and Ruppeiner metric respectively. Moreover, the Legendre invariant metric of the black hole has also been discussed. We find out the relationship between the curvature of those metrics and the phase transitions. Throughout the paper, the units $G = c = \hbar = k_B = 1$ are used.

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2. Temperature and heat capacity of the Kerr–Sen black hole

In the Boyer–Lindquist coordinates the line element of the Kerr–Sen black hole can be written as[31,32]

$$\begin{aligned} \mathrm{d}s^2 = & -\frac{\Delta}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right)^2 \\ & + \frac{\sin^2\theta}{\Sigma}\left[a\mathrm{d}t - \left(\Sigma + a^2\sin^2\theta\right)\mathrm{d}\phi\right]^2 \\ & + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right), \end{aligned} \tag{1}$$

where

$$\Delta = r^2 + 2\left(b - M\right)r + a^2 = \left(r - r_+\right)\left(r - r_-\right), \tag{2}$$

$$\Sigma = r^2 + 2br + a^2\cos^2\theta, \tag{3}$$

the specific angular momentum a and the twist parameter b are respectively given by

$$a = J/M, \quad b = \frac{Q^2}{2M}, \tag{4}$$

where M, Q and J are the mass, electric charge and angular momentum of the Kerr–Sen black hole respectively. The event horizon is located at

$$r_\pm = M - b \pm \sqrt{\left(M - b\right)^2 - a^2}, \tag{5}$$

it can be also expressed as

$$r_\pm = M - \frac{Q^2}{2M} \pm \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - \frac{J^2}{M^2}}. \tag{6}$$

Obviously, in the case of $|J| = M^2 - Q^2/2$, there is $r_+ = r_-$, that is just the extremal limit of the black hole. At the outer event horizon r_+, an important thermodynamic quantity corresponding to the black hole is the Hawking temperature T_{H}, which can be calculated at the pole as

$$\begin{aligned} T_{\mathrm{H}} &= \frac{1}{2\pi}\lim_{r\to r_+}\sqrt{g^{rr}}\partial_r\sqrt{-g_{tt}}\bigg|_{\theta=0} \\ &= \frac{\sqrt{\left(2M^2 - Q^2\right)^2 - 4J^2}}{4\pi M[2M^2 - Q^2 + \sqrt{\left(2M^2 - Q^2\right)^2 - 4J^2}]}. \end{aligned} \tag{7}$$

The area of the outer event horizon is given by

$$A = 8\pi M\left(M - \frac{Q^2}{2M} + \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - \frac{J^2}{M^2}}\right), \tag{8}$$

by the area law, the entropy of the black hole can be expressed as

$$\begin{aligned} S &= \frac{1}{4}A \\ &= 2\pi M\left(M - \frac{Q^2}{2M} + \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - \frac{J^2}{M^2}}\right). \end{aligned} \tag{9}$$

Then we can obtain

$$M\left(S, Q, J\right) = \sqrt{\frac{4\pi^2J^2 + 2\pi Q^2S + S^2}{4\pi S}}. \tag{10}$$

The first law of black hole thermodynamics can be written as

$$\mathrm{d}M = T\mathrm{d}S + \Phi\mathrm{d}Q + \Omega_{\mathrm{H}}\mathrm{d}J, \tag{11}$$

where T, Φ and Ω_{H} are the temperature, electrostatic potential and angular velocity of the black hole respectively. Among them, the temperature can be expressed in terms of the mass, electric charge and angular momentum as

$$T = \frac{\partial M}{\partial S} = \frac{S^2 - 4\pi^2J^2}{4S\sqrt{4\pi^3J^3S + 2\pi^2Q^2S^2 + \pi S^3}}, \tag{12}$$

which is equal to the Hawking temperature and we can prove $T = T_{\mathrm{H}}$ by substituting Eq. (9) into Eq. (12). Obviously, it is always positive except at the extremal limit, namely $S = 2\pi J$. And then, we obtain the corresponding heat capacity which can be expressed in terms of the entropy, electric charge and angular momentum as

$$C_Q = \frac{\partial M}{\partial T} = T\frac{\partial S}{\partial T} = \frac{2S\left(S - 2\pi J\right)\left(S + 2\pi J\right)\left(4\pi^2J^2 + 2\pi Q^2S + S^2\right)}{48\pi^4J^4 + 32\pi^3J^2Q^2S + 24\pi^2J^2S^2 - S^4}, \tag{13}$$

we find that it vanishes when $S = 2\pi J$, which is considered to be the first-order phase transition point. It is generally believed that the Davies' point, where the second-order phase transition takes place, corresponds to the divergence point of the heat capacity, so it may indicate that the second-order phase transition takes place when the denominator goes to zero, that is

$$48\pi^4 J^4 + 32\pi^3 J^2 Q^2 S + 24\pi^2 J^2 S^2 - S^4 = 0. \quad (14)$$

3. Weinhold geometry of the Kerr–Sen black hole

Now, we turn to discuss thermodynamic geometry of the Kerr–Sen black hole. According to Weinhold's viewpoint, the metric can be defined as the second derivative of internal energy.[9] In general, that is

$$g^W_{ij} = \partial_i \partial_j M\left(S, N^a\right), \quad (15)$$

where the index W denotes the Weinhold geometry and M, S, N^a are the mass, entropy and any other extensive variables of a black hole, respectively. For the Kerr–Sen black hole, the line element of the Weinhold geometry can be written as

$$\begin{aligned} \mathrm{d}s^2_W =& \, g^W_{SS}\mathrm{d}S^2 + 2g^W_{SQ}\mathrm{d}S\mathrm{d}Q + 2g^W_{SJ}\mathrm{d}S\mathrm{d}J \\ &+ g^W_{QQ}\mathrm{d}Q^2 + 2g^W_{QJ}\mathrm{d}Q\mathrm{d}J + g^W_{JJ}\mathrm{d}J^2. \end{aligned} \quad (16)$$

Substituting Eq. (10) into Eq. (15), we can obtain the Weinhold metric components of the Kerr–Sen black hole as

$$g^W_{SS} = \frac{48\pi^4 J^4 + 32\pi^3 J^2 Q^2 S + 24\pi^2 J^2 S^2 - S^4}{8\sqrt{\pi S^5\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)^3}}, \quad (17)$$

$$g^W_{SQ} = g^W_{QS} = \frac{4\sqrt{\pi^5} J^2 Q - \sqrt{\pi} Q S^2}{2\sqrt{S\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)^3}}, \quad (18)$$

$$\begin{aligned} g^W_{SJ} &= g^W_{JS} \\ &= -\frac{4\sqrt{\pi^7} J^3 + 4\sqrt{\pi^5} J Q^2 S + 3\sqrt{\pi^3} J S^2}{\sqrt{S^3\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)^3}}, \end{aligned} \quad (19)$$

$$g^W_{QQ} = \frac{4J^2\sqrt{\pi^5 S} + \sqrt{\pi S^5}}{\sqrt{\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)^3}}, \quad (20)$$

$$g^W_{QJ} = g^W_{JQ} = -\frac{4JQ\sqrt{\pi^5 S}}{\sqrt{\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)^3}}, \quad (21)$$

$$g^W_{JJ} = \frac{4Q^2\sqrt{\pi^5 S} + 2\sqrt{\pi^3 S^3}}{\sqrt{\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)^3}}. \quad (22)$$

The Ricci curvature tensor and the curvature scalar are respectively expressed as

$$\Re_{\mu\nu} = \Gamma^\kappa_{\mu\nu,\kappa} - \Gamma^\kappa_{\mu\kappa,\nu} + \Gamma^\kappa_{\rho\kappa}\cdot\Gamma^\rho_{\mu\nu} - \Gamma^\kappa_{\rho\nu}\cdot\Gamma^\rho_{\mu\kappa}, \quad (23)$$

$$\Re = g^{\mu\nu}\Re_{\mu\nu}, \quad (24)$$

where the Christoffel symbols $\Gamma^\kappa_{\mu\nu}$ can be worked out by

$$\Gamma^\kappa_{\mu\nu} = \frac{1}{2}g^{\alpha\kappa}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right). \quad (25)$$

From Eqs. (17)–(25), we calculate the Weinhold curvature scalar which can be written as

$$\Re_W = \frac{5\sqrt{\pi S}}{\sqrt{4\pi^2 J^2 + 2\pi Q^2 S + S^2}}. \quad (26)$$

Obviously, it is always positive for any value of the entropy, electric charge and angular momentum and no singular point comes into being. However, it implies no information about the phase transition of the Kerr–Sen black hole.

4. Ruppeiner geometry of the Kerr–Sen black hole

Because Weinhold geometry carries no information about the phase transition point, it has no direct effect on the Kerr–Sen black hole, so we turn to do some researches on the Ruppeiner metric which relates to the Weinhold metric with the following relation:[10]

$$\mathrm{d}s^2_{\mathrm{R}} = \frac{1}{T}\mathrm{d}s^2_W. \quad (27)$$

With Eq. (12) and Eqs. (17)–(22), the Ruppeiner metric of the Kerr–Sen black hole can be worked out, we have

$$g^{\mathrm{R}}_{SS} = \frac{48\pi^4 J^4 + 32\pi^3 J^2 Q^2 S + 24\pi^2 J^2 S^2 - S^4}{2S\left(S^2 - 4\pi^2 J^2\right)\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)}, \quad (28)$$

$$g^{\mathrm{R}}_{SQ} = g^{\mathrm{R}}_{QS} = -\frac{2\pi Q S}{4\pi^2 J^2 + 2\pi Q^2 S + S^2}, \quad (29)$$

$$\begin{aligned} g^{\mathrm{R}}_{SJ} &= g^{\mathrm{R}}_{JS} \\ &= \frac{4\pi^2 J\left(4\pi^2 J^2 + 4\pi Q^2 S + 3S^2\right)}{\left(4\pi^2 J^2 - S^2\right)\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)}, \end{aligned} \quad (30)$$

$$g^{\mathrm{R}}_{QQ} = \frac{4\pi S^2\left(4\pi^2 J^2 + S^2\right)}{\left(S^2 - 4\pi^2 J^2\right)\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)}, \quad (31)$$

$$\begin{aligned} g^{\mathrm{R}}_{QJ} &= g^{\mathrm{R}}_{JQ} \\ &= -\frac{16\pi^3 J Q S^2}{\left(S^2 - 4\pi^2 J^2\right)\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)}, \end{aligned} \quad (32)$$

$$g^{\mathrm{R}}_{JJ} = \frac{8\pi^2 S^2\left(2\pi Q^2 + S\right)}{\left(S^2 - 4\pi^2 J^2\right)\left(4\pi^2 J^2 + 2\pi Q^2 S + S^2\right)}. \quad (33)$$

Substituting them into Eqs. (23)–(25), we obtain the Ruppeiner curvature scalar, which can be expressed in terms of the entropy, electric charge and angular momentum as

$$\Re_{\mathrm{R}} = \frac{36\pi^2 J^2 S + S^3}{\left(4\pi^2 J^2 + S^2\right)\left(2\pi J + S\right)\left(2\pi J - S\right)}. \quad (34)$$

Obviously, this scalar curvature goes to positive infinity when $S = 2\pi J$, from which we can obtain a divergence of the curvature occuring along $|J| = M^2 - Q^2/2$, namely at $r_+ = r_-$, which is an extremal limit of this black hole.

5. Legendre invariant metric of the Kerr–Sen black hole

In the previous section, we show that the curvature scalar of Weinhold curvature implies no information about the phase transition of the Kerr–Sen black hole while the Ruppeiner one does. But they both fail to give the information about the second-order phase transition point. Nevertheless, a geometric approach proposed by Quevedo *et al.* can solve this problem. Considering the Legendre invariance, they presented a unified geometry and the metric structure can provide good descriptions of various types of black hole thermodynamics.[13−15] Now, we extend their work to the Kerr–Sen black hole and make sure whether it works. The Legendre invariant metric of the Kerr–Sen black hole can be expressed as

$$g=\left(S\frac{\partial M}{\partial S}+Q\frac{\partial M}{\partial Q}+J\frac{\partial M}{\partial J}\right)\begin{pmatrix}-\dfrac{\partial^2 M}{\partial S\partial S} & 0 & 0\\ 0 & \dfrac{\partial^2 M}{\partial Q\partial Q} & \dfrac{\partial^2 M}{\partial Q\partial J}\\ 0 & \dfrac{\partial^2 M}{\partial Q\partial J} & \dfrac{\partial^2 M}{\partial J\partial J}\end{pmatrix}. \tag{35}$$

Substituting Eq. (10) into Eq. (35) we obtain

$$g_{SS}=-\frac{\left(4\pi^2J^2+4\pi Q^2S+S^2\right)\left(48\pi^4J^4+32\pi^3J^2Q^2S+24\pi^2J^2S^2-S^4\right)}{32\pi S^3\left(4\pi^2J^2+2\pi Q^2S+S^2\right)^2}, \tag{36}$$

$$g_{QQ}=\frac{\left(4\pi^2J^2+S^2\right)\left(4\pi^2J^2+4\pi Q^2S+S^2\right)}{4\left(4\pi^2J^2+2\pi Q^2S+S^2\right)^2}, \tag{37}$$

$$g_{QJ}=g_{JQ}=\frac{16\pi^4Q^4S^2J-16\pi^6J^5-8\pi^4J^3S^2-\pi^2S^4J}{\left(4\pi^2J^2+2\pi Q^2S+S^2\right)^3}, \tag{38}$$

$$g_{JJ}=\frac{\left(2\pi^2Q^2+\pi S\right)\left(4\pi^2J^2+4\pi Q^2S+S^2\right)}{2\left(4\pi^2J^2+2\pi Q^2S+S^2\right)^2}. \tag{39}$$

Obviously, the metric components are rather cumbersome which cannot give a compact form of the scalar curvature. In order to avoid complicated calculation, the scalar curvature of the Legendre invariant metric can be expressed as

$$\Re_L=\frac{N_\Re}{D_\Re},\quad D_\Re=4\left(S\frac{\partial M}{\partial S}+Q\frac{\partial M}{\partial Q}+J\frac{\partial M}{\partial J}\right)^3\left[\left(\frac{\partial^2 M}{\partial Q\partial J}\right)^2-\frac{\partial^2 M}{\partial Q\partial Q}\cdot\frac{\partial^2 M}{\partial J\partial J}\right]^3\left(\frac{\partial^2 M}{\partial S\partial S}\right)^2. \tag{40}$$

Substituting Eq. (10) into the denominator $D_\Re$, we can obtain

$$D_\Re=-\frac{\sqrt{\pi^7}\left(4\pi^2J^2+4\pi Q^2S+S^2\right)^3\left(48\pi^4J^4+32\pi^3J^2Q^2S+24\pi^2J^2S^2-S^4\right)^2}{128\sqrt{S\left(4\pi^2J^2+2\pi Q^2S+S^2\right)^{21}}}, \tag{41}$$

whereas the numerator $N_\Re$ is too complicated to express in a compact form and not required to look for the points of divergence of $\Re_{\rm L}$. Obviously, a divergence of the scalar curvature occurs along $D_\Re=0$ besides $N_\Re\neq 0$, namely at $48\pi^4J^4+32\pi^3J^2Q^2S+24\pi^2J^2S^2-S^4=0$, which is the same as Eq. (14) and consists with the second-order phase transition point reproduced from the capacity.

6. Conclusions

This paper has studied the thermodynamic properties of the Kerr–Sen black hole from the viewpoint of geometry. The results show that the temperature of the Kerr–Sen black hole is equal to its Hawking temperature. In addition, we find that both the Weinhold geometry and Ruppeiner geometry of the black hole are curved, the scalar curvature of the Weinhold geom-

etry implies no information about the phase transition of the black hole while the Ruppeiner one does. But they both carry no information about the second-order phase transition point. In addition, we also studied the Legendre invariant metric of the Kerr–Sen black hole and found that its scalar curvature can give the information about the second-order phase transition point reproduced from the capacity.

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