Post-post-Newtonian deflection of light ray in multiple systems with PPN parameters*

Gong Yan-Xiang(宫衍香)† and Wu Xiao-Mei(吴晓梅)

College of Physics and Electronic Engineering, Taishan University, Tai'an 271021, *China*

(Received 16 April 2010; revised manuscript received 19 October 2010)

The post-Newtonian scheme in multiple systems with post-Newtonian parameters presented by Klioner and Soffel is extended to the post-post-Newtonian (PPN) order for light propagation problem in the solar system. Under considering the solar system experiment requirement, a new parameter ε is introduced. This extension does not change the virtue of the scheme on the linear partial differential equations of the potential and vector potential mentioned in previous work. Furthermore, this extension is based on the former work done by Richter and Matzner in one global system theory. As an application, we also consider the deflection of light ray in the global coordinates. And the deflection angle of light ray is obtained with post-Newtonian parameters.

Keywords: post-Newtonian approximation, deflection of light ray, multiple systems, parameterized metric

PACS: 04.25.Nx, 04.30.Nk, 95.45.+i, 04.80.Cc **DOI:** 10.1088/1674-1056/20/2/020403

1. Introduction

In some space missions advanced or implemented today, such as LISA (laser interferometer space antenna),[1,2] SIM (space interferometer mission),[3] GAIA (global astrometric interferometer for astrophysics)[4] and so on, the light propagation must be calculated to a more precise level, which is the post-post-Newtonian (2PN) level in solar system. Furthermore, in all of the discussion on the motion of artificial satellites, minor planets and photons in the solar system and on the astronomical measurements of the very long baseline interferometer (VLBI),[5] the global positioning system (GPS)[6] and so on, the orbital equation of the photons should also be taken into account to achieve a higher accuracy.[7,8] Therefore, the second-order contribution to light propagation is required to be considered. If we consider the post-post-Newtonian (PPN) problem, the post-Newtonian (1PN) quadrupole must be taken into account. When we calculate relativistic multipole moments and astrodynamics (equations of motion for planets), we must use the Damour–Soffel–Xu (DSX) scheme in which a complete 1PN general relativistic celestial mechanics for N arbitrarily composed and shaped, rotating deformable bodies is described.[9−12] In Ref. [7], the post-Newtonian scheme in multiple systems was extended to the second order contribution to light propagation without changing the virtue of the scheme on the linear partial differential equations of the potential and vector potential. The spatial components of the metric are extended to the second-order level both in a global coordinates (q_{ij}/c^4) and all local coordinates ($Q_{\rm ab}/c^4$). The equations of q_{ij} (or $Q_{\rm ab}$) are obtained from the field equations. The relationship between q_{ij} and $Q_{\rm ab}$ is also presented. In the special case of the solar system, where isotropic condition is applied ($q_{ij} = \delta_{ij}q$), the solution of q was obtained. Actually, this extension can also be used in the parametrized post-post-Newtonian (2PPN) formalism, and will have wide applications in the relativistic experiments and observation.[13,14] The DSX scheme established in 1994 did not consider the PN parameters. Until 2000, the relativistic celestial mechanics with PN parameters (or the parameterized DSX scheme (1PN)) was obtained by Klioner and Soffel, the parameterized metric was written as[13]

$$g_{00} = -1 + \frac{2w}{c^2} - \frac{2\beta w^2}{c^4} + O\left(6\right), \tag{1}$$

$$g_{0i} = -\frac{2\left(1+\gamma\right)w_i}{c^3} + O(5), \tag{2}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 10674099), the National Science Foundation for Young Scientists of China (Grant No. 10925313), and the Shandong Provincial Natural Science Foundation, China (Grant No. ZR2010AQ023).

†Corresponding author. E-mail: yxgong@sina.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

$$g_{ij} = \delta_{ij}\left(1 + \frac{2\gamma w}{c^2}\right) + O(4). \tag{3}$$

Here, β and γ are two PN parameters. How to extend this parameterized metric to the post-post-Newtonian order for the light propagation in solar systems and how to calculate the deflection angle in the global system are two important problems. Because the values of these parameters can be obtained in the space experiments, by comparing them with their corresponding theoretical values, the gravitation theory can be checked. For example, the values of β and γ are equal to 1 in the general relativity.

2. The 2PN metric tensor with PN parameters to light propagation

Now we extend the above metric to the post-post-Newtonian accuracy to light ray, namely, extend only g_{ij} to $O(4)$ (small quantities which contain c^{-4}). First of all, we find how many PN parameters are needed at least to satisfy solar system experiment. In the book "Theory and Experiment in Gravitational Physics",[15] Will has mentioned that five basic quantities are involved in the gravitational experiments inside the present solar system, they are β, γ, ε, W and J_2. Here, β, γ, and ε are PN parameters, and W and J_2 are two crucial parameters but not PN parameters. Actually, they or their combination needs to be measured from observational datum. Now there are four fundamental experiments to test whether the gravitational theory is correct. They are the time delay test, deflection of light ray, the precession of perihelion of Mercury, and the Lunar Eötvös experiments (Nordtvedt effect). Parameter W is related to the violation of the weak equivalence principle, although general relativity requires both the weak equivalence principle and the strong equivalence principle, but in some of the alternative gravitational theories the equivalence principle may not be valid, e.g., in the scalar-tensor theories the strong equivalence principle is violated. The degree of the violation of the equivalence principles (both strong equivalent principle and weak equivalent principle) may be expressed in the following formula:[16]

$$\frac{M_{\rm G}}{M_{\rm I}} = 1 + \eta\frac{E_{\rm G}}{Mc^2} + W, \tag{4}$$

in which $\eta = 4\beta - \gamma - 3$, $M_{\rm G}$ is the gravitational mass, $M_{\rm I}$ is the inertial mass, and $E_{\rm G}$ is the gravitational binding energy. The parameter η comes from the partial conservation theory of post-Newtonian theory which means the violations of the strong equivalence principle in the post-Newtonian equation of motion. Otherwise, the parameter W means the violations of weak equivalence principle. Considering Lunar Eötvös experiments (Nordtvedt effect) the relative difference between the Earth's and the Moon's accelerations toward the Sun is taken into account as ηW. The deflection of light ray does not involve this parameter either. Another parameter J_2 is the quadrupole moment of celestial body, which is not PN parameter either. J_2 comes from the multipolar expansion of gravitational potential and it is related to body oblateness. According to the above analysis there are at least three post-Newtonian parameters needed in the deflection of light ray. In Ref. [7], the author brought forward a kind of parameterized 2PN metric. As a tentative form of parameterized 2PN metric, there are five post-Newtonian parameters, i.e., β, γ, ε, δ and ξ. As we mentioned above, there are only three basic post-Newtonian parameters needed in the gravitational experiments inside the present solar system. In order to obtain a simplest parameterization scheme, we modified the parameterized metric in Ref. [17] and took

$$g_{ij} = \delta_{ij}\left(1 + \frac{2\gamma w}{c^2}\right) + \delta_{ij}\frac{\varepsilon}{c^4}\left(2w^2 + \frac{q}{3}\right) + \frac{\eta}{c^4}q_{\langle ij\rangle} + O(6). \tag{5}$$

Here, $q_{ij} = \delta_{ij}q/3 + q_{\langle ij\rangle}/c^4$ for the non-parameterized form, the coefficient $1/3$ comes from the tensor symmetric trace free (STF) and the symbol '$\langle\ \rangle$' is the STF symbol,[9] and η is another post-Newtonian parameter to symbolize the spatial anisotropic contribution in 2PN order. To be consistent with Ref. [7], we take $q_{ij} = \delta_{ij}q$ in this paper. So considering former work from Richter and Matzner (the expansion in one single system theory),[18] we give our extension for g_{ij} in the multiple systems as follows:

$$g_{ij} = \delta_{ij}\left(1 + \frac{2\gamma w}{c^2}\right) + \delta_{ij}\frac{\varepsilon}{c^4}\left(2w^2 + q\right) + O(6). \tag{6}$$

Here, we have considered only the light propagation problem in the solar system. What to be emphasized is that equation (6) does not violate all the content of Ref. [7]. In the special case of the solar system, we have $q_{\langle ij\rangle}/c^4 = 0$, because if the origin of the coordinate system is located in the centre of the solar (dipole always equals zero), the quadrupole terms (and higher multipole moments) in 2PN contribution to $Q_{\rm ab}$ can

be ignored, i.e. $Q_{\rm ab}=\delta_{ij}Q$, and in the global coordinates of the Sun $q_{ij}=\delta_{ij}q$. As for other local coordinates of every planet 2PN contribution is too small ($10^{-19}\sim 10^{-18}$) to be calculated.[7] When $\gamma=\beta=1$ and $\varepsilon=3/4$, the metric tensor returns to the case in general relativity (for further details see Ref. [8]). But in Ref. [8], the author took $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$ as the Minkowki flat space metric. Here, we take $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$ in order to be consistent with Refs. [7] and [9]–[12] (the DSX scheme). The difference of the coefficients before q is insignificant because they can be written in the definition of the gravitational potential function of q.

The g_{ij} introduced by Richter and Matzner in a single coordinate system is written as[18]

$$g_{ij}=\delta_{ij}\left(1+2\gamma U+\frac{3}{2}\Lambda U^2+Q\right)+O\left(6\right). \tag{7}$$

Here, $c=1$ and Q is the sum of other possible terms of $O(4)$ except $3\Lambda U^2/2$, and U is the Newtonian potential. Therefore, the parameter Λ introduced by them is corresponding to our ε in Eq. (6). If we ignore all of $1/c^4$ terms in g_{ij}, our results are in agreement with those of Klioner and Soffel.[13] On the other hand, this extension does not change the virtue of the scheme for the linear partial differential equations of the potential and vector potential in the DSX scheme (for further details see Ref. [7]).

3. Deflection of light ray in global coordinates

As an application of the parametrized 2PN metric, we consider the deflection of light ray in the global coordinates. Since in solar system the gravitational field produced by a main gravitational source (Sun) with centric mass M, the scalar potential w and the vector potential w_i of the gravitational field are, respectively,[8]

$$w=\frac{GM}{r}\left(1-\frac{K\left(3\cos^2\vartheta-1\right)}{2r^2}\right), \tag{8}$$

$$w_i=\frac{GM}{2r^3}(\boldsymbol{r}\times\boldsymbol{a})_i, \tag{9}$$

in which $K=J_2R^2$, R and J_2 are the radius and the quadrupole-moment of the stellar object (Sun) respectively, $\boldsymbol{a}$ is the angular momentum per unit mass (we assume that the directions of all parts of angular momentum are the same (i.e. rotational axis), $\boldsymbol{r}$ and ϑ are the radial vector of a given field point and its inclined angle to the rotational axis respectively. In the spherical coordinate system (r, θ, ϕ) with the stellar mass-centre taken as origin and the rotational axis taken as the Z axis (so that $\theta=\vartheta$), the 2PN line-element is simplified into

$$\begin{aligned}\mathrm{d}s^2 &= g_{tt}\mathrm{d}t^2+g_{rr}\left(\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\phi^2\right)\\ &\quad+2g_{t\phi}\sin\theta\mathrm{d}\phi\mathrm{d}t,\end{aligned} \tag{10}$$

in which

$$g_{tt}=c^2\left(-1+\frac{2w}{c^2}-\frac{2\beta w^2}{c^4}\right), \tag{11}$$

$$g_{rr}=1+\frac{2\gamma w}{c^2}+\frac{\varepsilon}{c^4}\left(2w^2+q\right), \tag{12}$$

$$g_{t\phi}=-\frac{1+\gamma}{2}\frac{R_g a_\theta}{r} \tag{13}$$

with $R_g=2GM/c^2$ being the Schwarzchild radius of the stellar object, a_θ is the component of $\boldsymbol{a}$ in the $\boldsymbol{e}_\theta$-direction, g_{tt} and g_{rr} are functions of (r, θ).

Let the Lagrange function be[8]

$$\begin{aligned}L &= \left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2\\ &= g_{tt}\dot{t}^2+g_{rr}\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right)\\ &\quad+2g_{t\phi}\sin\theta\dot{\phi}\dot{t},\end{aligned} \tag{14}$$

in which the symbol '$\cdot$' denotes the derivative with respect to the affine parameter λ. According to the Hamilton theorem, when the Lagrange function is only a function of x^μ and $\dot{x}^\mu$, the Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial\dot{x}^\mu}\right)-\frac{\partial L}{\partial x^\mu}=0. \tag{15}$$

We consider only the orbit of photon lying in the plane included in the rotational axis to be taken as OZ axis, then the Lagrange function of photon is independent of the coordinates t and ϕ. Then it follows that $\partial L/\partial\dot{t}$ and $\partial L/\partial\dot{\phi}$ are constants, and the following two simple integrals exist:

$$g_{tt}\dot{t}+g_{t\phi}\sin\theta\dot{\phi}=E, \tag{16}$$

$$-g_{rr}r^2\sin^2\theta\dot{\phi}-g_{t\phi}\sin\theta\dot{t}=J. \tag{17}$$

Here, E and J are the integral constants. From Eqs. (16) and (17), we can solve the equation

$$\dot{t}=\frac{g_{rr}r^2E+g_{t\phi}J}{g_{t\phi}^2-g_{tt}g_{rr}r^2},\quad\text{and}\quad\dot{\phi}=\frac{g_{t\phi}E+g_{tt}J}{g_{t\phi}^2-g_{tt}g_{rr}r^2}. \tag{18}$$

Considering θ component, the Lagrange equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2g_{rr}r^2\dot{\theta}\right)$$

$$
\begin{aligned}
&= \left(1-\frac{2\beta w}{c^2}\right)\frac{3GMK}{r^3}\sin 2\theta\dot{t}^2 \\
&+ \left(\frac{\gamma}{c^2}+\frac{2\varepsilon w}{c^4}\right)\frac{3GMK}{r^3} \\
&\times \sin 2\theta\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right) \\
&+ g_{rr}r^2\sin 2\theta\dot{\phi}^2+2g_{t\phi}\cos\theta\dot{\phi}\dot{t}. \qquad (19)
\end{aligned}
$$

For the motion of photons, from $\mathrm{d}s^2=0$ we can further obtain

$$
\begin{aligned}
&g_{tt}\dot{t}^2+g_{rr}\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right) \\
&+2g_{t\phi}\sin\theta\dot{\phi}\dot{t}=0. \qquad (20)
\end{aligned}
$$

Next, we discuss the special case of $\theta=\pi/2$. In this case, the right-hand side of Eq. (19) is zero, i.e., $g_{rr}r^2\dot{\theta}$ is a constant. So if the photon is initially in the plane of $\theta=\pi/2$ and has $\dot{\theta}=0$, then it will always stay in this plane. Inserting Eqs. (18) into Eq. (20), writing $u=1/r$ and omitting the high-order small quantities $O(c^{-6})$, we obtain the 2-dimensional orbit equation

$$
\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2+u^2=\left(\frac{E}{cJ}\right)^2\left[1+\frac{2\left(1+\gamma\right)w}{c^2}-\frac{2\left(\beta-\varepsilon-2\gamma-2\right)w^2}{c^4}-\frac{2\left(1+\gamma\right)GMua_\theta E}{c^4J}+\frac{\varepsilon q}{c^4}\right]. \qquad (21)
$$

Here, E and J are conserved quantities, and they can be determined by their values at $r\to\infty$. As the gravitational field disappears at infinity, E and J are the energy and angular momentum of photon respectively, and $\boldsymbol{P}$ is the momentum of photon. Thus we have[8]

$$
\frac{E}{cJ}=\frac{cP_\infty}{c\left|\boldsymbol{r}_\infty\times\boldsymbol{P}_\infty\right|}=\frac{P_\infty}{\left|r_\infty\times\sin\phi_\infty\right|}=\frac{1}{R_s}, \qquad (22)
$$

where R_s is the closest distance from the mass centre of the stellar object to the initial photon orbit extended to infinity, i.e., the so-called collision parameter. For $\theta=\pi/2$, we have

$$
w=GMu+\frac{1}{2}GMKu^3, \quad a_\theta=-a. \qquad (23)
$$

Inserting the above relations into Eq. (21), then differentiating, we obtain the standard differential equation of the photon orbit

$$
\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\frac{\left(1+\gamma\right)R_g}{2R_s^2}\left(1+\frac{a}{cR_s}\right)-\frac{\left(\beta-\varepsilon-2\gamma-2\right)}{2}\left(\frac{R_g}{R_s}\right)^2u+\frac{3\left(1+\gamma\right)KR_g}{4R_s^2}u^2+\frac{\varepsilon q_{,u}}{2c^4R_s^2}. \qquad (24)
$$

As for the last term on the right-hand side of Eq. (24), when $q_{,u}$ is a constant, this term can be merged into the first term, and when q is a constant, it is zero. Neglecting this term, we derive the deflection angle of light ray. When $M=0$ ($R_g=0$), its solution will degenerate into a linear equation, i.e., the zero order approximation solution is $u^{(0)}=\cos\phi/R_s$. Using the iterative method and the boundary condition ($r=\infty$), the deflection angle is deduced as follows:

$$
\begin{aligned}
\delta &= 2\left(\phi-\frac{\pi}{2}\right) \\
&= \frac{\left(1+\gamma\right)R_g}{R_s}\left(1+\frac{a}{cR_s}\right)\left[1-\frac{\left(\beta-\varepsilon-2\gamma-2\right)}{2}\left(\frac{R_g}{R_s}\right)^2\right]-\frac{\left(\beta-\varepsilon-2\gamma-2\right)\pi}{4}\left(\frac{R_g}{R_s}\right)^2 \\
&+ \frac{\left(1+\gamma\right)J_2R_gR^2}{R_s^3}. \qquad (25)
\end{aligned}
$$

4. Discussion

The first term on the right-hand side of Eq. (25) accords with the results from the Schwarzchild metric and 1PN metric; the second term is the 2PN correction to the spherically symmetric gravitational field; the third term is the contribution from the angular momentum of the gravitational source: when a is negative the deflection angle is decreasing, when a is positive the deflection angle is increasing; the last term originates mainly from the quadrupole moment of the gravitational source, which causes the deflection angle to increase. Furthermore, when the higher terms in Eq. (25) are ignored, our result is consistent with

the Kerr deflection, see Ref. [19].

In short, this parameterization work can reach the following four points. (i) In meeting the needs of the solar system gravity experiments, we have given a simplest parameterization scheme. (ii) It does not violate the requirements in Ref. [7]. (iii) It can be self-consistent with the extended work in one single system theory done by Richter and Matzner. (iv) The 2PN deflection angle of photon which contains three basic post-Newtonian parameters is deduced for the first time in this paper and it can be directly verified in space experiments. Future deep space laser ranging missions make the gravity theory such as gravitational waves (see Refs. [20]–[23]) testable. The 2PPN metric in the multiple systems has the great advantage of linear superposition. Using the results of this paper, the motion of photons in the multiple gravitational sources can be considered, and this is something that the Kerr metric field can not do.

Acknowledgement

We thank professor Xu Chong-Ming of the Shanghai Astronomical Observatory for his beneficial advice.

References

[1] Perryman M A C, de Boer K S, Gilmore G, Høg E, Lattanzi M G, Lindegren L, Luri X, Mignard F, Pace O and de Zeeuw P T 2001 *Astron. Astrophys.* **369** 339

[2] Bander P L 2009 *Proc. IAU Symp.* **261** 240

[3] Makarov V V 2009 *Proc. IAU Symp.* **261** 345

[4] Lindegren L 2009 *Proc. IAU Symp.* **261** 296

[5] Kramer M 2009 *Proc. IAU Symp.* **261** 366

[6] Ashby N and Nelson R A 2009 *Proc. IAU Symp.* **261** 22

[7] Xu C M and Wu X J 2003 *Chin. Phys. Lett.* **20** 195

[8] Xiao Y, Fei B J, Sun W J and Ji C X 2008 *Chin. Astron. Astrophys.* **32** 423

[9] Damour T, Soffel M and Xu C 1991 *Phys. Rev.* D **43** 3273

[10] Damour T, Soffel M and Xu C 1992 *Phys. Rev.* D **45** 1017

[11] Damour T, Soffel M and Xu C 1993 *Phys. Rev.* D **47** 3124

[12] Damour T, Soffel M and Xu C 1994 *Phys. Rev.* D **49** 618

[13] Klioner S A and Soffel M H 2000 *Phys. Rev.* D **62** 024019

[14] Qin B, Wu X and Zou Z 1997 *Chin. Phys. Lett.* **14** 155

[15] Will C 1993 *Theory and Experiment in Gravitational Physics* (Cambridge: Cambridge University Press) pp. 88–90

[16] Damour T 1999 *arXiv:gr-qc/*9904057

[17] Xu C M, Gong Y X, Wu X J, Soffel M and Klioner S 2005 *arXiv:gr-qc/*0510074

[18] Richter G W and Matzner R A 1982 *Phys. Rev.* D **26** 1219

[19] Edery A and Godin J 2006 *Gen. Relat. Grav.* **38** 1715

[20] Zeng X X, Yang S Z and Chen D Y 2008 *Chin. Phys.* B **17** 1629

[21] Peng J J and Wu S Q 2008 *Chin. Phys.* B **17** 825

[22] Li J, Li F Y and Zhong Y H 2009 *Chin. Phys.* B **18** 922

[23] Chen J H and Wang Y J 2007 *Chin. Phys.* **16** 3212