Comments on "Inverse scattering method and soliton solution family for the Einstein–Maxwell theory with multiple Abelian gauge fields"

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This paper points out that equations (18a) and (18b) in Ref. [7] [Gao Y J 2008 *Chin. Phys.* B **17** 3574] only possess the solutions $M = \pm\rho\tilde{\gamma}\varepsilon$. So, there does not exist the so-called soliton solution family for the Einstein–Maxwell theory with multiple Abelian gauge fields shown in Ref. [7].

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1. Introduction and review

The stationary axisymmetric (SAS) problem of the gravitational theory has been greatly studied.[1−6] When there exists the p Maxwell (EM-p) field, the essentially complete set of the SAS EM-p field equations can be written as[7−9]

$$\mathrm{d}(\rho^{-1}M\gamma^{*}\mathrm{d}M) = 0, \tag{1}$$

where the two-dimensional dual operation "$*$" is defined by ${}^{*}\mathrm{d}x = \mathrm{d}y$ and ${}^{*}\mathrm{d}y = -\mathrm{d}x$, the $(p+2)\times(p+2)$ ordinary complex matrix function $M = M(x;y)$ satisfies

$$M^{\dagger} = M, \tag{2}$$

$$M\gamma M = \rho^{2}\gamma, \tag{3}$$

$$\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{i}\,I_p & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{4}$$

with I_p being the $p\times p$ unit matrix. The matrix M is an ordinary complex matrix function related to an ordinary complex Ernst potential and the p ordinary complex electromagnetic potentials.

According to Ref. [7], introducing the matrix twist potential $Q = Q(x,y)$ with $(x,y) = (\rho,z)$ by $\mathrm{d}Q = -\rho^{-1}M\gamma^{*}\mathrm{d}M$, then one has $Q - Q^{\dagger} = -2z\gamma$, where z is introduced by ${}^{*}\mathrm{d}\rho = \mathrm{d}z$ with $\mathrm{d}^{*}\mathrm{d}\rho = 0$. Defining the so-called extended hyperbolic complex (EHC) matrix potential as

$$H = M + \varepsilon Q \tag{5}$$

with ε being the so-called extended hyperbolic imaginary unit satisfying $\varepsilon^2 = 1$ ($\varepsilon \neq \pm 1$),[7] then one can obtain the equation for H (see also Eq. (7) in Ref. [7]) as

$$2(z+\rho^{*})\mathrm{d}H = (H + H^{\ddagger})\Omega\mathrm{d}H, \tag{6}$$

where $\Omega = \varepsilon\gamma$ and $H^{\ddagger} = (H^{\dagger})^{\star}$ with $\dagger$ being the Hermitian conjugation operation and $\star$ denoting the so-called EHC conjugation.

Introducing the ordinary complex parameter w, then equation (6) is rewritten as[8,9]

$$w\mathrm{d}H = A(w)\Gamma(w), \tag{7}$$

where

$$A(w) = w - (H + H^{\ddagger})\Omega, \tag{8}$$

$$\Gamma(w) = w\Lambda^{-1}(w)\mathrm{d}H, \tag{9}$$

$$\Lambda(w) = w - 2(z+\rho^{*}). \tag{10}$$

According to Ref. [7], from Eq. (6) and Eqs. (8)–(10) one can obtain (see Eq. (12) in Ref. [7])

$$\mathrm{d}\Gamma(w) = \Gamma(w)\Omega\wedge\Gamma(w), \tag{11}$$

where "$\wedge$" denotes the exterior product of differential forms. Equation (11) is the complete compatibility condition of the following EHC linear differential equation (see Eq. (13) in Ref. [7])

$$\mathrm{d}F(w) = F(w)\Omega F(w), \tag{12}$$

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where $F(w) = F(x, y; w; \varepsilon)$ satisfies the following conditions (see Eqs. (15a)–(16b) in Ref. [7])

$$F^{(0)} = I, \tag{13}$$

$$F^{(1)} = H\Omega, \tag{14}$$

$$\lambda(w)F(w)^{+}\Omega F(w) = w\Omega, \tag{15}$$

$$F(w)^{\times}\Omega A(w)F(w) = w\Omega, \tag{16}$$

with + and × being defined by $F(w)^{+} = F(\overline{w})^{\dagger}$ and $F(w)^{\times} = F(\overline{w})^{\ddagger}$ and $\overline{w}$ denoting the complex conjugate of w. Equation (12) with $F(w)$ satisfying Eqs. (13)–(16) may be called the so-called HE-type EHC linear system for the EM-p theories with parameter t being replaced by w^{-1}.[8] The $F(w)$ is assumed to take the following form (see Eq. (14) in Ref. [7])

$$F(w) = F^{(0)}(\rho, z, \varepsilon) + w^{-1}F^{(1)}(\rho, z, \varepsilon) + \cdots, \tag{17}$$

and $A(w)$ is a linear function of w.

Introducing the complex variables $\eta = z + \mathrm{i}\,\rho$ and $\xi = z - \mathrm{i}\,\rho$, then one can rewrite Eq. (6) as (see Eqs. (18a) and (18b) in Ref. [7])

$$[2\eta - (H + H^{\ddagger})\Omega]\partial_{\eta}H = 0, \tag{18}$$

$$[2\xi - (H + H^{\ddagger})\Omega]\partial_{\xi}H = 0. \tag{19}$$

We find that equations (18) and (19) have only the two simple solutions and thus there is not the soliton solution family based on the HE-type EHC linear system shown in Ref. [7].

2. Analysis on the fundamental equations in Ref. [7]

According to Ref. [7], equation (6) has been rewritten as Eqs. (18) and (19). Putting $Q - Q^{\dagger} = -2z\gamma$ in Eqs. (18) and (19) gives

$$(\mathrm{i}\,\rho - M\Omega)\partial_{\eta}H = 0, \tag{20}$$

$$(\mathrm{i}\,\rho + M\Omega)\partial_{\xi}H = 0. \tag{21}$$

In the following, let us analyse the solutions of Eqs. (20) and (21). Assuming that $M\Omega \neq \pm\,\mathrm{i}\,\rho$, then we can reduce Eqs. (20) and (21) to

$$\partial_{\eta}H = 0, \tag{22}$$

$$\partial_{\xi}H = 0. \tag{23}$$

Equations (22) and (23) require that H must be a constant matrix. Note that the constant matrix M (corresponding to a constant matrix H) does not satisfy the condition (3). So, the possible solutions of Eqs. (20) and (21) are

$$M\Omega = \pm\,\mathrm{i}\,\rho = \pm\frac{\eta - \xi}{2}. \tag{24}$$

Putting $M\Omega = (\eta - \xi)/2$ in Eq. (21) leads to

$$2\,\mathrm{i}\,\rho\left(\frac{1}{2}\varepsilon\gamma + \varepsilon\partial_{\xi}Q\right) = 0, \tag{25}$$

namely,

$$\partial_{\xi}Q = -\frac{1}{2}\gamma, \tag{26}$$

which yields

$$Q = -\frac{1}{2}\gamma\xi + Q_0(\eta), \tag{27}$$

where $Q_0(\eta)$ is determined by $Q - Q^{\dagger} = -(\eta + \xi)\gamma$. Putting $M\Omega = -(\eta - \xi)/2$ in Eq. (20) gives

$$2\,\mathrm{i}\,\rho\left(\frac{1}{2}\varepsilon\gamma + \varepsilon\partial_{\xi}Q\right) = 0, \tag{28}$$

which yields

$$Q = -\frac{1}{2}\gamma\eta + \tilde{Q}_0(\xi), \tag{29}$$

where $\tilde{Q}_0(\xi)$ is determined by $Q - Q^{\dagger} = -(\eta + \xi)\gamma$.

The solutions of Eqs. (18) and (19) may be written as

$$M = \mp\,\mathrm{i}\,\rho\gamma\varepsilon = \pm\rho\tilde{\gamma}\varepsilon \tag{30}$$

with

$$\tilde{\gamma} = \begin{pmatrix} 0 & 0 & -\,\mathrm{i} \\ 0 & I_p & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}. \tag{31}$$

The matrix operation † on M shown in Eq. (30) yields

$$M^{\dagger} = \pm\rho\varepsilon^{\dagger}\tilde{\gamma}^{\dagger} = \pm\rho\varepsilon\tilde{\gamma}, \tag{32}$$

where the two relations $\varepsilon^{\dagger} = \bar{\varepsilon} = \varepsilon$ and

$$\tilde{\gamma}^{\dagger} = \tilde{\gamma} \tag{33}$$

have been used. Note that the orders of ε and $\tilde{\gamma}$ on the right-hand sides of Eqs. (30) and (32) are different. Assuming that $\varepsilon\tilde{\gamma} = \tilde{\gamma}\varepsilon$ (or $\varepsilon\,\mathrm{i} = \mathrm{i}\,\varepsilon$), then we have $M^{\dagger} = M$.

3. Remarks and conclusions

Whether the relation $\varepsilon \mathrm{i} = \mathrm{i}\varepsilon$ holds for the ordinary imaginary unit i and the extended hyperbolic imaginary unit ε is still a question. In Eq. (30) there are both i and ε. The physical meaning of such a matrix is unclear since M should be an ordinary complex matrix.

Equation (46) in Ref. [7] shows the N-soliton EHC H-potentials, H_N, with the undetermined ordinary complex constants a_k and c_k, the ordinary complex constant column vector b_k and μ_k. The purpose of Ref. [7] is to obtain the solutions of Eqs. (18a) and (18b) which are equivalent to Eq. (7) in Ref. [7]. Whether equaltion (46) in Ref. [7] can provide the new solution of Eq. (7) in Ref. [7] depends on the values of a_k, c_k and μ_k, as well as the form of b_k. In other words, one should substitute H_N into Eq. (7) in Ref. [7] to check it.

In fact, equations (18a) and (18b) in Ref. [7] have only the solutions shown in Eq. (30). It is clear that the procedure generating the soliton solution given in Ref. [7] does not work since the solutions of Eqs. (18a) and (18b) in Ref. [7] must take the forms shown in Eq. (30).

From the above analyses, we conclude that there does not exist the soliton solution family based on the HE-type EHC linear system shown in Ref. [7] for the Einstein–Maxwell theory with multiple Abelian gauge fields.

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